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NONEXISTENCE OF GLOBAL SOLUTIONS IN TIME FOR REACTION-DIFFUSION SYSTEMS WITH INHOMOGENEOUS TERMS IN CONES

By

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Abstract. We consider initial-boundary value problems for the reaction-diffusion systems with inhomogeneous terms in cones. In this paper we show the nonexistence of global solutions of the problems in time.

1. Introduction

We consider nonnegative solutions of initial-boundary value problems for the reaction-diffusion systems of the form

$$\begin{cases} u_t = \Delta u + K_1(x,t)v^{p_1}, & x \in D,\ t > 0, \\ v_t = \Delta v + K_2(x,t)u^{p_2}, & x \in D,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in D, \end{cases} \tag{1}$$

where $p_1, p_2 \geq 1$ with $p_1 p_2 > 1$. The domain D is a cone in $\mathbf{R}^N$, such as

$$D = \{x \in \mathbf{R}^N; x \neq 0 \text{ and } x/|x| \in \Omega\}, \tag{2}$$

where Ω is some region on S^{N-1} smooth enough.

The initial data $u_0(x)$ and $v_0(x)$ are nonnegative, bounded and continuous in $\bar{D}$, and $u_0(x) = v_0(x) = 0$ on ∂D. The inhomogeneous terms K_i $(i = 1, 2)$ are nonnegative continuous functions in $D \times (0, \infty)$.

In this paper we denote by BC the set of all bounded continuous functions in $\bar{D}$. The "nontrivial solution" denotes the solution u satisfying $(u, v) \not\equiv 0$ in $D \times (0, T)$ with some $T > 0$, it thus means that $(u_0, v_0) \not\equiv 0$ with the condition $(u_0, v_0) \in BC$.

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For the Laplace-Beltrami operator with homogeneous Dirichlet boundary condition on $\Omega \in S^{N-1}$, define ω_n as Dirichlet eigenvalues and $\psi_n(\theta)$ as the Dirichlet eigenfunctions corresponding to ω_n which is normalized so that

$$\int_\Omega \psi_n(\theta)\,d\theta = 1.$$

It is following that

$$\int_\Omega \psi_m(\theta)\psi_n(\theta)\,d\theta = 0$$

for $m \neq n$. We introduce the Green's function $G(x,y,t) = G(r,\theta,\rho,\phi,t)$ for the linear heat equation in the cone D, where

$$r = |x|, \quad \rho = |y|, \quad \theta = x/|x| \quad \text{and} \quad \phi = y/|y| \in \Omega \tag{3}$$

The Green's function is expressed to

$$G(r,\theta,\rho,\phi,t) = \frac{1}{2t}(r\rho)^{-(N-2)/2}\exp\left(-\frac{\rho^2+r^2}{4t}\right)\sum_{n=1}^{\infty} I_{v_n}\left(\frac{r\rho}{2t}\right)\psi_n(\theta)\psi_n(\phi), \tag{4}$$

where $v_n = [(N-2)^2/4 + \omega_n]^{1/2}$, and I_v is the modified Bessel function or

$$I_v(z) = \left(\frac{z}{2}\right)^v \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\Gamma(v+k+1)} \tag{5}$$

with the Gamma function $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}\,ds$ (see Watson [27, p. 395]).

For our first theorem we shall give the conditions of the inhomogeneous terms K_i $(i = 1,2)$ as following:

$$\left.\begin{array}{l}\text{there exist } C_U,\ \hat{\sigma}_i \text{ and } \hat{q}_i \geq 0 \text{ such that}\\ K_i(x,t) \leq C_U\langle x\rangle^{\hat{\sigma}_i}(t+1)^{\hat{q}_i} \text{ for any } x \in D,\ t \geq 0,\end{array}\right\} \tag{6}$$

where $\langle x\rangle = (|x|^2+1)^{1/2}$.

Let L_a^∞ be a Banach space of L^∞-functions in D with the norm

$$\|\xi\|_{\infty,a} \equiv \operatorname{esssup}_{x\in D}(\langle x\rangle^a|\xi(x)|).$$

For $T > 0$, set

$$E_T = \{(u,v) : [0,T] \to L_{\delta_1}^\infty \times L_{\delta_2}^\infty; \|(u,v)\|_{E_T} < \infty\} \tag{7}$$

with the norm

$$\|(u,v)\|_{E_T} := \sup_{t\in[0,T]}\{\|u(t)\|_{\infty,\delta_1} + \|v(t)\|_{\infty,\delta_2}\},$$

where

$$\delta_i = \frac{\hat{\sigma}_j p_i + \hat{\sigma}_i}{p_i p_j - 1} \quad ((i,j) = (1,2),(2,1)). \tag{8}$$

It is easily seen that E_T is a Banach space.

We begin with stating the existence of the local solution for (1).

