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THE LIFTING OF ELLIPTIC MODULAR FORMS TO HILBERT MODULAR FORMS AND PETERSSON INNER PRODUCTS

By

Hisashi KOJIMA

Introduction

Let K be a totally real number field of degree l and discriminant q which is a cyclic extension of $\mathbf{Q}$ and satisfies certain conditions and Γ_K the full Hilbert modular group over K. We denote by $S_k(\Gamma_K)$ the space of Hilbert cusp forms of weight k with respect to Γ_K. Put $S_k(SL_2(\mathbf{Z})) = S_k(\Gamma_{\mathbf{Q}})$. Furthemore, we set $S_k = S_k(SL_2(\mathbf{Z})) \oplus_\chi S_k(\Gamma_0(q), \chi)$, where χ is the non-trivial primitive characters attached to K and $S_k(\Gamma_0(q), \chi)$ is the space of all cusp forms of weight k, level q and character χ.

Applying Selberg-Eichler trace formula, Saito [4] proved the existence of a linear mapping Ψ_k of S_k to $S_k(\Gamma_K)$ satisfying the following conditions:

$\Psi_k(f)$ is a Hecke eigen form for every primitive form f in S_k and the Fourier coefficient $C_f(\mathfrak{P})$ of $\Psi_k(f)$ at any prime ideal $\mathfrak{P}$ is determined by those of f at prime p satisfying $\mathfrak{P}|p$.

The first purpose of this note is to express explicitly the Fourier coefficients of $\Psi_k(f)$ in terms of those of f for an arbitrary f in S_k. Kohnen [2] had proved such relations of Fourier coefficients in the case of Ikeda lifting of Siegel modular forms of even degree.

Our second purpose is to determine a relation between the Petersson norm $\langle \Psi_k(f), \Psi_k(f) \rangle$ and that $\langle f, f \rangle$ of f. Such relations of another liftings of modular forms were discussed by several authors [1], [3] and [10] in the case where the lifts are constructed by kernel functions. Our proof is based only on a relation between the Rankin's convolution of L-functions and the Petersson inner products and the results in [8]. Next we explain contents of each section in precise form.

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Section 0 is a preliminary section. In Section 1, we shall express explicitly the Fourier coefficients $C_f(\mathfrak{A})$ of $\Psi_k(f)$ at any ideal $\mathfrak{A}$ in terms of those of f for every f in S_k. From this we may reformulate the lifting in terms of linear relations among Fourier coefficients of modular forms.

In Section 2, using the Rankin method of Dirichlet series, we shall deduce that the ratio $\dfrac{\langle \Psi_k(f), \Psi_k(f) \rangle}{\langle f, f \rangle}$ of Petersson norms is equal to the critical value of product of certain zeta functions attached to f at $s = k$ for every eigen form $f \in S_k$. As a corollary of it, using a theorem in Strum [9], we may deduce that the ratio $\dfrac{\langle \Psi_k(f), \Psi_k(f) \rangle}{\langle f, f \rangle^l}$ is an algebraic number for $f \in S_k(SL_2(\mathbf{Z}))$ and $l \neq 2$.

We mention that Zagier [10] proved the algebraicity properties of the Petersson norms in the case where $l = 2$ and $f \in S_k(\Gamma_0(q), \chi)$.

§ 0 Notation and Preliminaries

We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For an associative commutative ring R with identity element we denote by $M_{m,n}(R)$ the set of $m \times n$ matrices entries in R. We set $M_n(R) = M_{n,n}(R)$ and $R^n = M_{1,n}(R)$. Put $SL_n(R) = \{g \in M_n(R) \mid \det g = 1\}$ and $GL_n(R) = \{g \in M_n(R) \mid \det g \in R^\times\}$, where $R^\times$ denotes the group of all invertible elements of R. Let $\mathfrak{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}$ be the complex upper half plane.

§ 1 The Lifting of Modular Forms to Hilbert Modular Forms

For a totally real number field K of degree l, we denote by O_K, $\mathfrak{d}$, d_F, E the ring of integers, the different, the discriminant and the unit group, respectively. We consider the set of the isomorphims τ_i $(1 \leqq i \leqq l)$ of K into $\mathbf{R}$. For $\alpha \in K$, we put $\alpha^{(i)} = \tau_i(\alpha)$ $(1 \leqq i \leqq l)$. Introduce a Hilbert modular group Γ_K defined by

$$\Gamma_K = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K) \middle| a, d \in o_K, b \in \mathfrak{d}^{-1}, c \in \mathfrak{d}, \det \gamma \in E^+ \right\},$$

where $E^+ = \{\varepsilon \in E \mid \varepsilon^{(i)} > 0 \ (1 \leqq i \leqq l)\}$. The group Γ_K acts on $\mathfrak{H}^l$ by

$$z = (z_1, \dots, z_l) \in \mathfrak{H}^l \to \gamma(z) = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \dots, \frac{a^{(l)} z_l + b^{(l)}}{c^{(l)} z_l + d^{(l)}} \right) \in \mathfrak{H}^l$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$. For a positive integer k, let $S_k(\Gamma_K)$ be the space of all Hilbert cusp forms $F(z_1, \dots, z_l) = \sum_{\lambda \in O_K, \lambda \gg 0} C_\lambda e^{2\pi i tr \lambda z}$ of weight k with respect to Γ_K such that

$$F(\gamma(z)) = \prod_{i=1}^{l} (c^{(i)} z_i + d^{(i)})^k F(z) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K \text{ and}$$

$z = (z_1, \dots, z_l) \in \mathfrak{H}^l$, where $tr\, \lambda z = \lambda^{(1)} z_1 + \cdots \lambda^{(l)} z_l$. Here we impose the following conditions on K:

(1-1) $l = [K : \mathbf{Q}]$ is prime, the class number of K is one, K is a tamely ramified cyclic extenstion of $\mathbf{Q}$ and $[E, E^+] = 2^l$.

