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ON THE CARTIER DUALITY OF CERTAIN FINITE GROUP SCHEMES OF TYPE (p^n, p^n)

By

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Abstract. In this paper we show that the finite subgroup scheme $\operatorname{Spec} A[X, Y]/(X^{p^l}, Y^{p^l})$ of $\mathscr{E}^{(\lambda,\mu,D)} \in \operatorname{Ext}^1(\mathscr{G}^{(\lambda)}, \mathscr{G}^{(\mu)})$ is a Cartier dual of a certain finite subgroup scheme of the fiber product $W_{l,A} \times_{\operatorname{Spec} A} W_{l,A}$ of Witt vectors of length l in positive characteristic p. After this, we treat the kernel of the type $F^2 + [a]F + [b] : W_{l,A} \to W_{l,A}$, where F is the Frobenius endomorphism and $[a]$ is the Teichmüller lifting of $a \in A$, respectively.

1. Introduction

Throughout this paper, we denote by p a prime number. Let A be a commutative ring with unit and λ a suitable arbitrary element of A. T. Sekiguchi, F. Oort and N. Suwa [SOS] have introduced a group scheme $\mathscr{G}^{(\lambda)} = \operatorname{Spec} A[T, 1/(1+\lambda T)]$ which is a deformation of the additive group scheme $\mathbf{G}_a$ to the multiplicative group scheme $\mathbf{G}_m$ determined by λ. (for the group structure see section 3.1.) Let l be a positive integer. If A is of characteristic p, the following morphism ψ is a surjective homomorphism:

$$\psi : \mathscr{G}^{(\lambda)} \to \mathscr{G}^{(\lambda^{p^l})}; \quad x \mapsto x^{p^l}.$$

Put $N_{1,l} = \operatorname{Ker} \psi$. Let $W_{l,A}$ be the Witt ring scheme of length l over A. Let $F : W_{l,A} \to W_{l,A}$ be the Frobenius endomorphism of $W_{l,A}$ and $[\lambda]$ the Teichmüller lifting of $\lambda \in A$. Set $F^{(\lambda)} = F - [\lambda^{p-1}]$. Then we have the following:

Theorem 1 ([A]). *Assume that A is of characteristic p. Then the Cartier dual of $N_{1,l}$ is canonically isomorphic to* $\operatorname{Ker}[F^{(\lambda)} : W_{l,A} \to W_{l,A}]$.

First Y. Tsuno [T] proved the case $l = 1$ of Theorem 1 by skillful calculations. Next M. Amano [A] proved Theorem 1 for any l by using the deformations of Artin-Hasse exponential series.

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Our main purpose in this paper is to more generally treat the above argument as follows. Let λ, μ be elements of A. T. Sekiguchi and N. Suwa [SS1] have introduced a group scheme $\mathscr{E}^{(\lambda,\mu,D)} = \mathrm{Spec}\, A[X, Y, 1/(1+\lambda X), 1/(D(X)+\mu Y)]$ which is an extension of $\mathscr{G}^{(\lambda)}$ by $\mathscr{G}^{(\mu)}$, where $D(X)$ denotes a deformation of Artin-Hasse exponential series. (for the group structure of $\mathscr{E}^{(\lambda,\mu,D)}$ and the definition of $D(X)$, see section 3.3.) For a group scheme G, let $\hat{G}$ be the formal completion along the zero section. We consider an endomorphism φ of $\hat{\mathbf{G}}_{m,A}^2 = \hat{\mathbf{G}}_{m,A} \times_{\mathrm{Spec}\, A} \hat{\mathbf{G}}_{m,A}$ defined by

$$\varphi : \hat{\mathbf{G}}_{m,A}^2 \to \hat{\mathbf{G}}_{m,A}^2; \quad (t, u) \mapsto (t^{p^l}, u^{p^l}).$$

We determine a morphism $\psi_2^{(l)} : \hat{\mathscr{E}}^{(\lambda,\mu,D)} \to \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l},D')}$ so that the following diagram is commutative:

$$\begin{array}{ccc} \hat{\mathscr{E}}^{(\lambda,\mu,D)} & \xrightarrow{\alpha^{(\lambda,\mu)}} & \hat{\mathbf{G}}_{m,A}^2 \\ {\scriptstyle \psi_2^{(l)}} \downarrow & & \downarrow {\scriptstyle \varphi} \\ \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l},D')} & \xrightarrow{\alpha^{(\lambda^{p^l},\mu^{p^l})}} & \hat{\mathbf{G}}_{m,A}^2, \end{array}$$

where a morphism $\alpha^{(\lambda,\mu)}$ is the following homomorphism:

$$\alpha^{(\lambda,\mu)} : \hat{\mathscr{E}}^{(\lambda,\mu,D)} \to \hat{\mathbf{G}}_{m,A}^2; \quad (x, y) \mapsto (1+\lambda x, D(x)+\mu y).$$

By the commutativity of this diagram, $\psi_2^{(l)}$ should be given by

$$\psi_2^{(l)} : \hat{\mathscr{E}}^{(\lambda,\mu,D)} \to \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l},D')}; \quad (x, y) \mapsto (x^{p^l}, y^{p^l}).$$

Then the morphism $\psi_2^{(l)}$ is a surjective homomorphism. Put $N_{2,l} = \mathrm{Ker}\, \psi_2^{(l)}$. The formal group scheme $N_{2,l}$ is nothing but the finite subgroup scheme $\mathrm{Spec}\, A[X, Y]/(X^{p^l}, Y^{p^l})$ of $\mathscr{E}^{(\lambda,\mu,D)}$, i.e., we have the following short exact sequence:

$$0 \to N_{2,l} \to \mathscr{E}^{(\lambda,\mu,D)} \to \mathscr{E}^{(\lambda^{p^l},\mu^{p^l},D')} \to 0.$$

$$(x, y) \mapsto (x^{p^l}, y^{p^l})$$

This fact is important in the proof of the following theorem. Set $U = \begin{pmatrix} F^{(\lambda)} & -T_{\boldsymbol{b}} \\ 0 & F^{(\mu)} \end{pmatrix}$ which is an endomorphism of $W_A^2 = W_A \times_{\mathrm{Spec}\, A} W_A$. (for the definition of $T_{\boldsymbol{b}}$ see section 2.) Put $U_l = U|_{W_{l,A}}$ which is an endomorphism

$W_{l,A}^2 = W_{l,A} \times_{\mathrm{Spec}\,A} W_{l,A}$. Then the one of the results of this paper is the following:

Theorem 2. *Assume that A is of characteristic p. Then the Cartier dual of $N_{2,l}$ is canonically isomorphic to* $\mathrm{Ker}[U_l : W_{l,A}^2 \to W_{l,A}^2]$.