THEOREM 1. *Assume that* $u_0, v_0 \in BC$, $u_0 \equiv v_0 \equiv 0$ *on* ∂D, *and* $\langle x \rangle^{\delta_1} u_0(x)$, $\langle x \rangle^{\delta_2} v_0(x)$ *are bounded in* $\bar{D}$. *Suppose that* $K_i(x,t)$ $(i = 1,2)$ *satisfy* (6). *Then there exists a nonnegative solution* $(u,v) \in E_T$ *which solves* (1) *in* $D \times (0,T)$ *for some* $T > 0$.

For given initial values (u_0, v_0), let $T^* = T^*(u_0, v_0)$ be a maximal existence time of the solution of (1). If $T^* = \infty$, the solutions are global in time. On the other hand, if $T^* < \infty$, then the solutions are not global in time. If the solution blows up in finite time such that

$$\limsup_{t \to T^*} \|u(\cdot,t)\|_\infty + \limsup_{t \to T^*} \|v(\cdot,t)\|_\infty = \infty, \tag{9}$$

then the solution is not global, where $\|\cdot\|_\infty$ denotes the L^∞-norm with respect to space variable.

For our second theorem we shall define a region $\tilde{D}$ such that

$$\left.\begin{array}{l} \text{there exist } k > 0 \text{ and } \{x_m\}_{m=1}^\infty \text{ satisfying } 0 < |x_m| < |x_{m+1}|, \\ B(x_m, k|x_m|) \subset \tilde{D} \subset D \text{ for any } m, \text{ and } \lim_{m\to\infty}|x_m| = \infty, \end{array}\right\} \tag{10}$$

where $B(x,r)$ denotes the ball with radius r centered at x. We let the inhomogeneous terms K_i $(i = 1,2)$ satisfy

$$\left.\begin{array}{l} \text{there exist } C_L > 0,\ \check{\sigma}_i, \check{q}_i \ge 0 \text{ and } \tilde{D} \text{ satisfying (10) such that} \\ K_i(x,t) \ge C_L |x|^{\check{\sigma}_i} t^{\check{q}_i} \text{ for any } x \in \tilde{D},\ t \ge 0. \end{array}\right\} \tag{11}$$

For the theorem we should define γ_+ denoting the positive root of the equation $\gamma(\gamma + N - 2) = \omega_1$,

$$\alpha_i = \frac{(2 + \check{\sigma}_i + 2\check{q}_i) + (2 + \check{\sigma}_j + 2\check{q}_j) p_i}{p_i p_j - 1} \quad ((i,j) = (1,2),(2,1)), \tag{12}$$

and

$$H_a = \{\xi \in C(\bar{D}); \xi(x) \ge M \langle x \rangle^{-a} \psi_1(x/|x|) \text{ for } x \in \tilde{D} \text{ with some } M > 0\}.$$

The main result of this paper is summarized in the following theorem.

Theorem 2. *Assume that* $u_0, v_0 \in BC$, $u_0 \equiv v_0 \equiv 0$ *on* ∂D, *and* $K_i(x,t)$ $(i = 1,2)$ *satisfy* (11). *Suppose that one of the following two conditions holds*;

(i) $\max\{\alpha_1, \alpha_2\} \geq N + \gamma_+$.
(ii) $u_0 \in H_{a_1}$ *with* $a_1 < \alpha_1$ *or* $v_0 \in H_{a_2}$ *with* $a_2 < \alpha_2$.

Then, there exists no nontrivial nonnegative global solution of (1).

It is expected that if (6) holds, $\max\{\hat{\alpha}_1, \hat{\alpha}_2\} < N + \gamma_+$, $u_0 \leq c\langle x\rangle^{-a_1}\psi_1(x/|x|)$ and $v_0 \leq c\langle x\rangle^{-a_2}\psi_1(x/|x|)$ with $c > 0$ small enough, $a_i > \hat{\alpha}_i$ $(i = 1,2)$, then the solution of (1) is global in time, where $\hat{\alpha}_i = \{(2 + \hat{\sigma}_i + 2\hat{q}_i) + (2 + \hat{\sigma}_j + 2\hat{q}_j)p_i\}/(p_ip_j - 1)$ $((i,j) = (1,2),(2,1))$. However, we have not proved it yet.

The method using the sequence of balls in (11) was used in [4, 22] and other papers.

Remark. (i) *It is easily seen that* $\gamma_+ = \nu_1 - (N-2)/2$.

(ii) *If both* (6) *and* (11) *hold, then it is necessarily that* $C_U \geq C_L$, $\hat{\sigma}_i \geq \check{\sigma}_i$ *and* $\hat{q}_i \geq \check{q}_i$.