From those conditions we see that the conductor of $K/\mathbf{Q}$ is equal to a prime q. Furthermore we assume that

(1-2) $$(l, q) = 1.$$

By those assumptions, every form $F(z)$ of $S_k(\Gamma_K)$ has the following Fourier expansion

$$F(z_1, \dots, z_l) = \sum_{\mathfrak{A}} C(\mathfrak{A}) \sum_{\lambda \in O_K, \lambda \gg 0, \mathfrak{A} = (\lambda)} e^{2\pi i tr \lambda z},$$

where $\mathfrak{A}$ runs over all integral ideals of K. We call $C(\mathfrak{A})$ the Fourier coeffcient of F at $\mathfrak{A}$. For two $F, G \in S_k(\Gamma_K)$, we define the Petersson inner product $\langle F, G \rangle$ by

$$\langle F, G \rangle = \mathrm{vol}(\Gamma_K \backslash \mathfrak{H}^l)^{-1} \int_{\Gamma_K \backslash \mathfrak{H}^l} \overline{F(z)} G(z) \left(\prod_{i=1}^{l} y^{(i)} \right)^k dz,$$

where $z = (z_1, \dots, z_l) \in \mathfrak{H}^l$, $z_i = x^{(i)} + \sqrt{-1} y^{(i)}$ $(1 \leqq i \leqq l)$ and $dz = \prod_{i=1}^{l} \dfrac{dx^{(i)} dy^{(i)}}{(y^{(i)})^2}$. For a positive integer M, put

$$\Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \,\middle|\, c \equiv 0 \pmod{M} \right\}.$$

Let ψ be a Dirichlet character modulo M. For a positive integer k, we denote by $S_k(\Gamma_0(M), \psi)$ the set of all cusp forms $f(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}$ such that

$$f\left(\frac{az + b}{cz + d}\right) = \psi(d)(cz + d)^k f(z) \quad \text{for every } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M).$$

For two $f, g \in S_k(\Gamma_0(M), \psi)$, we define the Petersson inner product $\langle f, g\rangle$ by

$$\langle f, g\rangle = \mathrm{vol}(\Gamma_0(M)\backslash\mathfrak{H})^{-1}\int_{\Gamma_0(M)\backslash\mathfrak{H}} \overline{f(z)}g(z)y^k\,dz,$$

where $z \in \mathfrak{H}$, $z = x + \sqrt{-1}y$ and $dz = \dfrac{dxdy}{y^2}$. Applying twisted trace formula, Saito [4, Theorem 3] proved the following theorem.

Theorem 1.1. *Let k be an even integer such that $k > 3$ and χ a primitive character modulo q attached to the abelian extension K of* $\mathbf{Q}$. *Then, there exist normalized Hecke eigen forms $F_1, \dots, F_d$ (resp. $F'_1, \dots, F'_{d'}$) in $S_k(\Gamma_K)$ for a basis of consisting of normalized eigen form $f_1, \dots, f_d$ (resp. $f'_1, \dots, f'_{d'}$) in $S_k(SL_2(\mathbf{Z}))$ (resp. $S_k(\Gamma_0(q), \chi)$) such that*

$$f_i(z) = \sum_{n=1}^{\infty} c_i(n)e^{2\pi inz} \quad (1 \leqq i \leqq d) \tag{1-3}$$

$$\left(\mathit{resp.}\ f'_i(z) = \sum_{n=1}^{\infty} c'_i(n)e^{2\pi inz} \ (1 \leqq i \leqq d')\right)$$

are a basis of $S_k(SL_2(\mathbf{Z}))$ (resp. $S_k(\Gamma_0(q), \chi)$) and the Fourier coefficient of F_i (resp. F'_i) at every $\mathfrak{P}$ $((\mathfrak{P}, q) = 1)$ is given by

$$C_i(\mathfrak{P}) = \begin{cases} c_i(p) & \text{if } N(\mathfrak{P}) = p \text{ for some prime } p, \\ c_i(p^l) - p^{k-1}c_i(p^l/p^2) & \text{if } N(\mathfrak{P}) = p^l \text{ for some prime } p. \end{cases} \tag{1-4}$$

$$\left(\mathit{resp.}\ C'_i(\mathfrak{P}) = \begin{cases} c'_i(p) & \text{if } N(\mathfrak{P}) = p \text{ for some prime } p, \\ c'_i(p^l) - \chi(p)p^{k-1}c'_i(p^l/p^2) & \text{if } N(\mathfrak{P}) = p^l \text{ for some prime } p. \end{cases}\right)$$