The proof of Theorem 2 is almost similar to the previous paper [A]. To prove Theorem 2, we make use of the deformations of Artin-Hasse exponential series introduced by T. Sekiguchi and N. Suwa [SS2] and a duality between $\mathrm{Ker}[U : W(A)^2 \to W(A)^2]$ with $\hat{\mathscr{E}}^{(\lambda,\mu,D)}$ proved by them [Ibid.].

Let K be a perfect field of characteristic p. We set Dieudonné ring as follows:

$$\mathbf{D}_K = W(K)[F,V]/(FV - p, VF - p, F\boldsymbol{a} - \boldsymbol{a}^{(p)}F, V\boldsymbol{a}^{(p)} - \boldsymbol{a}V, \text{ for any } \boldsymbol{a} \in W(K)),$$

where $\boldsymbol{a}^{(p)} = (a_0^p, a_1^p, \dots)$ $(\boldsymbol{a} = (a_0, a_1, \dots) \in W(A))$. Then there are the isomorphism $\mathbf{D}_K/\mathbf{D}_K V^l \simeq \mathrm{Hom}(W_{l,K}, W_{l,K})$. (cf. [DG, p. 550]) From this point of view, we are interested in that which element of Dieudonné ring is Cartier dual of the subgroup scheme $N_{2,l}$ of $\mathscr{E}^{(\lambda,\mu;D)}$. We have already seen that the type of $F - [a^{p-1}] \in \mathbf{D}_K/\mathbf{D}_K V^n$ was Cartier dual of the subgroup scheme $N_{1,l}$ of $\mathscr{G}^{(\lambda)}$. In Section 5 we consider the kernel of the type $F^2 + [a]F + [b] : W_{l,A} \to W_{l,A}$. If the morphism $T_{\boldsymbol{b}}$ is invertible, we have the following isomorphism:

$$\mathrm{Ker}[U_l : W_{l,A}^2 \to W_{l,A}^2] \simeq \mathrm{Ker}[F^{(\mu)} T_{\boldsymbol{b}}^{-1} F^{(\lambda)} : W_{l,A} \to W_{l,A}].$$

This together with some further arguments will prove the following:

Theorem 3. *Assume that the base ring is of characteristic p.*

(1) *If $T_{\boldsymbol{b}}$ can be chosen to be the identity map we have the following isomorphism*:

$$\begin{aligned}&\mathrm{Ker}[U_l : W_{l,A}^2 \to W_{l,A}^2]\\ &\qquad \simeq \mathrm{Ker}[F^2 - ([\mu^{p-1}] + [\lambda^{p(p-1)}])F + [(\lambda\mu)^{p-1}] : W_{l,A} \to W_{l,A}].\end{aligned}$$

(2) *After a suitable faithful base extension $A \hookrightarrow A'$, $T_{\boldsymbol{b}}$ can be chosen to be the identity map over A'.*

The contents of the paper is as follows. The next two sections are devoted to recalling the definitions and the some basic properties of the Witt schemes and of the deformed Artin-Hasse exponential series. In Section 4 and Section 5 we give our proofs of Theorem 2 and Theorem 3.

Notation

$$\mathbf{G}_{a,A} : \text{additive group scheme over } A$$

$$\mathbf{G}_{m,A} : \text{multiplicative group scheme over } A$$

$$W_{n,A} : \text{group scheme of Witt vectors of length } n \text{ over } A$$

$$W_A : \text{group scheme of Witt vectors over } A$$

$$\hat{\mathbf{G}}_{m,A} : \text{multiplicative formal group scheme over } A$$

$$F : \text{Frobenius endomorphism of } W_A$$

$$V : \text{Verschiebung endomorphism of } W_A$$

$$R_n : \text{restriction homomorphism of } W_A \text{ to } W_{n,A}$$

$$[\lambda] : \text{Teichmüller lifting } (\lambda, 0, 0, \dots) \in W(A) \text{ of } \lambda \in A$$

$$\boldsymbol{a}^{(p)} := (a_0^p, a_1^p, \dots) \quad (\boldsymbol{a} = (a_0, a_1, \dots) \in W(A))$$

$$F^{(\lambda)} := F - [\lambda^{p-1}]$$

$$W(A)^{F^{(\lambda)}} := \mathrm{Ker}[F^{(\lambda)} : W(A) \to W(A)]$$

$$W(A)/F^{(\lambda)} := \mathrm{Coker}[F^{(\lambda)} : W(A) \to W(A)]$$

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2. Witt Vectors

In this short section we recall necessary facts on Witt vectors for this paper. For details, see [DG, Chap. V] or [HZ, Chap. III].

2.1 Let $\mathbf{X} = (X_0, X_1, \dots)$ be a sequence of variables. For each $n \geq 0$, we denote by $\Phi_n(\mathbf{X}) = \Phi_n(X_0, X_1, \dots, X_n)$ the Witt polynomial

$$\Phi_n(\mathbf{X}) = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n$$

in $\mathbf{Z}[\mathbf{X}] = \mathbf{Z}[X_0, X_1, \dots]$. Let $W_{n,\mathbf{Z}} = \operatorname{Spec} \mathbf{Z}[X_0, X_1, \dots, X_{n-1}]$ be the n-dimensional affine space over $\mathbf{Z}$. We define a morphism $\Phi^{(n)}$ by

$$\Phi^{(n)} : W_{n,\mathbf{Z}} \to \mathbf{A}^n_{\mathbf{Z}}; \quad \boldsymbol{x} \mapsto (\Phi_0(\boldsymbol{x}), \Phi_1(\boldsymbol{x}), \dots, \Phi_{n-1}(\boldsymbol{x})),$$

where $\mathbf{A}^n_{\mathbf{Z}}$ is the usual n-dimensional affine space over $\mathbf{Z}$. We call $\Phi^{(n)}$ the phantom map. The scheme $\mathbf{A}^n_{\mathbf{Z}}$ has a natural ring scheme structure. It is well-known that $W_{n,\mathbf{Z}}$ has a unique commutative ring scheme structure over $\mathbf{Z}$ so that the phantom map $\Phi^{(n)}$ is a homomorphism of commutative ring schemes over $\mathbf{Z}$. Then the points of $W_{n,\mathbf{Z}}$ are called Witt vectors of length n over $\mathbf{Z}$.