We briefly recall a history of the study on global nonexistence of solutions to the system (1). First, the global nonexistence of solutions in the case $D = \mathbf{R}^N$ $(\Omega = S^{N-1})$, $u = v$, $p_i = p$ and $K_i(x,t) = 1$ $(i = 1,2)$, that is

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^N, \end{cases} \tag{13}$$

was studied by Fujita [3]. Fujita proved that when $p < 1 + 2/N$ the solution of (13) is not global in time for any nonnegative bounded and continuous initial data $u_0 \not\equiv 0$. Fujita's results were also extended by some researcher. Hayakawa [8], Kobayashi-Sirao-Tanaka [11] and Weissler [28] proved that when $p = 1 + 2/N$, the solution of (13) blows up in finite time for any $u_0 \not\equiv 0$. For the case $p > 1 + 2/N$, Lee-Ni [12] studied that if $\|u_0\|_\infty$ is large enough or $\liminf_{|x|\to\infty} |x|^a u_0(x) > 0$ with $a < 2/(p-1)$, the solution of (13) is not global in time. When D is a cone, that is

$$\begin{cases} u_t = \Delta u + u^p, & x \in D,\ t > 0, \\ u(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in D, \end{cases} \tag{14}$$

Levine-Meier [14], [15] proved that if $p \leq 1 + 2/(N + \gamma_+)$, there is no global solution of (14).

Fujita's results were extended to the case $D = \mathbf{R}^N$, $u = v$, $p_i = p$ and $K_i(x,t) = K(x,t)$ for $i = 1,2$, that is

$$\begin{cases} u_t = \Delta u + K(x,t)u^p, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^N. \end{cases} \tag{15}$$

In the case $K(x,t) = |x|^\sigma$ with $\sigma \geq 0$, Bandle-Levine [1] had that when $p < 1 + (2+\sigma)/N$ the solution of (15) is not global in time for any $u_0 \not\equiv 0$. Hamada [6] had the same result for $p = 1 + (2+\sigma)/N$ (see also [18]). Suzuki [23] extended to the case $\sigma \in \mathbf{R}$ for the quasilinear parabolic equations. Thereafter, Qi [20] extended the result to the case $K(x,t) = t^q|x|$ with $q \geq 0$, $\sigma \geq 0$. He proved that when $p \leq 1 + (2+\sigma+2q)/N$ there exists no global solution of (15). When D is a cone, that is

$$\begin{cases} u_t = \Delta u + K(x,t)u^p, & x \in D,\ t > 0, \\ u(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in D, \end{cases} \tag{16}$$

in the case $K(x,t) = |x|^\sigma$ with $\sigma \geq 0$, Levine-Meier [14], [15] and Hamada [6] had that if $p \leq 1 + (2+\sigma)/(N+\gamma_+)$, there is no global solution of (16). For the case $p > 1 + (2+\sigma)/(N+\gamma_+)$, Hamada [7] studied that if $u_0(x) \geq M\langle x\rangle^{-a}\psi_1(x/|x|)$ with $a \leq (2+\sigma)/(p-1)$, $0 \leq \sigma \leq (p-1)(N-2)$ and some $M > 0$, the solution of (16) is not global. In the case $K(x,t) \sim t^q$ with $q > -1$ as $t \to \infty$, Levine-Meier [15] had that if $p \leq 1 + (2+2q)/(N+\gamma_+)$, there exists no global solution of (16).

In the case $D = \mathbf{R}^N$, our results are reduced to Escobedo-Herrero [2] and Mochizuki [16] with $K_i(x,t) = 1$ $(i = 1,2)$, to Uda [24] with $K_i(x,t) = t^{q_i}$ $(i = 1,2)$, and to Mochizuki-Huang [17] with $K_i(x,t) = |x|^{\sigma_i}$ with $\sigma_i \in [0, n(p_i - 1))$ $(i = 1,2)$. Additionally, Guedda-Kirane [5] and Kirane-Qafsaoui [10] studied in this field. They studied the case $K_i(x,t) \sim t^{q_i}|x|^{\sigma_i}$ as $t \to \infty$ and $|x| \to \infty$. However, they needed the condition $\max\{2q_i, \sigma_i\} < n(p_i - 1)$ $(i = 1,2)$. Moreover, when $K_i(x,t)$ $(i = 1,2)$ satisfy (6) and (11) with $D = \mathbf{R}^N$, the system (1) was studied by Igarashi-Umeda [9]. When D is a cone, in the case $K_i(x,t) = 1$, the condition (i) of Theorem 2 is reduced to Levine [13].

The rest of the paper is organized as follows. In section 2, we state the proof of the local existence (Theorem 1). The proof of global nonexistence (Theorem 2) is given in section 3. For the change of variable as (3), we decide $\eta(x,y,t) = \eta(r,\theta,\rho,\phi,t)$, $\eta(x,t) = \eta(r,\theta,t)$ or $\eta_0(x) = \eta_0(r,\theta)$ for any function.