Remark. This theorem is a part of Saito [4, Theorem 3]. In fact, he showed that $F_1, \dots, F_d$ are linearly independent (resp. $F'_1, \dots, F_{d'}$ generate a vector space of dimension $\frac{1}{2}d'$) and characterized the subspace of $S_k(\Gamma_K)$ generated by $\Psi_k(f_i)$ $(1 \leqq i \leqq d)$ and $\Psi_k(f'_i)$ $(1 \leqq i \leqq d')$ as the space of $S_k(\Gamma_K)$ generated by eigen forms which are invariant under the action of $Gal(K/\mathbf{Q})$, where f'_i $(1 \leqq i \leqq d')$ are all primitive forms in $\bigoplus_\chi S_k(\Gamma_0(q), \chi)$, χ runs over all primitive characters of the conductor q associated with the abelian extension K of $\mathbf{Q}$ and $\Psi_k : S_k(SL_2(\mathbf{Z})) \oplus_\chi S_k(\Gamma_0(q), \chi) \to S_k(\Gamma_K)$ is a linear mapping given by $F_i = \Psi_k(f_i)$ and $F'_j = \Psi_k(f'_j)$ $(1 \leqq i \leqq d, 1 \leqq j \leqq d')$. Let $\{\chi_i\}_{i=1}^{l}$ be the

Dirichlet characters modulo q determined by the cyclic extension K over $\mathbf{Q}$, where $\chi_1(n) = 1$ for every $n \in \mathbf{Z}$. For convenience, we write X_K as $\{\chi_i\}_{i=1}^{l}$. Put

$$L(s, f_i) = \sum_{n=1}^{\infty} c_i(n) n^{-s}, \quad L(s, f_i, \chi_h) = \sum_{n=1}^{\infty} \chi_h(n) c_i(n) n^{-s} \tag{1-5}$$

and

$$L(s, F_i) = \sum_{\mathfrak{A}} C_i(\mathfrak{A}) N(\mathfrak{A})^{-s} \quad (1 \leqq i \leqq d).$$

Put

$$L(s, f_i') = \sum_{n=1}^{\infty} c_i'(n) n^{-s}, \quad L(s, (f_i')_\rho) = \sum_{n=1}^{\infty} \overline{c_i'(n)} n^{-s}, \quad L(s, f_i', \chi) = \sum_{n=1}^{\infty} \chi(n) c_i'(n) n^{-s}$$

for every $\chi \in X_K$ and $L(s, F_i') = \sum_{\mathfrak{A}} C_i'(\mathfrak{A}) N(\mathfrak{A})^{-s}$ $(1 \leqq i \leqq d')$. Then the following proposition is given in [5].

Proposition 1.2. *Under the same notation in Theorem* 1.1, *let* $\mathfrak{B}$ *be the prime ideal in* K *such that* $\mathfrak{B} | q$. *Then*

$$C_i(\mathfrak{B}) = c_i(q), \quad L(s, F_i) = \prod_{h=1}^{l} L(s, f_i, \chi_h) \quad (1 \leqq i \leqq d), \tag{1-6}$$

$$C_i'(\mathfrak{B}) = c_i'(q) + \overline{c_i'(q)} \tag{1-7}$$

and

$$L(s, F_i') = L(s, f_i') L(s, (f_i')_\rho) \prod_{\chi' \in X_K, \chi' \neq 1, \chi' \neq \bar{\chi}} L(s, f_i, \chi') \quad (1 \leqq i \leqq d'),$$

where $(f_i')_\rho(z) = \sum_{n=1}^{\infty} \overline{c_i'(n)} e^{2\pi i n z}$.

For any integral ideal $\mathfrak{A}$ and positive integer d such that $d \,|\, N(\mathfrak{A})$, a function $\phi_k(d; \mathfrak{A})$ (resp. $\phi_k'(d; \mathfrak{A})$) is determined by the relations:

(1-8) If $\mathfrak{A} = \mathfrak{A}_1 \mathfrak{A}_2$ $((\mathfrak{A}_1, \mathfrak{A}_2) = 1)$, $d = d_1 d_2$ $(d_1 \,|\, N(\mathfrak{A}_1), d_2 \,|\, N(\mathfrak{A}_2))$, then

$\phi_k(d_1 d_2; \mathfrak{A}) = \phi_k(d_1; \mathfrak{A}) \phi_k(d_2; \mathfrak{A})$ (resp. $\phi_k'(d_1 d_2; \mathfrak{A}) = \phi_k'(d_1; \mathfrak{A}) \phi_k'(d_2; \mathfrak{A})$), (1-9) If $\mathfrak{A}=\mathfrak{P}^{\lambda}$ and $N(\mathfrak{P})=p^{l}$ for some prime ideal $\mathfrak{P}$ and some prime p, then

$$\phi_k(d;\mathfrak{P}^{\lambda})=\phi'_k(d;\mathfrak{P}^{\lambda})\quad\text{and}$$

$$\phi_k(d;\mathfrak{P}^{\lambda})=\begin{cases}(-1)^i d^{k-1} & \text{if } l\neq 2,\ \frac{d}{p^i}\in S \text{ for some } i\in\{0,1\},\\ (-1)^i d^{k-1} & \text{if } l=2,\ \frac{d}{p^i}\in S \text{ for some } i\in\{0,1\} \text{ and } d^2\neq N(\mathfrak{P}^{\lambda}),\\ \frac{1+(-1)^{\lambda}}{2}d^{k-1} & \text{if } l=2,\ d^2=N(\mathfrak{P}^{\lambda}),\\ 0 & \text{otherwise,}\end{cases}$$

with $S=\{N(\mathfrak{A}')\,|\,\text{an ideal } \mathfrak{A}' \text{ such that } \mathfrak{A}'\supset\mathfrak{P}^{\lambda}\}$,

If $\mathfrak{A}=\mathfrak{P}^{\lambda}$ and $N(\mathfrak{P})=p$ for some prime ideal $\mathfrak{P}$ and some prime p and $(\mathfrak{P},q)=1$, then

$$\phi_k(d;\mathfrak{P}^{\lambda})=\phi'_k(d;\mathfrak{P}^{\lambda})\quad\text{and}\quad\phi_k(d;\mathfrak{P}^{\lambda})=\begin{cases}1 & \text{if } d=1,\\ 0 & \text{otherwise}\end{cases}$$

and

If $\mathfrak{A}=\mathfrak{P}^{\lambda}$ and $N(\mathfrak{P})=q$, then $\phi_k(d;\mathfrak{P}^{\lambda})=\begin{cases}1 & \text{if } d=1,\\ 0 & \text{otherwise}\end{cases}$ and $\phi'_k(d;\mathfrak{P}^{\lambda})=d^{k-1}$. We may deduce the following theorem.