2.2 The Verschiebung homomorphism V is defined by

$$V : W(A) \to W(A); \quad \boldsymbol{x} = (x_0, x_1, \dots) \mapsto V(\boldsymbol{x}) = (0, x_0, x_1, \dots).$$

The restriction homomorphism R_n is defined by

$$R_n : W(A) \to W_n(A); \quad \boldsymbol{x} = (x_0, x_1, \dots) \mapsto \boldsymbol{x}_n = (x_0, x_1, \dots, x_{n-1}).$$

We define a morphism $F : W_n(A) \to W_{n-1}(A)$ by

$$\Phi_i(F\boldsymbol{x}) = \Phi_{i+1}(\boldsymbol{x})$$

for $\boldsymbol{x} \in W_n(A)$. If A is of characteristic p, F is nothing but the usual Frobenius endomorphism. For $\lambda \in A$, $[\lambda]$ and $F^{(\lambda)}$ denote the Teichmüller lifting $[\lambda] = (\lambda, 0, 0, \dots) \in W(A)$ and the endomorphism $F - [\lambda^{p-1}]$ of $W(A)$, respectively.

For $\boldsymbol{a} = (a_0, a_1, \dots) \in W(A)$, we also define a morphism $T_{\boldsymbol{a}} : W(A) \to W(A)$ by

$$\Phi_n(T_{\boldsymbol{a}}\boldsymbol{x}) = a_0^{p^n}\Phi_n(\boldsymbol{x}) + pa_1^{p^{n-1}}\Phi_{n-1}(\boldsymbol{x}) + \cdots + p^n a_n \Phi_0(\boldsymbol{x})$$

for $\boldsymbol{x} \in W(A)$. Then it is known that this morphism has the equality $T_{\boldsymbol{a}} = \sum_{k \geq 0} V^k \cdot [a_k]$. (cf. [SS2, Chap. 4, p. 20])

3. Deformed Artin-Hasse Exponential Series

In this short section we recall necessary facts on the deformed Artin-Hasse exponential series for this paper.

3.1 Let A be a ring and λ an element of A. Put $\mathscr{G}^{(\lambda)} = \operatorname{Spec} A[X, 1/(1 + \lambda X)]$. We define a morphism $\alpha^{(\lambda)}$ by

$$\alpha^{(\lambda)} : \mathscr{G}^{(\lambda)} \to \mathbf{G}_{m,A}; \quad x \mapsto 1 + \lambda x.$$

It is well-known that $\mathscr{G}^{(\lambda)}$ has a unique group scheme structure so that the morphism $\alpha^{(\lambda)}$ is a homomorphism over A. Then the group scheme structure of $\mathscr{G}^{(\lambda)}$ is given by $x \cdot y = x + y + \lambda xy$. If λ is invertible in A, $\alpha^{(\lambda)}$ is an A-isomorphism. On the other hand, if $\lambda = 0$, $\mathscr{G}^{(\lambda)}$ is nothing but the additive group scheme $\mathbf{G}_{a,A}$.

3.2 The Artin-Hasse exponential series $E_p(X)$ is given by

$$E_p(X) = \exp\left(\sum_{r \geq 0} \frac{X^{p^r}}{p^r}\right) \in \mathbf{Z}_{(p)}[[X]].$$

We define a formal power series $E_p(U, \Lambda; X)$ in $\mathbf{Q}[U, \Lambda][[X]]$ by

$$E_p(U, \Lambda; X) = (1 + \Lambda X)^{U/\Lambda} \prod_{k=1}^{\infty} (1 + \Lambda^{p^k} X^{p^k})^{(1/p^k)((U/\Lambda)^{p^k} - (U/\Lambda)^{p^{k-1}})}.$$

As in [SS1, Corollary 2.5.] or [SS2, Lemma 4.8.], we see that this formal power series $E_p(U, \Lambda; X)$ is integral over $\mathbf{Z}_{(p)}$. Note that $E_p(1, 0; X) = E_p(X)$.

Let A be a $\mathbf{Z}_{(p)}$-algebra. Let $\lambda \in A$ and $\boldsymbol{v} = (v_0, v_1, \dots) \in W(A)$. We define a formal power series $E_p(\boldsymbol{v}, \lambda; X)$ in $A[[X]]$ by

$$\begin{aligned} E_p(\boldsymbol{v}, \lambda; X) &= \prod_{k=0}^{\infty} E_p(v_k, \lambda^{p^k}; X^{p^k}) \\ &= (1 + \lambda X)^{v_0/\lambda} \prod_{k=1}^{\infty} (1 + \lambda^{p^k} X^{p^k})^{(1/p^k \lambda^{p^k}) \Phi_{k-1}(F^{(\lambda)} \boldsymbol{v})}. \end{aligned}$$

Moreover we define a formal power series $F_p(\boldsymbol{v}, \lambda; X, Y)$ as follows:

$$F_p(\boldsymbol{v}, \lambda; X, Y) = \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{(1/p^k \lambda^{p^k}) \Phi_{k-1}(\boldsymbol{v})}.$$

As in [SS1, Lemma 2.16.] or [SS2, Lemma 4.9.], we see that the formal power series $F_p(\boldsymbol{v}, \lambda; X, Y)$ is integral over $\mathbf{Z}_{(p)}$. For the formal power series $F_p(F^{(\lambda)} \boldsymbol{v}, \lambda; X, Y)$, we have the following equalities:

$$\begin{aligned} F_p(F^{(\lambda)} \boldsymbol{v}, \lambda; X, Y) &= \prod_{k=1}^{\infty} \left(\frac{(1 + \lambda^{p^k} X^{p^k})(1 + \lambda^{p^k} Y^{p^k})}{1 + \lambda^{p^k} (X + Y + \lambda XY)^{p^k}} \right)^{(1/p^k \lambda^{p^k}) \Phi_{k-1}(F^{(\lambda)} \boldsymbol{v})} \\ &= \frac{E_p(\boldsymbol{v}, \lambda; X) E_p(\boldsymbol{v}, \lambda; Y)}{E_p(\boldsymbol{v}, \lambda; X + Y + \lambda XY)}. \end{aligned}$$

3.3 Let λ, μ be elements of A. Put $\mathscr{E}^{(\lambda,\mu;D)} = \operatorname{Spec} A[X, Y, 1/(1+\lambda X), 1/(D(X)+\mu Y)]$, where $D(X) = E_p(\boldsymbol{a}, \lambda; X)$ $(\boldsymbol{a} \in W(A_\mu)^{F^{(\lambda)}})$. We define a morphism $\alpha^{(\lambda,\mu)}$ by

$$\alpha^{(\lambda,\mu)} : \mathscr{E}^{(\lambda,\mu;D)} \to \mathbf{G}_{m,A}^2; \quad (x, y) \mapsto (1+\lambda x, D(x)+\mu y).$$

It is known that $\mathscr{E}^{(\lambda,\mu;D)}$ has a unique group scheme structure so that the morphism $\alpha^{(\lambda,\mu)}$ is a homomorphism over A. Then the group structure of $\mathscr{E}^{(\lambda,\mu;D)}$ is

$$(x_1, y_1) \cdot (x_2, y_2) = \left(x_1 + x_2 + \lambda x_1 x_2, \textstyle\sum_1\right),$$

where $\sum_1 = y_1 D(x_2) + y_2 D(x_1) + \mu y_1 y_2 + \mu^{-1}(D(x_1)D(x_2) - D(x_1 + x_2 + \lambda x_1 x_2))$.