2. Local Existence in Time

In this section we use $\sigma_i = \hat{\sigma}_i$ and $q_i = \hat{q}_i$ for $i = 1, 2$. In order to show the local solvability of the Cauchy problem (1), we consider the associated integral system

$$u(x,t) = S(t)u_0(x) + \int_0^t S(t-s)K_1(x,s)v(x,s)_1^p \, ds, \tag{17}$$

$$v(x,t) = S(t)v_0(x) + \int_0^t S(t-s)K_2(x,s)u(x,s)_2^p \, ds, \tag{18}$$

where

$$S(t)\xi(x) = \int_D G(x,y,t)\xi(y) \, dy \tag{19}$$

with G defined by (4). Define

$$\Psi(u,v) = (S(t)u_0(x) + \Phi_1(v), S(t)v_0(x) + \Phi_2(u)), \tag{20}$$

where

$$(\Phi_1(v), \Phi_2(u)) = \int_0^t S(t-s)(K_1(x,s)v(x,s)^{p_1}, K_2(x,s)u(x,s)^{p_2}) \, ds.$$

LEMMA 2.1. *Let* $\delta \geq 0$ *and* $\alpha := \max\{0, -\delta(N-2-\delta)/2\}$. *If we take* $0 < T \leq (\log 2)/\alpha$, *then for* $0 \leq t < T$

$$\|S(t)\xi\|_{\infty,\delta} \leq 2\|\xi\|_{\infty,\delta}.$$

Moreover for $0 \leq t < T$

$$\|S(t)\langle\cdot\rangle^{-\delta}\|_{\infty,\delta} \leq 2.$$

PROOF. Let $w(x,t) := S(t)\xi(x) - \|\xi\|_{\infty,\delta}\langle x\rangle^{-\delta}\exp(\alpha t)$, then we have

$$\begin{aligned}\Delta w - w_t &= [\alpha|x|^4 + \{2\alpha + \delta(N-2-\delta)\}|x|^2 + N\delta + \alpha]\|\xi\|_{\infty,\delta}\langle x\rangle^{-\delta-4}\exp(\alpha t) \\ &\geq 0.\end{aligned}$$

Combining this with Protter-Weinberger [19, Theorem 10, pp. 183–184], we get $w(x,t) \leq 0$; that is,

$$\langle x\rangle^{\delta} S(t)\xi(x) \leq \|\xi\|_{\infty,\delta}\exp(\alpha t).$$

Then we obtain $\|S(t)\xi\|_{\infty,\delta} \le \|\xi\|_{\infty,\delta} \exp(\alpha t)$. Moreover, if we take $0 < T \le (\log 2)/\alpha$, then for $0 < t < T$

$$\|S(t)\xi\|_{\infty,\delta} \le 2\|\xi\|_{\infty,\delta}. \qquad \square$$

Lemma 2.2. (i) *Assume that* $\langle x\rangle^{\delta_1} u_0(x)$ *and* $\langle x\rangle^{\delta_2} v_0(x)$ *are bounded in* $\bar{D}$. $(S(\cdot)u_0, S(\cdot)v_0) \in E_T$ *for* $0 < T \le (\log 2)/\alpha$, *and we have*

$$\|(S(\cdot)u_0, S(\cdot)v_0)\|_{E_T} \le 2\{\|u_0\|_{\infty,\delta_1} + \|v_0\|_{\infty,\delta_2}\},$$

where E_T *is defined in* (7).

(ii) *Let* $(u,v) \in E_T$. *Suppose that* $K_i(x,t)$ $(i = 1,2)$ *satisfy* (6). *Then* $(\Phi_1(v), \Phi_2(u)) \in E_T$ *for some* $T > 0$, *and we have*

$$\|(\Phi_1(v), \Phi_2(u))\|_{E_T} \le 2C_U(\tilde{T}_1(T) + \tilde{T}_2(T))\{\|(0,v)\|_{E_T}^{p_1} + \|(u,0)\|_{E_T}^{p_2}\},$$

where the constant C_U *is appeared in* (6), *and* $\tilde{T}_i(t) = \{(t+1)^{q_i+1} - 1\}/(q_i+1)$ $(i = 1,2)$.

Proof. (i) It is obvious from Lemma 2.1 with $\delta = \delta_i$ $(i = 1,2)$.