THEOREM 1.3. *Let* $f(z)=\sum_{n=1}^{\infty}c(n)e^{2\pi inz}$ *be an element of* $S_k(SL_2(\mathbf{Z}))$ (*resp.* $S_k(\Gamma_0(q),\chi)$). *Then the Fourier coefficient* $C_f(\mathfrak{A})$ *of* $\Psi_k(f)$ *at any integral ideal* $\mathfrak{A}$ *equals*

(1-10)
$$C_f(\mathfrak{A})=\sum_{d|N(\mathfrak{A})}\phi_k(d;\mathfrak{A})c(N(\mathfrak{A})/d^2)$$
$$\left(\textit{resp.}\quad=\sum_{d|N(\mathfrak{A})}\phi'_k(d;\mathfrak{A})c(N(\mathfrak{A})/d^2)\right).$$

PROOF. We assume that $f(z)$ is a normalized Hecke eigen form in $S_k(SL_2(\mathbf{Z}))$. We put

(1-11)
$$L(s,f)=\prod_p(1-c(p)p^{-s}+p^{k-1-2s})^{-1}$$
$$=\prod_p((1-\alpha_p p^{(k-1)/2}p^{-s})(1-\alpha_p^{-1}p^{(k-1)/2}p^{-s}))^{-1}$$

and

$$L(s, \Psi_k(f)) = \prod_{\mathfrak{P}} (1 - C_f(\mathfrak{P}) N(\mathfrak{P})^{-s} + N(\mathfrak{P})^{k-1-2s})^{-1}$$
$$= \prod_{\mathfrak{P}} ((1 - \alpha_{\mathfrak{P}} N(\mathfrak{P})^{(k-1)/2} N(\mathfrak{P})^{-s})(1 - \alpha_{\mathfrak{P}}^{-1} N(\mathfrak{P})^{(k-1)/2} N(\mathfrak{P})^{-s}))^{-1}.$$

The following is a key relation for our later arguments:

$$\alpha_{\mathfrak{P}} = \begin{cases} \alpha_p^l \text{ or } \alpha_p^{-l} & \text{if } N(\mathfrak{P}) = p^l, \\ \alpha_p \text{ or } \alpha_p^{-1} & \text{if } N(\mathfrak{P}) \text{ is prime.} \end{cases} \tag{1-12}$$

First we treat the case where $N(\mathfrak{P}) = p^l$. To prove (1-10), we need the Laurent polynomial defined by

$$Y_n(x) = \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}} = x^{-n} + x^{-n+2} + \cdots + x^{n-2} + x^n.$$

From (1-11), we have

$$\sum_{n=1}^{\infty} c(p^n) x^n = \frac{1}{1 - c(p)x + p^{k-1}x^2} = \sum_{n=1}^{\infty} p^{((k-1)/2)n} Y_n(\alpha_p) x^n, \tag{1-13}$$

which implies that

$$c(p^n) = p^{((k-1)/2)n} Y_n(\alpha_p).$$

By our definition, we may check that

$$C_f(\mathfrak{P}) = p^{((k-1)/2)l} (Y_l(\alpha_p) - Y_{l-2}(\alpha_p)) = N(\mathfrak{P})^{(k-1)/2} (\alpha_{\mathfrak{P}} + \alpha_{\mathfrak{P}}^{-1}), \tag{1-14}$$

which yields (1-12). Similarly, from (1-11), we obtain

$$C_f(\mathfrak{P}^{\lambda}) = (N(\mathfrak{P})^{(k-1)/2})^{\lambda} Y_{\lambda}(\alpha_{\mathfrak{P}}).$$

If λ is even, we have

$$Y_{\lambda}(\alpha_{\mathfrak{P}}) = Y_{\lambda}(\alpha_p^l) = \sum_{i=0}^{\lambda/2-1} (\alpha_p^{-l(\lambda-2i)} + \alpha_p^{l(\lambda-2i)}) + 1.$$

Since $x^{-h} + x^h = Y_h(x) - Y_{h-2}(x)$ $(h \geqq 2)$, we find that

$$Y_{\lambda}(\alpha_{\mathfrak{P}}) = \sum_{i=0}^{\lambda/2-1} (Y_{l(\lambda-2i)}(\alpha_p) - Y_{l(\lambda-2i)-2}(\alpha_p)) + 1. \tag{1-15}$$

Similarly, if λ is odd, we conclude that

$$Y_\lambda(\alpha_{\mathfrak{P}}) = \sum_{i=0}^{(\lambda-1)/2} (Y_{l(\lambda-2i)}(\alpha_p) - Y_{l(\lambda-2i)-2}(\alpha_p)).$$

Thus we may deduce that

$$C_f(\mathfrak{P}^\lambda) = \sum_{d|N(\mathfrak{P}^\lambda)} \phi_k(d;\mathfrak{P}^\lambda)c(N(\mathfrak{P}^\lambda)/d^2).$$