We define formal power series $E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); x, y)$, $H_1(x, y)$ and $G_p(\boldsymbol{v}, \mu; F)$ as follows:

$$E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); x, y) = E_p(\boldsymbol{v}_1, \lambda; x) E_p\left(\boldsymbol{v}_2, \mu; \frac{y}{D(x)}\right),$$

$$H_1(x, y) = \frac{1}{\mu}(F_p(F^{(\lambda)}\boldsymbol{v}, \lambda; x, y) - 1),$$

$$G_p(\boldsymbol{v}, \mu; F) = \prod_{l \geq 1} \left(\frac{1 + (F-1)^{p^l}}{[p]^l F}\right)^{(1/p^l \mu^{p^l})\Phi_{l-1}(\boldsymbol{v})}.$$

Moreover we define a formal power series $F_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1), (x_2, y_2))$ as follows:

$$\begin{aligned} &F_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1), (x_2, y_2)) \\ &\quad = F_p(\boldsymbol{v}_1, \lambda; x_1, y_1) \cdot F_p\left(\boldsymbol{v}_2, \mu; \frac{y_1}{D(x_1)}, \frac{y_2}{D(x_2)}\right) \\ &\quad\quad \times F_p\left(\boldsymbol{v}_2, \mu; H_1, \frac{y_1}{D(x_1)} \dot{+} \frac{y_2}{D(x_2)}\right) \cdot G_p(\boldsymbol{v}_2, \mu; F_p(\boldsymbol{v}_1, \lambda; x_1, y_2))^{-1}, \end{aligned}$$

where the symbol $\dot{+}$ denotes the multiplication of the group scheme $\mathscr{G}^{(\lambda)}$.

4. Proof of the Theorem

In this section we give the proof of Theorem 2.

Suppose A is a ring of characteristic p. Let λ, μ be elements of A. Let $\mathscr{E}^{(\lambda,\mu;D)}$ be a group scheme defined in Section 3.3 and $\hat{\mathscr{E}}^{(\lambda,\mu;D)}$ be the formal completion of $\mathscr{E}^{(\lambda,\mu;D)}$ along the zero section. We consider the following homomorphism:

$$\psi_2^{(l)} : \hat{\mathscr{E}}^{(\lambda,\mu;D)} \to \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}; \quad (x,y) \mapsto (x^{p^l}, y^{p^l}),$$

where $D'(X) = E_p(\boldsymbol{a}^{(p^l)}, \lambda^{p^l}; X)$ for $\boldsymbol{a} \in W(A_\mu)^{F^{(\lambda)}}$. Note that $\boldsymbol{a}^{(p^l)} \in W(A_{\mu^{p^l}})^{F^{(\lambda^{p^l})}}$. For the kernel of the homomorphism $\psi_2^{(l)}$, we have

$$N_{2,l} = \mathrm{Ker}\, \psi_2^{(l)} = \mathrm{Spf}\, A[[X,Y]]/(X^{p^l}, Y^{p^l}) = \mathrm{Spec}\, A[X,Y]/(X^{p^l}, Y^{p^l}).$$

Note that the formal scheme $N_{2,l}$ is nothing but the finite group scheme of order p^{2l}, since the classes $\overline{X}$ and $\overline{Y}$ are nilpotents in the coordinate ring of $N_{2,l}$. The following exact sequence is induced by the homomorphism $\psi_2^{(l)}$:

$$0 \longrightarrow N_{2,l} \xrightarrow{\iota} \hat{\mathscr{E}}^{(\lambda,\mu;D)} \xrightarrow{\psi_2^{(l)}} \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')} \longrightarrow 0, \tag{1}$$

where ι is a canonical inclusion. This exact sequence (1) deduces the following long exact sequence:

$$\begin{aligned} (2) \qquad & 0 \longrightarrow \mathrm{Hom}(\hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}, \hat{\mathbf{G}}_{m,A}) \xrightarrow{(\psi_2^{(l)})^*} \mathrm{Hom}(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A}) \\ & \xrightarrow{(\iota)^*} \mathrm{Hom}(N_{2,l}, \hat{\mathbf{G}}_{m,A}) \xrightarrow{\partial} \mathrm{Ext}^1(\hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}, \hat{\mathbf{G}}_{m,A}) \\ & \xrightarrow{(\psi_2^{(l)})^*} \mathrm{Ext}^1(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A}) \longrightarrow \cdots. \end{aligned}$$

Since we can directly check that the base schemes of the images of the boundary map ∂ and the map $(\psi_2^{(l)})^*$ are given by direct products of schemes, we can replace $\mathrm{Ext}^1(\hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}, \hat{\mathbf{G}}_{m,A})$ and $\mathrm{Ext}^1(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A})$ with $H_0^2(\hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}, \hat{\mathbf{G}}_{m,A})$ and $H_0^2(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A})$, respectively. Here $H_0^2(G,H)$ denotes the Hochschild cohomology group consisting of symmetric 2-cocycles of G with coefficients in H for formal group schemes G and H. (cf. [DG, Chap. II.3 and Chap. III.6]) Therefore we have the following exact sequence:

$$\begin{aligned} (3) \qquad & 0 \longrightarrow \mathrm{Hom}(\hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}, \hat{\mathbf{G}}_{m,A}) \\ & \xrightarrow{(\psi_2^{(l)})^*} \mathrm{Hom}(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A}) \xrightarrow{(\iota)^*} \mathrm{Hom}(N_{2,l}, \hat{\mathbf{G}}_{m,A}) \\ & \xrightarrow{\partial} H_0^2(\hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}, \hat{\mathbf{G}}_{m,A}) \xrightarrow{(\psi_2^{(l)})^*} H_0^2(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A}). \end{aligned}$$