(ii) Note that

$$\int_0^t S(t-s)K_1(x,s)v(x,s)^{p_1}\,ds$$
$$\le \int_0^t S(t-s)C_U(s+1)^{q_1}\langle x\rangle^{\sigma_1-\delta_2 p_1}\,ds \sup_{s\in[0,t]} \|v(\cdot,s)\|_{\infty,\delta_2}^{p_1}.$$

A simple calculation gives $-\sigma_1 + \delta_2 p_1 = \delta_1$. Then it follows from Lemma 2.1 that

$$\|S(t-s)\langle\cdot\rangle^{\sigma_1-\delta_2 p_1}\|_{\infty,\delta_1} \le 2.$$

Thus we have

$$\|\Phi_1(v)\|_{\infty,\delta_1} \le 2C_U\tilde{T}_1(t) \sup_{s\in[0,t]} \|v(s)\|_{\infty,\delta_2}^{p_1}.$$

Similarly, we have

$$\|\Phi_2(u)\|_{\infty,\delta_2} \le 2C_U\tilde{T}_2(t) \sup_{s\in[0,t]} \|u(s)\|_{\infty,\delta_1}^{p_2}.$$

We conclude from these inequations. $\square$

PROOF OF THEOREM 1. Let $B_R = \{(u,v) \in E_T; \|(u,v)\|_{E_T} \le R\}$ and $P_T = \{(u,v) \in E_T; u \ge 0, v \ge 0\}$, and define $\tilde{T}_i$ same as in Lemma 2.2 (ii). For (u_1,v_1), $(u_2,v_2) \in B_R \cap P_T$ with $R \ge 1$ sufficient large, we have

$$\|\Psi(u_1,v_1) - \Psi(u_2,v_2)\|_{E_T} = \|(\Phi_1(v_1) - \Phi_1(v_2), \Phi_2(u_1) - \Phi_2(u_2))\|_{E_T}. \tag{21}$$

We consider

$$\begin{aligned}&|\Phi_1(v_1) - \Phi_1(v_2)|\langle x\rangle^{\delta_1}\\ &\quad \le \int_0^t S(t-s)C_U(s+1)^{q_1}\langle x\rangle^{\sigma_1}|v_1(x,s)^{p_1} - v_2(x,s)^{p_1}|\,ds\langle x\rangle^{\delta_1}.\end{aligned}$$

Then, since $(u_2,v_2) \in B_R$ we obtain

$$\begin{aligned}&|\Phi_1(v_1) - \Phi_1(v_2)|\langle x\rangle^{\delta_1}\\ &\quad \le 2^{p_1}C_U\tilde{T}_1(T)\sup_{s\in[0,t]}\|R^{p_1-1}p_1(v_1(\cdot,s) - v_2(\cdot,s))\|_{\infty,\delta_2}.\end{aligned} \tag{22}$$

By the same argument we have

$$\begin{aligned}&|\Phi_2(u_1) - \Phi_2(u_2)|\langle x\rangle^{\delta_2}\\ &\quad \le 2^{p_2}C_U\tilde{T}_2(T)\sup_{s\in[0,t]}\|R^{p_2-1}p_2(u_1(\cdot,s) - u_2(\cdot,s))\|_{\infty,\delta_1}.\end{aligned} \tag{23}$$

Substitute (22) and (23) into (21). Since $\max\{p_1,p_2\} \le p_1p_2$, we obtain

$$\begin{aligned}&\|\Psi(u_1,v_1) - \Psi(u_2,v_2)\|_{E_T}\\ &\quad \le 2^{p_1p_2}C_U(\tilde{T}_1(T) + \tilde{T}_2(T))R^{p_1p_2-1}p_1p_2\|(u_1-u_2, v_1-v_2)\|_{E_T}.\end{aligned}$$

Taking $T > 0$ small enough, we have

$$\|\Psi(u_1,v_1) - \Psi(u_2,v_2)\|_{E_T} \le \rho\|(u_1,v_1) - (u_2,v_2)\|_{E_T}$$

for some $\rho < 1$. Then Ψ is a strict contraction of $B_R \cap P_T$ into itself, whence there exists a unique fixed point $(u,v) \in B_R \cap P_T$ which solves (1). □

3. Nonexistence of Global Solution

In this section we treat the nonexistence of global solutions in time of (1). Here, we take the same strategy as in [17], [18], [25] and [26]. Let $\sigma_i = \check{\sigma}_i$ and $q_i = \check{q}_i$ for $i = 1,2$ through this section.

First, we should consider only the case $k \in (0,1/2)$ by comparison. Let

$\lambda_m > 0$ denote the principal eigenvalue of $-\Delta$ with Dirichlet problem in $B(x_m, k|x_m|)$, and let $\zeta_m(x) > 0$ denote the corresponding positive eigenfunction, normalized by $\int_{B(x_m, k|x_m|)} \zeta_m(x)\, dx = 1$. Define

$$F_m(t) = \int_{B(x_m, k|x_m|)} u(x,t)\zeta_m(x)\, dx, \quad G_m(t) = \int_{B(x_m, k|x_m|)} v(x,t)\zeta_m(x)\, dx. \quad (24)$$

By applying Green's formula and Jensen's inequality, we see that $(F_m(t), G_m(t))$ satisfies

$$\begin{cases} F_m'(t) \geq -c_1|x_m|^{-2}F_m(t) + c_2 t^{q_1}|x_m|^{\sigma_1}G_m(t)^{p_1}, \\ G_m'(t) \geq -c_1|x_m|^{-2}G_m(t) + c_2 t^{q_2}|x_m|^{\sigma_2}F_m(t)^{p_2} \end{cases}$$