We may also verify the case where $N(\mathfrak{P})$ is a prime. Next we assume that $f'(z) = \sum_{n=1}^{\infty} c'(n)e^{2\pi inz}$ is a normalized Hecke eigen form in $S_k(\Gamma_0(q),\chi)$. We put

$$L(s,f') = \prod_p (1 - c'(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1} \quad \text{and} \tag{1-16}$$

$$\begin{aligned} L(s,\Psi_k(f')) &= \prod_{\mathfrak{P}} (1 - C_{f'}(\mathfrak{P})N(\mathfrak{P})^{-s} + N(\mathfrak{P})^{k-1-2s})^{-1} \\ &= \prod_{\mathfrak{P}} ((1 - \alpha_{\mathfrak{P}}N(\mathfrak{P})^{(k-1)/2}N(\mathfrak{P})^{-s})(1 - \alpha_{\mathfrak{P}}^{-1}N(\mathfrak{P})^{(k-1)/2}N(\mathfrak{P})^{-s}))^{-1}. \end{aligned}$$

Furthermore, we define two complex numbers α_p and β_p by

$$\begin{cases} \alpha_p + \beta_p = (p^{(k-1)/2})^{-1}c'(p) \text{ and } \alpha_p\beta_p = \chi(p) & \text{if } p \neq q, \\ \alpha_p = (p^{(k-1)/2})^{-1}c'(p) \text{ and } \beta_p = (p^{(k-1)/2})^{-1}\overline{c'(p)} & \text{if } p = q. \end{cases} \tag{1-17}$$

We can derive the following relation

$$\alpha_{\mathfrak{P}} = \begin{cases} \alpha_p^l \text{ or } \beta_p^l & \text{if } N(\mathfrak{P}) = p^l, \\ \alpha_p \text{ or } \beta_p & \text{if } N(\mathfrak{P}) \text{ is prime.} \end{cases} \tag{1-18}$$

First we treat the case where $N(\mathfrak{P}) = p^l$. Define a Laurent polynomial $Y_n(x,y)$ by

$$Y_n(x,y) = \frac{x^{n+1} - y^{n+1}}{x - y}. \tag{1-19}$$

Then we obtain

$$c'(p^n) = p^{((k-1)/2)n}Y_n(\alpha_p,\beta_p).$$

From the definition, we have

(1-20) $$C_{f'}(\mathfrak{P}) = c'(p^l) - \chi(p)p^{k-1}c'(p^{l-2}) = p^{((k-1)/2)l}(Y_l(\alpha_p,\beta_p) - \chi(p)Y_{l-2}(\alpha_p,\beta_p)).$$

Now we can check that

$$Y_l(\alpha_p,\beta_p) = \alpha_p^l + \beta_p^l + \alpha_p\beta_p(\alpha_p^{l-2} + \alpha_p^{l-1}\beta_p + \cdots + \beta_p^{l-2}) = \alpha_p^l + \beta_p^l + \chi(p)Y_{l-2}(\alpha_p,\beta_p).$$

Since $C_{f'}(\mathfrak{P}) = N(\mathfrak{P})^{(k-1)/2}Y_1(\alpha_{\mathfrak{P}},\alpha_{\mathfrak{P}}^{-1})$, we get our desired relation (1-18). We see that

(1-21) $$C_{f'}(\mathfrak{P}^\lambda) = (N(\mathfrak{P})^{(k-1)/2})^\lambda Y_\lambda(\alpha_{\mathfrak{P}},\alpha_{\mathfrak{P}}^{-1}) = (N(\mathfrak{P})^{(k-1)/2})^\lambda((\alpha_p^l)^\lambda + (\alpha_p^l)^{\lambda-1}\beta_p^l + \cdots + (\alpha_p^l)(\beta_p^l)^{\lambda-1} + (\beta_p^l)^\lambda).$$

If $2|\lambda$, then we have

$$\begin{aligned} Y_\lambda(\alpha_{\mathfrak{P}},\alpha_{\mathfrak{P}}^{-1}) &= Y_1(\alpha_p^{l\lambda},\beta_p^{l\lambda}) + (\alpha_p\beta_p)^l Y_1(\alpha_p^{l(\lambda-2)},\beta_p^{l(\lambda-2)}) + (\alpha_p\beta_p)^{2l} \\ &\quad \times Y_1(\alpha_p^{l(\lambda-4)},\beta_p^{l(\lambda-4)}) + \cdots + (\alpha_p\beta_p)^{(\lambda/2-1)l}Y_1(\alpha_p^{2l},\beta_p^{2l}) + 1 \\ &= \sum_{i=0}^{\lambda/2-1}(\alpha_p^{l(\lambda-2i)} + \beta_p^{l(\lambda-2i)}) + 1 \\ &= \sum_{i=0}^{\lambda/2-1}(Y_{l(\lambda-2i)}(\alpha_p,\beta_p) - Y_{l(\lambda-2i)-2}(\alpha_p,\beta_p)) + 1. \end{aligned}$$

If $2\nmid\lambda$, we have

$$\begin{aligned} Y_\lambda(\alpha_{\mathfrak{P}},\alpha_{\mathfrak{P}}^{-1}) &= \sum_{i=0}^{(\lambda-1)/2}(\alpha_p^{l(\lambda-2i)} + \beta_p^{l(\lambda-2i)}) \\ &= \sum_{i=0}^{(\lambda-1)/2}(Y_{l(\lambda-2i)}(\alpha_p,\beta_p) - Y_{l(\lambda-2i)-2}(\alpha_p,\beta_p)). \end{aligned}$$

Consequently, we may deduce

$$C_{f'}(\mathfrak{P}^\lambda) = \sum_{d|N(\mathfrak{P}^\lambda)}\phi_k'(d,\mathfrak{P}^\lambda)c(N(\mathfrak{P}^\lambda)/d^2).$$

We may also verify the case where $N(\mathfrak{P})$ is a prime. This proves our assertion.