On the other hand, as in the case $n = 2$ of [SS2, Theorem 5.1.], the following morphisms are isomorphic:

$$\text{(4)} \qquad \operatorname{Ker}[U : W(A)^2 \to W(A)^2] \to \operatorname{Hom}(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A});$$
$$(\boldsymbol{v}_1, \boldsymbol{v}_2) \mapsto E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); x, y)$$

$$\text{(5)} \qquad \operatorname{Coker}[U : W(A)^2 \to W(A)^2] \to H_0^2(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A});$$
$$(\boldsymbol{w}_1, \boldsymbol{w}_2) \mapsto F_p((\boldsymbol{w}_1, \boldsymbol{w}_2), (\lambda, \mu); (x_1, y_1), (x_2, y_2)).$$

We put $U = \begin{pmatrix} F^{(\lambda)} & -T_{\boldsymbol{b}} \\ 0 & F^{(\mu)} \end{pmatrix}$ and $U' = \begin{pmatrix} F^{(\lambda^{p^l})} & -T_{\mathbf{B}} \\ 0 & F^{(\mu^{p^l})} \end{pmatrix}$, where $\boldsymbol{b} = \mu^{-1} F^{(\lambda)} \boldsymbol{a}$ and $\mathbf{B} = \mu^{-p^l} F^{(\lambda^{p^l})} \boldsymbol{a}^{(p^l)}$. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W(A)^2 & \xrightarrow{V^l \times V^l} & W(A)^2 & \xrightarrow{R_l \times R_l} & W_l(A)^2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle U'} & & \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U_l} & & \\
0 & \longrightarrow & W(A)^2 & \xrightarrow{V^l \times V^l} & W(A)^2 & \xrightarrow{R_l \times R_l} & W_l(A)^2 & \longrightarrow & 0,
\end{array}$$

where U_l is the restriction morphism of U to $W_l(A)^2$. The exactness of the horizontal sequences are obvious. The commutativity of the second square of this diagram rather obvious, and the one of the first square is given as follows. For $\boldsymbol{v} = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} \in W(A)^2$, we have

$$U \circ (V^l \times V^l)(\boldsymbol{v}) = \begin{pmatrix} F^{(\lambda)} V^l \boldsymbol{v}_1 - T_{\boldsymbol{b}} V^l \boldsymbol{v}_2 \\ F^{(\mu)} V^l \boldsymbol{v}_2 \end{pmatrix} = \begin{pmatrix} V^l F^{(\lambda^{p^l})} \boldsymbol{v}_1 - V^l T_{\mathbf{B}} \boldsymbol{v}_2 \\ V^l F^{(\mu^{p^l})} \boldsymbol{v}_2 \end{pmatrix}$$
$$= (V^l \times V^l) \circ U'(\boldsymbol{v}).$$

Here we must show the equality $T_{\boldsymbol{b}} V^l = V^l T_{\mathbf{B}}$. This equality is proved as follows. Since $\mathbf{B} = \mu^{-p^l} F^{(\lambda^{p^l})} \boldsymbol{a}^{(p^l)} = (\mu^{-1} F^{(\lambda)} \boldsymbol{a})^{(p^l)} = \boldsymbol{b}^{(p^l)}$, we have

$$T_{\boldsymbol{b}} V^l = \sum_{k \geq 0} V^k [b_k] \circ V^l = V^l \left(\sum_{k \geq 0} V^k [b_k^{p^l}] \right) = V^l \left(\sum_{k \geq 0} V^k [B_k] \right) = V^l T_{\mathbf{B}}.$$

Thus, by the snake lemma for this diagram, we have the following exact sequence:

$$\text{(6)} \qquad 0 \longrightarrow \operatorname{Ker} U' \xrightarrow{V^l \times V^l} \operatorname{Ker} U \xrightarrow{R_l \times R_l} \operatorname{Ker} U_l$$
$$\xrightarrow{\partial} \operatorname{Coker} U' \xrightarrow{V^l \times V^l} \operatorname{Coker} U \xrightarrow{R_l \times R_l} \operatorname{Coker} U_l \longrightarrow 0.$$

Now, by combining the exact sequences (3), (6) and the isomorphisms (4), (5), we have the following diagram consisting of exact horizontal lines and vertical isomorphisms except for ϕ:

(7)

$$\begin{array}{ccccc}
\mathrm{Hom}(\hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{(\psi_2^{(l)})^*} & \mathrm{Hom}(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{(\iota)^*} & \mathrm{Hom}(N_{2,l}, \hat{\mathbf{G}}_{m,A}) \\
\phi_1 \uparrow & & \phi_2 \uparrow & & \phi \uparrow \\
\mathrm{Ker}\ U' & \xrightarrow{V^l \times V^l} & \mathrm{Ker}\ U & \xrightarrow{R_l \times R_l} & \mathrm{Ker}\ U_l
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\partial} & H_0^2(\hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}, \hat{\mathbf{G}}_{m,A}) & \xrightarrow{(\psi_2^{(l)})^*} & H_0^2(\hat{\mathscr{E}}^{(\lambda,\mu;D)}, \hat{\mathbf{G}}_{m,A}) \\
 & \phi_3 \uparrow & & \phi_4 \uparrow \\
\xrightarrow{\partial} & \mathrm{Coker}\ U' & \xrightarrow{V^l \times V^l} & \mathrm{Coker}\ U,
\end{array}$$

where ϕ is the following homomorphism induced from the short exact sequence (1) and the isomorphism (4):

$$\phi : \mathrm{Ker}[U_l : W_l^2(A) \to W_l^2(A)] \to \mathrm{Hom}(N_{2,l}, \hat{\mathbf{G}}_{m,A});$$
$$(\boldsymbol{v}_1, \boldsymbol{v}_2) \mapsto E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); x, y).$$

If the diagram (7) is proved to commute, then the five lemma shows that ϕ is isomorphism, i.e., $\mathrm{Ker}[U_l : W_l(A)^2 \to W_l(A)^2] \simeq \mathrm{Hom}(N_{2,l}, \hat{\mathbf{G}}_{m,A})$. Hence we obtain the Theorem 2. Therefore it is sufficient to prove that the diagram (7) is commutative.

Lemma 1. $(\psi_2^{(l)})^* \circ \phi_1 = \phi_2 \circ (V^l \times V^l)$.