(see [9, §3]). We will show that for an appropriate choice of k, $(F_m(t), G_m(t))$ is not global in time, thereby contradicting the assumption that (u, v) is a global solution. By the same arguments as in [9, §3], [13] and [21], we have the following proposition:

Proposition 3.1. *Let $(F_m(t), G_m(t))$ by (24) for some $t_0 \in (0, t]$ and $m \in \mathbf{N}$. If*

$$F_m(c_1|x_m|^2) > A|x_m|^{-\alpha_1} \quad or \quad G_m(c_1|x_m|^2) > B|x_m|^{-\alpha_2}$$

with some $A, B > 0$ and some $c_1 > 0$, then $(F_m(t), G_m(t))$ is not global in time.

Lemma 3.1. *Let u_0 and v_0 are BC and $(u_0, v_0) \not\equiv 0$, and let (u, v) be a solution of (1). Then for any $\tau > 0$ and $x \in D$ there exist constants $\mu \geq 1$ and $C = C(N, \tau, u_0, v_0, K_1, K_2, p_1, p_2, \mu) > 0$ such that*

$$u(x,\tau) \geq C|x|^{\gamma_+}e^{-\mu|x|^2}\psi_1(x/|x|) \quad and \quad v(x,\tau) \geq C|x|^{\gamma_+}e^{-\mu|x|^2}\psi_1(x/|x|).$$

Proof. We may let $u_0(x) \not\equiv 0$ without loss of generality. Since $u(x,t) \geq S(t)u_0(x)$, $I_\nu(z) \geq Cz^\nu$ and $\gamma_+ = \nu_1 - (N-2)/2$, we obtain

$$u(x,t) \geq \frac{C}{(2t)^{1+\nu_1}} r^{\gamma_+}e^{-r^2/4t}\psi_1(\theta)$$
$$\times \int_0^\infty \int_\Omega \rho^{\gamma_+ + N-1}e^{-\rho^2/4t}\psi_1(\phi)u_0(\rho,\phi)\, d\phi d\rho.$$

Then we have, for every $\tau_1 > 0$,

$$u(x,\tau_1) \geq C_1 r^{\gamma_+}e^{-\mu_1 r^2}\psi_1(\theta) \quad (25)$$

with $\mu_1 = \max\{1, 1/4\tau_1\}$ and

$$C_1 = C_1(\tau_1, N, u_0) = \frac{C}{(2\tau_1)^{1+\nu_1}} \int_0^\infty \int_\Omega \rho^{\gamma_+ + N - 1} e^{-\rho^2/4\tau_1} \psi_1(\phi) u_0(\rho, \phi)\, d\phi d\rho.$$

From (18) and the fact that $I_\nu \geq C z^\nu$ with some $C > 0$, we have

$$v(x,t) \geq C r^{\gamma_+} \psi_1(\theta) \int_0^t \int_0^\infty \int_\Omega \frac{1}{(2(t-s))^{\nu_1+1}} e^{-r^2/4(t-s)}$$
$$\times \rho^{\gamma_+ + N - 1} e^{-\rho^2/4(t-s)} \psi_1(\phi) K_2(\rho,\phi,s) u^{p_2}(\rho,\phi,s)\, d\phi d\rho ds.$$

Then by (25) we obtain for $\tau_2 > 2\tau_1$

$$v(x,\tau_2) \geq C_2 r^{\gamma_+} \psi_1(\theta) \int_{\tau_2/2}^{\tau_2} \frac{1}{(2(\tau_2 - s))^{\nu_1+1}} e^{-r^2/4(\tau_2 - s)}\, ds$$
$$\geq C_2 r^{\gamma_+} \psi_1(\theta) \frac{1}{\tau_2^{\nu_1+1}} e^{-r^2/2\tau_2} \int_{\tau_2/2}^{\tau_2} ds = C_2 r^{\gamma_+} \psi_1(\theta) \frac{1}{2\tau_2^{\nu_1}} e^{-r^2/2\tau_2}$$

with

$$C_2 = C_2(\tau_2, N, u_0, K_2)$$
$$= \inf_{s \in (\tau_2/2, \tau_2)} C \int_0^\infty \int_\Omega \rho^{\gamma_+ + N - 1} e^{-\rho^2/4(\tau_2 - s)} \psi_1(\phi) K_2(\rho,\phi,s) u^{p_2}(\rho,\phi,s)\, d\phi d\rho.$$