§ 2 The Petersson Inner Product of the Lifting

Let $f(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ be a normalized Hecke eigen form in $S_k(SL_2(\mathbf{Z}))$. Define the Dirichlet series $D(s, f, \chi')$ by

(2-1) $D(s, f, \chi')$

$$= \prod_p ((1 - \chi'(p)\alpha_p^2 p^{k-1-s})(1 - \chi'(p)\alpha_p\beta_p p^{k-1-s})(1 - \chi'(p)\beta_p^2 p^{k-1-s}))^{-1}$$

$$= L(s - k + 1, \chi')^{-1} L(2s - 2k + 2, (\chi')^2) \sum_{n=1}^{\infty} \chi'(n)c(n)^2 n^{-s}$$

for every $\chi' \in X_K$, where $L(*, \psi)$ means the Dirichlet series attached to a Dirichlet character ψ and α_p and β_p are given by

$$L(s, f) = \sum_{n=1}^{\infty} c(n)n^{-s} = \prod_p ((1 - \alpha_p p^{(k-1)/2} p^{-s})(1 - \beta_p p^{(k-1)/2} p^{-s}))^{-1}.$$

Then the following theorem was proved by Shimura [6, Theorem 1 and 2].

Theorem 2.1. *Set*

$$H(s, f, \chi') = \pi^{-3s/2}\Gamma(s/2)\Gamma((s+1)/2)\Gamma((s-k+2-\lambda_0)/2)D(s, f, \chi'),$$

where $\lambda_0 = \begin{cases} 0 & \text{if } \chi'(-1) = 1, \\ 1 & \text{if } \chi'(-1) = -1. \end{cases}$ *Then, $H(s, f, \chi')$ can be continued to a meromorphic function on the whole s-plane, which is holomorphic except for possible simple poles at $s = k$ and $s = k - 1$. Moreover, the following is equivalent:*

(2-2) *$H(s, f, \chi')$ has a simple pole at $s = k$.*

χ' is a non-trivial character of order 2 and

$$\int_{\Gamma_0(r^2)\backslash\mathfrak{H}} f\bar{g}y^{k-1}\, dxdy \neq 0,$$

where r is the conductor of χ' and $g(z) = \sum_{n=1}^{\infty} \overline{\chi'(n)c(n)}e^{2\pi inz}$.

This theorem is proved in a more general form in [6]. Put $F = \Psi_k(f)$ for a normalized Hecke eigenform f in S_k. Define the function $\tilde{L}(s, F)$ by

$$\tilde{L}(s, F) = \sum_{\mathfrak{A}} C_f(\mathfrak{A})\overline{C_f(\mathfrak{A})}N(\mathfrak{A})^{-s}. \tag{2-3}$$

We see that $\tilde{L}(s,F) = D(s,F_\rho,F)$, where $D(s,F_\rho,F)$ is the function given in [8, p. 661]. By [7, Lemma 1], we have

(2-4) $$\tilde{L}(s,F) = \zeta_K(2s-2k+2)^{-1}$$
$$\prod_{\mathfrak{P}} \frac{1}{(1-\alpha_{\mathfrak{P}}\overline{\alpha_{\mathfrak{P}}}N(\mathfrak{P})^{k-1-s})(1-\alpha_{\mathfrak{P}}^{-1}\overline{\alpha_{\mathfrak{P}}}N(\mathfrak{P})^{k-1-s})}$$
$$\times \frac{1}{(1-\alpha_{\mathfrak{P}}\overline{\alpha_{\mathfrak{P}}}^{-1}N(\mathfrak{P})^{k-1-s})(1-\alpha_{\mathfrak{P}}^{-1}\overline{\alpha_{\mathfrak{P}}}^{-1}N(\mathfrak{P})^{k-1-s})},$$

where ζ_K means the Dedekind zeta function of K and $\alpha_{\mathfrak{P}}$ is given by

$$L(s,F) = \prod_{\mathfrak{P}} (1 - C_f(\mathfrak{P})N(\mathfrak{P})^{-s} + N(\mathfrak{P})^{k-1-2s})^{-1}$$
$$= \prod_{\mathfrak{P}} ((1-\alpha_{\mathfrak{P}}N(\mathfrak{P})^{(k-1)/2}N(\mathfrak{P})^{-s})(1-\alpha_{\mathfrak{P}}^{-1}N(\mathfrak{P})^{(k-1)/2}N(\mathfrak{P})^{-s})^{-1}.$$

We may derive the following.