Proof. By the definition and (10) of Sublemma 1 (see the end of this section.), we have the following equalities for $\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} \in \mathrm{Ker}\ U'$:

$$\begin{aligned}
E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda^{p^l}, \mu^{p^l}); x^{p^l}, y^{p^l}) &= E_p(\boldsymbol{v}_1, \lambda^{p^l}; x^{p^l}) E_p\left(\boldsymbol{v}_2, \mu^{p^l}; \frac{y^{p^l}}{D'(x^{p^l})}\right) \\
&= E_p(V^l \boldsymbol{v}_1, \lambda; x) E_p\left(V^l \boldsymbol{v}_2, \mu; \frac{y}{D(x)}\right) \\
&= E_p((V^l \boldsymbol{v}_1, V^l \boldsymbol{v}_2), (\lambda, \mu); x, y). \qquad \square
\end{aligned}$$

LEMMA 2. $(\iota)^* \circ \phi_2 = \phi \circ (R_l \times R_l)$.

PROOF. This follows from the definitions of ϕ and $(\iota)^*$. $\square$

LEMMA 3. $\partial \circ \phi = \phi_3 \circ \partial$.

PROOF. For $\begin{pmatrix} R_l \boldsymbol{v}_1 \\ R_l \boldsymbol{v}_2 \end{pmatrix} \in \operatorname{Ker} U_l$, we can calculate $\partial E_p((R_l \boldsymbol{v}_1, R_l \boldsymbol{v}_2), (\lambda, \mu); x, y)$ on the direct product $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda^{p^l}, \mu^{p^l}; D')}$ so that the following diagram is commutative:

(8)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{2,l} & \longrightarrow & \hat{\mathscr{E}}^{(\lambda,\mu;D)} & \xrightarrow{\psi_2^{(l)}} & \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle E_p((R_l\boldsymbol{v}_1, R_l\boldsymbol{v}_2),(\lambda,\mu);x,y)} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\| & & \\
0 & \longrightarrow & \hat{\mathbf{G}}_{m,A} & \longrightarrow & \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')} & \longrightarrow & \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')} & \longrightarrow & 0.
\end{array}$$

By the commutativity of the diagram (8), Φ should be given by:

$$\Phi : \hat{\mathscr{E}}^{(\lambda,\mu;D)} \to \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}; \quad (x, y) \mapsto (E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); x, y), \psi_2^{(l)}(x, y))$$

(Note that $E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); x, y) : \hat{\mathscr{E}}^{(\lambda,\mu;D)} \to \hat{\mathbf{G}}_{m,A}$ is not a homomorphism.) We endow $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}$ with a group scheme structure so that $\Phi : \hat{\mathscr{E}}^{(\lambda,\mu;D)} \to \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}$ is a homomorphism. i.e., the following equality should be satisfied:

$$\Phi((x_1, y_1) \cdot (x_2, y_2)) = \Phi(x_1, y_1) \cdot \Phi(x_2, y_2) \quad ((x_1, y_1), (x_2, y_2) \in \hat{\mathscr{E}}^{(\lambda,\mu;D)}),$$

where

$$\begin{aligned}
&\Phi((x_1, y_1) \cdot (x_2, y_2)) \\
&\quad = (E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1) \cdot (x_2, y_2)), \psi_2^{(l)}((x_1, y_1) \cdot (x_2, y_2))), \\
&\Phi(x_1, y_1) \cdot \Phi(x_2, y_2) \\
&\quad = (E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); x_1, y_1), \psi_2^{(l)}(x_1, y_1)) \\
&\qquad \cdot (E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); x_2, y_2), \psi_2^{(l)}(x_2, y_2)).
\end{aligned}$$

For elements $(t_1, (z_1, w_1))$ and $(t_2, (z_2, w_2))$ of $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}$, we choose (x_1, y_1) and (x_2, y_2) in the inverse images of (z_1, w_1) and (z_2, w_2) for the homomorphism $\psi_2^{(l)}$, respectively. Then the group structure of $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda^{p^l}, \mu^{p^l}; D')}$ should be given by

$$(t_1, (z_1, w_1)) \cdot (t_2, (z_2, w_2))$$
$$= \left(t_1 t_2 \cdot \frac{E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1) \cdot (x_2, y_2))}{E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1)) \cdot E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_2, y_2))}, (z_1, w_1) \cdot (z_2, w_2) \right).$$

Hence, by the above argument and Theorem 4.16 of [SS2], the boundary map ∂ should be given by the following formal power series;

$$\frac{E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1) \cdot (x_2, y_2))}{E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1)) \cdot E_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_2, y_2))}$$
$$= F_p(U(\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1), (x_2, y_2)).$$

To prove the equality of Lemma 3, we must show the following equality of the formal power series:

$$F_p(U(\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1), (x_2, y_2)) = F_p((\boldsymbol{z}_1, \boldsymbol{z}_2), (\lambda^{p^l}, \mu^{p^l}); (x_1^{p^l}, y_1^{p^l}), (x_2^{p^l}, y_2^{p^l})),$$

where $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are elements of inverse images of the boundary $\partial \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix}$ for $W(A)^2 \to \mathrm{Coker}\, U'$. This equality is proved by (11) and (12) of Sublemma 1 (see the end of this section.) as follows:

$$F_p(U(\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1), (x_2, y_2))$$
$$= F_p(F^{(\lambda)}\boldsymbol{v}_1 - T_{\boldsymbol{b}}\boldsymbol{v}_2, \lambda; x_1, y_1) \cdot F_p\left(F^{(\mu)}\boldsymbol{v}_2, \mu; \frac{y_1}{D(x_1)}, \frac{y_2}{D(x_2)} \right)$$
$$\times F_p\left(F^{(\mu)}\boldsymbol{v}_2, \mu; H_1, \frac{y_1}{D(x_1)} \dot{+} \frac{y_2}{D(x_2)} \right) \cdot G_p(F^{(\mu)}\boldsymbol{v}_2, \mu; F)^{-1}$$
$$= F_p(V^l\boldsymbol{z}_1, \lambda; x_1, y_1) \cdot F_p\left(V^l\boldsymbol{z}_2, \mu; \frac{y_1}{D(x_1)}, \frac{y_2}{D(x_2)} \right)$$
$$\times F_p\left(V^l\boldsymbol{z}_2, \mu; H_1, \frac{y_1}{D(x_1)} \dot{+} \frac{y_2}{D(x_2)} \right) \cdot G_p(V^l\boldsymbol{z}_2, \mu; F)^{-1}$$
$$= F_p(\boldsymbol{z}_1, \lambda^{p^l}; x_1^{p^l}, x_2^{p^l}) \cdot F_p\left(\boldsymbol{z}_2, \mu^{p^l}; \left(\frac{y_1}{D(x_1)}\right)^{p^l}, \left(\frac{y_2}{D(x_2)}\right)^{p^l} \right)$$
$$\times F_p\left(\boldsymbol{z}_2, \mu^{p^l}; H_1^{(p^l)}, \left(\frac{y_1}{D(x_1)} \dot{+} \frac{y_2}{D(x_2)}\right)^{p^l} \right) \cdot G_p(\boldsymbol{z}_2, \mu^{p^l}; F^{(p^l)})^{-1}$$
$$= F_p((\boldsymbol{z}_1, \boldsymbol{z}_2), (\lambda^{p^l}, \mu^{p^l}); (x_1^{p^l}, y_1^{p^l}), (x_2^{p^l}, y_2^{p^l})). \qquad \square$$

Lemma 4. $(\psi_2^{(l)})^* \circ \phi_3 = \phi_4 \circ (V^l \times V^l)$.