Then we have

$$v(x,\tau_2) \geq C_3 r^{\gamma_+} e^{-\mu_2 r^2} \psi_1(\theta)$$

with $\mu_2 = \max\{1, 1/2\tau_2\}$ and $C_3 = C_2/2\tau_2^{\nu_1}$. Put $C = \min\{C_1, C_2, C_3\}$ and $\mu = \max\{\mu_1, \mu_2\}$ and $\tau = \tau_2$. Then we have

$$u(x,\tau) \geq C r^{\gamma_+} e^{-\mu r^2} \psi_1(\theta) \quad \text{and} \quad v(x,\tau) \geq C r^{\gamma_+} e^{-\mu r^2} \psi_1(\theta). \qquad \square$$

LEMMA 3.2. *For $\sigma \geq 0$, $\mu \geq 1$, $x \in D$ and $t \geq \tau$ with some $\tau > 0$, we have*

$$S(t)\chi_B(x)|x|^\sigma e^{-\mu|x|^2} \geq C t^{(\sigma-\gamma_+)/2} (1 + 4\mu t)^{-(N+\sigma+\gamma_+)/2} |x|^{\gamma_+} e^{-|x|^2/4t} \psi_1(x/|x|)$$

with some $C > 0$ and $B = B(b,a) \subset D$ with $a > 0$ and $b \in D$, where χ_B is a characteristic function of B such that $\chi_B(x) = 1$ for $x \in B$ and $= 0$ for $x \in D \backslash B$. The domain $B(b,a)$ denotes the open ball of radius a centered at b.

PROOF. We can put positive constants a_1, a_2 and domain $\Omega' \subset \Omega$ satisfying $0 < a_1 < a_2 < \infty$, $|\Omega'| \neq 0$ and $D_B = \{x; |x| \in (a_1, a_2), x/|x| \in \Omega'\} \subset B$. By (19) and $\int_{\Omega'} \psi_1(\phi)\, d\phi = C$ with some $C \in (0, 1]$, we have

$$
\begin{aligned}
&S(t)\chi_B(x)|x|^{\sigma}e^{-\nu|x|^2}\\
&\quad\geq \int_{a_1}^{a_2}\int_{\Omega'} G(r,\theta,\rho,\phi,t)\rho^{\sigma}e^{-\mu\rho^2}\rho^{N-1}\,d\phi d\rho\\
&\quad\geq \frac{C}{(2t)^{1+\nu_1}}r^{\gamma_+}e^{-r^2/4t}\psi_1(\theta)\int_{a_1}^{a_2}\int_{\Omega'}\rho^{\gamma_++\sigma+N-1}e^{-(1+4\mu t)\rho^2/4t}\psi_1(\phi)\,d\phi d\rho\\
&\quad\geq \frac{Cr^{\gamma_+}e^{-r^2/4t}\psi_1(\theta)}{(2t)^{1+\nu_1}\tilde{\mu}(t)^{\gamma_++\sigma+N}}\int_{\tilde{\mu}(t)a_1}^{\tilde{\mu}(t)a_2} s^{\gamma_++\sigma+N-1}e^{-s^2}\,ds
\end{aligned}
$$

where $\tilde{\mu}(t)=\sqrt{(1+4\mu t)/4t}$. Since $1\leq\sqrt{\mu}\leq\tilde{\mu}(t)\leq\tilde{\mu}(\tau)$ for $t\geq\tau$, we have

$$S(t)\chi_B(x)|x|^{\sigma}e^{-\nu|x|^2}\geq Ct^{(\sigma-\gamma_+)/2}(1+4\mu t)^{-(N+\sigma+\gamma_+)/2}r^{\gamma_+}e^{-r^2/4t}\psi_1(\theta). \qquad \square$$

By Lemma 3.1, we can assume

$$u_0(x)\geq C|x|^{\gamma_+}e^{-\mu|x|^2}\psi_1(x/|x|)$$

for some $C>0$ and $\mu>0$. Then we have, for $t\geq\tau$

$$u(x,t)\geq C(1+4\mu t)^{-N/2-\gamma_+}|x|^{\gamma_+}e^{-|x|^2/4t}\psi_1(x/|x|). \tag{26}$$

Lemma 3.3. *Let v be the second element of the solution of* (1). *Then we have*

$$v(x,t)\geq Ct^{((p_2-1)\gamma_++\sigma_2+2q_2+2)/2}(t+1)^{-\gamma_+p_2-Np_2/2}|x|^{\gamma_+}e^{-|x|^2/2t}\psi(x/|x|)^{p_2+1}$$

for $t\geq\tau$ with some $\tau>0$ and $C=C(\tau,u_0,v_0,K_1,K_2,p_1,p_2)>0$.