PROPOSITION 2.2. *Under the above notation, for a normalized Hecke eignform* $f \in S_k(SL_2(\mathbf{Z}))$ *(resp.* $f' \in S_k(\Gamma_0(q),\chi)$*), one has*

(2-5) $$\tilde{L}(s,F) = \zeta_K(s-k+1)\zeta_K(2s-2k+2)^{-1} \prod_{\chi' \in X_K} D(s,f,\chi').$$

$$\left(\textit{resp.} \;\; = \zeta_K(2s-2k+2)^{-1} \prod_{\chi' \in X_K} \prod_p ((1-\chi'(p)|\alpha_p|^2 p^{k-1-s}) \left(1-\chi'(p)\frac{\alpha_p}{\overline{\alpha_p}}p^{k-1-s}\right)\right.$$
$$\left. \left(1-\chi'(p)\frac{\overline{\alpha_p}}{\alpha_p}p^{k-1-s}\right)(1-\chi'(p)|\alpha_p|^{-2}p^{k-1-s}))^{-1} \right),$$

where $f'(z) = \sum_{n=1}^\infty c'(n)e^{2\pi inz}$ *and* $\sum_{n=1}^\infty c'(n)n^{-s} = (1-\alpha_q q^{(k-1)/2}q^{-s})^{-1} \cdot \prod_{p \neq q}((1-\alpha_p p^{(k-1)/2}p^{-s})(1-\beta_p p^{(k-1)/2}p^{-s}))^{-1}$.

PROOF. Let f be a Hecke eigen form in $S_k(SL_2(\mathbf{Z}))$. Then we see that the Fourier coefficients of $\Psi_k(f)$ are all real number. Therefore $\alpha_{\mathfrak{P}} = \overline{\alpha_{\mathfrak{P}}}$ for all prime ideals $\mathfrak{P}$. We assume that $N(\mathfrak{P}) = p^l$. We see that

(2-6) $$((1-\alpha_{\mathfrak{P}}^2 N(\mathfrak{P})^{k-1-s})(1-\alpha_{\mathfrak{P}}^{-2}N(\mathfrak{P})^{k-1-s}))^{-1}$$
$$= ((1-(\alpha_p^2 p^{k-1-s})^l)(1-(\alpha_p^{-2}p^{k-1-s})^l))^{-1}.$$

Since $\{\chi'(p)\,|\,\chi'\in X_K,\chi'\neq 1\}=\{\zeta\,|\,\zeta^l=1,\zeta\neq 1\}$, the right hand side of (2-6) is equal to

$$\text{(2-7)}\qquad \prod_{\chi'\in X_K}((1-\chi'(p)\alpha_p^2p^{k-1-s})(1-\chi'(p)\alpha_p^{-2}p^{k-1-s}))^{-1}$$
$$=\zeta_K(s-k+1)_p^{-1}\prod_{\chi'\in X_K}D_p(s,f,\chi'),$$

where $D_p(s,f,\chi')$ and $\zeta_K(s-k+1)_p$ are p-factors of $D(s,f,\chi')$ and $\zeta_K(s-k+1)$, respectively. Next consider the case where $p=\mathfrak{P}_1\cdots\mathfrak{P}_l$. Then

$$((1-\alpha_{\mathfrak{P}_i}^2N(\mathfrak{P}_i)^{k-1-s})(1-\alpha_{\mathfrak{P}_i}^{-2}N(\mathfrak{P}_i)^{k-1-s}))^{-1}=((1-\alpha_p^2p^{k-1-s})(1-\alpha_p^{-2}p^{k-1-s}))^{-1}.$$

Since $\chi'(p)=1$ for every $\chi'\in X_K$, we find that

$$\text{(2-8)}\qquad \prod_{i=1}^{l}((1-\alpha_{\mathfrak{P}_i}^2N(\mathfrak{P}_i)^{k-1-s})(1-\alpha_{\mathfrak{P}_i}^{-2}N(\mathfrak{P}_i)^{k-1-s}))^{-1}$$
$$=\prod_{\chi'\in X_K}((1-\chi'(p)\alpha_p^2p^{k-1-s})(1-\chi'(p)\alpha_p^{-2}p^{k-1-s}))^{-1}$$
$$=\zeta_K(s-k+1)_p^{-1}\prod_{\chi'\in X_K}D_p(s,f,\chi').$$

Thus we conclude the proof in this case. We can also treat the case where $f\in S_k(\Gamma_0(q),\chi)$ in the same manner. So we check only the case where $q=\mathfrak{P}^l$.

$$\zeta_K(2s-2k+2)_q^{-1}\prod_{\mathfrak{P}}\frac{1}{(1-\alpha_{\mathfrak{P}}\overline{\alpha_{\mathfrak{P}}}N(\mathfrak{P})^{k-1-s})(1-\alpha_{\mathfrak{P}}^{-1}\overline{\alpha_{\mathfrak{P}}}N(\mathfrak{P})^{k-1-s})}$$
$$\times\frac{1}{(1-\alpha_{\mathfrak{P}}\overline{\alpha_{\mathfrak{P}}}^{-1}N(\mathfrak{P})^{k-1-s})(1-\alpha_{\mathfrak{P}}^{-1}\overline{\alpha_{\mathfrak{P}}}^{-1}N(\mathfrak{P})^{k-1-s})}$$
$$=\zeta_K(2s-2k+2)_q^{-1}\prod_{\chi'\in X_K}\frac{1}{(1-\chi'(q)|\alpha_q|^2q^{k-1-s})(1-\chi'(q)\alpha_q^{-1}\overline{\alpha_q}q^{k-1-s})}$$
$$\times\frac{1}{(1-\chi'(q)\alpha_q\overline{\alpha_q}^{-1}q^{k-1-s})(1-\chi'(q)|\alpha_q|^{-2}q^{k-1-s})}.$$

This proves our assertion.

We may deduce the following theorem.