Proof. For $\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} \in \operatorname{Coker} U'$, we can calculate the direct image

$$(\psi_2^{(l)})^* F_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda^{p^l}, \mu^{p^l}); (z_1, w_1), (z_2, w_2))$$

on the direct product $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda,\mu;D)}$ so that the following diagram is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{\mathbf{G}}_{m,A} & \longrightarrow & \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda,\mu;D)} & \longrightarrow & \hat{\mathscr{E}}^{(\lambda,\mu;D)} & \longrightarrow & 0 \\ & & \| & & \downarrow \Psi & & \downarrow \psi_2^{(l)} & & \\ 0 & \longrightarrow & \hat{\mathbf{G}}_{m,A} & \longrightarrow & \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')} & \longrightarrow & \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')} & \longrightarrow & 0. \end{array} \tag{9}$$

By the commutativity of the diagram (9), Ψ should be given by

$$\Psi : \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda,\mu;D)} \to \hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda^{p^l},\mu^{p^l};D')}; \quad (t,(x,y)) \mapsto (t, \psi_2^{(l)}(x,y)).$$

We endow $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda,\mu;D)}$ with a group scheme structure so that Ψ is a homomorphism. For local sections $(t_1,(x_1,y_1))$ and $(t_2,(x_2,y_2))$ in $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda,\mu;D)}$, suppose that the product $(t_1,(x_1,y_1)) \cdot (t_2,(x_2,y_2))$ is written as

$$(t_1,(x_1,y_1)) \cdot (t_2,(x_2,y_2)) = (t_1 t_2 G((x_1,y_1),(x_2,y_2)), (x_1,y_1) \cdot (x_2,y_2)),$$

where $G((x_1,y_1),(x_2,y_2))$ is a cocycle on $\hat{\mathbf{G}}_{m,A} \times \hat{\mathscr{E}}^{(\lambda,\mu;D)}$. Then we have

$$\begin{aligned} \Psi((t_1,(x_1,y_1)) \cdot (t_2,(x_2,y_2))) &= \Psi(t_1 t_2 G((x_1,y_1),(x_2,y_2)), (x_1,y_1) \cdot (x_2,y_2)) \\ &= (t_1 t_2 G((x_1,y_1),(x_2,y_2)), \psi_2^{(l)}(x_1,y_1) \cdot \psi_2^{(l)}(x_2,y_2)), \end{aligned}$$

on the other hand, we have

$$\begin{aligned} \Psi(t_1,(x_1,y_1)) \cdot \Psi(t_2,(x_2,y_2)) &= (t_1, \psi_2^{(l)}(x_1,y_1)) \cdot (t_2, \psi_2^{(l)}(x_2,y_2)) \\ &= (t_1 t_2 F_p((\boldsymbol{v}_1,\boldsymbol{v}_2),(\lambda^{p^l},\mu^{p^l}); \psi_2^{(l)}(x_1,y_1), \\ &\qquad \psi_2^{(l)}(x_2,y_2)), \psi_2^{(l)}(x_1,y_1) \cdot \psi_2^{(l)}(x_2,y_2)). \end{aligned}$$

Hence, for Ψ being a homomorphism, the following equality is necessary:

$$G((x_1,y_1),(x_2,y_2)) = F_p((\boldsymbol{v}_1,\boldsymbol{v}_2),(\lambda^{p^l},\mu^{p^l}); \psi_2^{(l)}(x_1,y_1), \psi_2^{(l)}(x_2,y_2)).$$

To prove the equality of Lemma 4, we must show the following equality:

$$F_p((\boldsymbol{v}_1, \boldsymbol{v}_2), (\lambda^{p^l}, \mu^{p^l}); \psi_2^{(l)}(x_1, y_1), \psi_2^{(l)}(x_2, y_2))$$
$$= F_p((V^l \boldsymbol{v}_1, V^l \boldsymbol{v}_2), (\lambda, \mu); (x_1, y_1), (x_2, y_2)),$$

but this equality has been already proved in Lemma 3. □

If we prove the following Sublemma 1 which was used above (see the proofs of Lemmas 1 and 3), the proof of Theorem 2 will complete.

Sublemma 1.

$$E_p(V^l \boldsymbol{v}, \lambda; x) = E_p(\boldsymbol{v}, \lambda^{p^l}; x^{p^l}), \tag{10}$$

$$F_p(V^l \boldsymbol{v}, \lambda; x_1, x_2) = F_p(\boldsymbol{v}, \lambda^{p^l}; x_1^{p^l}, x_2^{p^l}) \pmod{p}, \tag{11}$$

$$G_p(V^l \boldsymbol{v}, \mu; F) = G_p(\boldsymbol{v}, \mu^{p^l}; F^{(p^l)}) \pmod{p}. \tag{12}$$

Proof. The equality (11) has been proved as Lemma 3 in the previous paper [A]. The proof of the equality (12) is similar to the proof of the equality (11). Therefore we prove only the equality (10). Since $\Phi_{l-1}(F^{(\lambda)} V^l \boldsymbol{v}) = \Phi_l(V^l \boldsymbol{v}) - \Phi_{l-1}([\lambda^{p-1}] V^l \boldsymbol{v}) = p^l v_0$, we have the following equalities:

$$E_p(V^l \boldsymbol{v}, \lambda; x) = (1 + \lambda x)^{v_0/\lambda} \prod_{k=1}^{\infty} (1 + \lambda^{p^k} x^{p^k})^{(1/p^k \lambda^{p^k}) \Phi_{k-1}(F^{(\lambda)} V^l \boldsymbol{v})}$$

$$= (1 + \lambda x)^{v_0/\lambda} \prod_{k \geq l} (1 + \lambda^{p^k} x^{p^k})^{(1/p^k \lambda^{p^k}) \Phi_{k-1}(V^l F^{(\lambda^{p^l})} \boldsymbol{v})}$$

(Here we put $r = k - l$.)

$$= (1 + \lambda^{p^l} x^{p^l})^{v_0/\lambda^{p^l}} \prod_{r=1}^{\infty} (1 + \lambda^{p^{l+r}} x^{p^{l+r}})^{(1/p^r \lambda^{p^{r+l}}) \Phi_{r-1}(F^{(\lambda^{p^l})} \boldsymbol{v})}$$

$$= E_p(\boldsymbol{v}, \lambda^{p^l}; x^{p^l}).$$ □

5. The Kernel of the Type $F^2 + [a]F + [b]$

In this section we consider the case that the morphism T_b is invertible. And we give the proof of Theorem 3.