THEOREM 2.3. *Suppose that* $f = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ *be a normalized Hecke eigen form in* $S_k(SL_2(\mathbf{Z}))$ (*resp.* $S_k(\Gamma_0(q),\chi)$). *Then*

$$\langle \Psi_k(f), \Psi_k(f)\rangle = ((4\pi)^{-k}(k-1)!)^{l-1}|d_F|^{-1/2}[E^+ : E^2]\langle f,f\rangle \times \prod_{\chi' \in X_K, \chi' \neq 1} D(k,f,\chi')L(2,\chi')^{-1}. \tag{2-9}$$

$$(\textit{resp.}\ = 2^{1-l}((4\pi)^{-k}(k-1)!)^{l-1}[E^+ : E^2]R_K^{-1}\langle f,f\rangle\zeta_K(2)^{-1}\tilde{D}(k,f)),$$

where $E^2 = \{\varepsilon^2 | \varepsilon \in E\}$,

$$\tilde{D}(s,f) = \prod_{\chi' \in X_K - \{1\}} \prod_p ((1-\chi'(p)|\alpha_p|^2 p^{k-1-s})\left(1-\chi'(p)\frac{\alpha_p}{\overline{\alpha_p}}p^{k-1-s}\right) \left(1-\chi'(p)\frac{\overline{\alpha_p}}{\alpha_p}p^{k-1-s}\right)(1-\chi'(p)|\alpha_p|^{-2}p^{k-1-s}))^{-1}$$

and R_K *is the regulator of* K.

PROOF. Let f be a normalized Hecke eigen form of $S_k(SL_2(\mathbf{Z}))$. By [6, Theorem 2], we see that $D(s,f,1)$ is holomorphic at $s=k$. Furthermore, by [8, Proposition 4.13], the Dirichlet series $\sum_{n=1}^{\infty} c(n)^2 n^{-s}$ has a simple pole at $s=k$. It follows from (2.1) that $D(k,f,1)$ is non vanishing. By virtue of [6, Theorem 2], $D(s,f,\chi')$ is holomorphic at $s=k$ in the case where $l \neq 2$. When $l=2$, we assume that $D(s,f,\chi')$ has a simple pole at $s=k$. Then by Proposition 2.2, $\tilde{L}(s,F)$ has a pole of order 2 at $s=k$. This contradicts the assertion in [8, Proposition 4.13]. Therefore $D(s,f,\chi')$ is holomorphic at $s=k$. Using [8, Proposition 4.13], we have

$$\lim_{s\to k}(s-k)\tilde{L}(s,F) = 2^{l-1}(4\pi)^{kl}\Gamma(k)^{-l}R_K[E^+ : E^2]^{-1}\langle F,F\rangle.$$

By Proposition 2.2, we obtain

$$\begin{aligned}\lim_{s\to k}(s-k)\tilde{L}(s,F) &= \lim_{s\to k}(s-k)\zeta_K(s-k+1)\zeta_K(2)^{-1}\prod_{\chi' \in X_K} D(k,f,\chi') \\ &= 2^l(R_K/2|d|^{1/2})\zeta_K(2)^{-1}\prod_{\chi' \in X_K} D(k,f,\chi') \\ &= 2^{l-1}R_K|d|^{-1/2}\zeta_K(2)^{-1}\prod_{\chi' \in X_K} D(k,f,\chi').\end{aligned}$$

Now by (2-2) and [8, Proposition 4.13], we have

$$D(k,f,1) = \lim_{s\to k} \zeta(s-k+1)^{-1}\zeta(2s-2k+2)\sum_{n=1}^{\infty} c(n)^2 n^{-s}$$
$$= \zeta(2)\lim_{s\to k}\frac{1}{(s-k)\zeta(s-k+1)}(s-k)\sum_{n=1}^{\infty} c(n)^2 n^{-s}$$
$$= \zeta(2)(4\pi)^k\Gamma(k)^{-1}\langle f,f\rangle.$$

Therefore we deduce

$$\langle F,F\rangle = ((4\pi)^{-k}(k-1)!)^{l-1}|d_F|^{-1/2}[E^+ : E^2]\langle f,f\rangle \times \prod_{\chi'\in X_K,\chi'\neq 1} D(k,f,\chi')L(2,\chi')^{-1}. \tag{2-9}$$

We obtain the proof of the case where $f \in S_k(SL_2(\mathbf{Z}))$. Similarly, we may derive the proof of the case where f belongs to $S_k(\Gamma_0(q),\chi)$. This completes our proof.

By virtue of Strum [9, Theorem 1], we may deduce the following corollary.

COROLLARY. *Let f be an element of $S_k(SL_2(\mathbf{Z}))$. Then $D(s,f,\chi')$ is non vanishing at $s=k$ for every $\chi' \in X_K - \{1\}$. Moreover, the value of $\langle \Psi_k(f),\Psi_k(f)\rangle \cdot \langle f,f\rangle^{-l}$ is an algebraic number for $l>2$.*

PROOF. By [9, Theorem 1], we have the value of $\pi^{-k-2}D(k,f,\chi')\langle f,f\rangle^{-1}$ is an algebraic number for $\chi' \in X_K - \{1\}$. On the other hand, the value of $L(2,\chi')\pi^{-2}$ is an algebraic number for $\chi' \in X_K - \{1\}$. Therefore, by Theorem 2.3, we obtain our assertion.

We mention that the author does not know whether Corollary is ture in the case where $l=2$.

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Department of Mathematics
Faculty of Science
Saitama University, Urawa, 338-8570
Japan
e-mail: hkojima@rimath.saitama-u.ac.jp