For $\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} \in \operatorname{Ker} U_l$, we have the following equalities:

$$F^{(\lambda)}\boldsymbol{v}_1 - T_{\boldsymbol{b}}\boldsymbol{v}_2 = 0, \quad F^{(\mu)}\boldsymbol{v}_2 = 0.$$

If $T_{\boldsymbol{b}}^{-1}$ exists, we have $\boldsymbol{v}_2 = T_{\boldsymbol{b}}^{-1}F^{(\lambda)}\boldsymbol{v}_1$. Since $0 = F^{(\mu)}\boldsymbol{v}_2 = F^{(\mu)}T_{\boldsymbol{b}}^{-1}F^{(\lambda)}\boldsymbol{v}_1$, we have the following isomorphism:

$$\operatorname{Ker}[F^{(\mu)}T_{\boldsymbol{b}}^{-1}F^{(\lambda)} : W_{l,A} \to W_{l,A}] \simeq \operatorname{Ker}[U_l : W_{l,A}^2 \to W_{l,A}^2]; \quad \boldsymbol{v} \mapsto \begin{pmatrix} \boldsymbol{v} \\ T_{\boldsymbol{b}}^{-1}F^{(\lambda)}\boldsymbol{v} \end{pmatrix},$$

where the inverse morphism is $\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{pmatrix} \mapsto \boldsymbol{v}_1$. By Proposition 4.4 of [SS2], if $\boldsymbol{b} = (1, 0, 0, \ldots)$, $T_{\boldsymbol{b}}$ is the identity map. At this point we have the following equalities:

$$\begin{aligned} F^{(\mu)}T_{\boldsymbol{b}}^{-1}F^{(\lambda)} = F^{(\mu)} \circ F^{(\lambda)} &= (F - [\mu^{p-1}]) \circ (F - [\lambda^{p-1}]) \\ &= F^2 - ([\mu^{p-1}] + [\lambda^{p(p-1)}])F + [(\mu\lambda)^{p-1}]. \end{aligned}$$

Here we must construct a base ring which the morphism $T_{\boldsymbol{b}}$ is the identity map, i.e., we make a ring extension of the base ring A that there exist a vector $\boldsymbol{a} = (a_0, a_1, \ldots)$ such that $F^{(\lambda)}\boldsymbol{a} = \boldsymbol{a}'$ for a vector $\boldsymbol{a}' = (a_0', a_1', \ldots)$ of $W(A)$. We construct this ring extension inductively as follows. First we have the equality $a_0^p - \lambda^{p-1}a_0 = a_0'$ by $\Phi_0(F^{(\lambda)}\boldsymbol{a}) = \Phi_0(\boldsymbol{a}')$, where Φ_i's are the Witt polynomials. We put $A_1 = A[X]/(X^p - \lambda^{p-1}X - a_0')$. Note that the ring A_1 is a faithful ring extension of A because of the polynomial $X^p - \lambda^{p-1}X - a_0'$ is monic. Then, as $a_0 = \overline{X}$, we can take an element a_0 of A_1 such that $a_0^p - \lambda^{p-1}a_0 = a_0'$. We assume that a_{i-1} is an element of A_i and A_i is a faithful ring extension of A_{i-1} for each i $(< n)$. By the following equality:

$$\Phi_n(F(\boldsymbol{a})) - \Phi_n([\lambda^{p-1}])\Phi_n(\boldsymbol{a}) = \Phi_n(\boldsymbol{a}'),$$

we have the following equality:

$$a_n' = a_n^p - \lambda^{p^n(p-1)}a_n + \frac{1}{p^n}\{\Phi_{n-1}(\boldsymbol{a}^{(p)}) - \Phi_{n-1}([\lambda^{p-1}])\Phi_{n-1}(\boldsymbol{a}^{(p)}) - \Phi_{n-1}(\boldsymbol{a}'^{(p)})\}.$$

By the assumption of the induction,

$$\Phi_{n-1}(\boldsymbol{a}^{(p)}) - \Phi_{n-1}([\lambda^{p-1}])\Phi_{n-1}(\boldsymbol{a}^{(p)}) - \Phi_{n-1}(\boldsymbol{a}'^{(p)})$$

is the polynomial of variables $a_0, a_1, \ldots, a_{n-1}$. Hence we have the following:

$$a_n' = a_n^p - \lambda^{p^n(p-1)}a_n + (\text{the terms of } a_0, a_1, \ldots, a_{n-1}).$$

Here we put

$$A_{n+1} = A_n[X]/(X^p - \lambda^{p^n(p-1)}X + (\text{the terms of } a_0, a_1, \dots, a_{n-1})).$$

Then, since $X^p - \lambda^{p^n(p-1)}X + (\text{the terms of } a_0, a_1, \dots, a_{n-1})$ is a monic polynomial, A_{n+1} is a faithful ring extension of A_n and, as $a_n = \overline{X}$, we can take an element a_n of A_{n+1}. Therefore we have following ring extensions:

$$A_0 = A \subset A_1 \subset A_2 \subset \cdots \subset A_{n+1}.$$

For the above sequence of ring extensions, we take inductive limit $A_\infty = \varinjlim A_i$. Then we can take a vecter $\boldsymbol{a} \in W(A_\infty)$ such that $F^{(\lambda)}\boldsymbol{a} = \boldsymbol{a}'$ for a vector $\boldsymbol{a}' = (\mu, 0, 0, \dots) \in W(A_\infty)$. Since $\boldsymbol{b} = \frac{1}{\mu}\boldsymbol{a}' = (1, 0, 0, \dots)$, we have $T_{\boldsymbol{b}} = 1$. Hence we obtain the Theorem 3. □

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