Proof. It follows from (11), (18) and (26), we obtain

$$
\begin{aligned}
v(x,t)\geq C\int_0^t & S(t-s)\chi_{\tilde{B}_{k,1}}(x)|x|^{\sigma_2+p_2\gamma_+}s^{q_2}\\
&\times(4s+1/\mu)^{-Np_2/2-p_2\gamma_+}e^{-p_2|x|^2/4s}\psi_1^{p_2}(x/|x|)\,ds.
\end{aligned}
$$

By Lemma 3.2, we then have

$$
\begin{aligned}
v(x,t)\geq C(t/2)^{(p_2-1)\gamma_+/2+\sigma_2/2}&(t/4)^{q_2}(2t+1/\mu)^{-\gamma_+p_2-Np_2/2}\\
&\times|x|^{\gamma_+}e^{-|x|^2/2t}\psi_1^{p_2+1}(x/|x|)\int_{t/4}^{t/2}ds.
\end{aligned}
$$

Thus, the inequality of the lemma holds. $\square$

LEMMA 3.4. *Let u be first element of the solution of* (1) *and* $\alpha_1 \geq N + \gamma_+$. *Then for* $t \geq a$

$$u(x,t) \geq \begin{cases} Ct^{-N/2-\gamma_+}|x|^{\gamma_+}e^{-|x|^2/2t}\psi_1(x/|x|)^{p_1p_2+p_1+1}\log(t/2a), & \text{if } \alpha_1 = N + \gamma_+, \\ Ct^{-N/2-\gamma_+}|x|^{\gamma_+}e^{-|x|^2/2t}\psi_1(x/|x|)^{p_1p_2+p_1+1}(t^{\tilde{p}} - (2a)^{\tilde{p}}), & \text{if } \alpha_1 > N + \gamma_+ \end{cases}$$

with $C = C(a, u_0, v_0, K_1, K_2, p_1, p_2, N) > 0$, *where* $a > 0$ *is a small constant and* $\tilde{p} = (p_1p_2 - 1)(\alpha_1 - N - \gamma_+)/2$.

PROOF. By Lemma 3.3, we have

$$\begin{aligned} u(x,t) \geq C \int_a^t & S(t-s)\chi_{\tilde{B}_{k,1}}(x)|x|^{\sigma_1}s^{q_1}s^{(1+\sigma_2/2+q_2)p_1+p_1(p_2-1)\gamma_+/2} \\ & \times (s+1)^{-Np_1p_2/2-\gamma_+p_1p_2}|x|^{p_1\gamma_+}e^{-p_1|x|^2/2s}\psi_1(x/|x|)^{p_1(p_2+1)}\,ds. \end{aligned}$$

It follows from Lemma 3.2 that

$$\begin{aligned} u(x,t) \geq\; & C(t/2)^{(\sigma_1+p_1\gamma_+-\gamma_+)/2}t^{-(N+\sigma_1+p_1\gamma_++\gamma_+)/2}|x|^{\gamma_+}e^{-|x|^2/2t} \\ & \times \psi_1(x/|x|)^{p_1p_2+p_1+1}\int_a^{t/2} s^{\{(-N-\gamma_+)(p_1p_2-1)+(2+\sigma_2+2q_2)p_1+\sigma_1+2q_1\}/2}\,ds \end{aligned}$$

for small $a > 0$. Since

$$\begin{aligned} & \{(-N-\gamma_+)(p_1p_2-1) + (2+\sigma_2+2q_2)p_1 + \sigma_1 + 2q_1\}/2 \\ & \quad = \{(p_1p_2-1)(\alpha_1 - N - \gamma_+)\}/2 - 1, \end{aligned}$$

this proves the inequality of the lemma. □

PROOF OF THEOREM 2. First we consider the case (i). We may assume $\alpha_1 \geq \alpha_2$. Put $Y_m = \sqrt{c_1}|x_m|$. From the definition, we have $\alpha_1 \geq N + \gamma_+$. By Lemma 3.4, since $x \in B(x_m, k|x_m|)$, we have

$$\begin{aligned} F_m(Y_m^2) \geq\; & CY_m^{-N-\gamma_+}h_m \\ & \times \int_{B(x_m,k|x_m|)} \frac{|x|^{\gamma_+}}{|x_m|^{\gamma_+}} \exp\left(-\frac{|x|^2}{2Y_m^2}\right)\zeta_m(x)\psi_1(x/|x|)^{p_1p_2+p_1+1}\,dx \\ \geq\; & C|x_m|^{-N-\gamma_+}(c_1)^{-(N+\gamma_+)/2}h_m(1+k)^{\gamma_+}\exp(-(1+k)^2/2c_1), \end{aligned}$$

where $h_m = \log(Y_m^2/2a)$ for $\alpha_1 = N + \gamma_+$ and $h_m = Y_m^{2\tilde{p}} - (2a)^{\tilde{p}}$ for $\alpha_1 > N + \gamma_+$ with $C = C(a, u_0, v_0, K_1, K_2, p_1, p_2, N) > 0$ and $\tilde{p}$ defined in Lemma 3.4. Since $\alpha_1 \geq N + \gamma_+$, it follows that

$$|x_m|^{\alpha_1} F_m(Y_m^2) \geq C|x_m|^{\alpha_1 - N - \gamma_+} c_1^{-(N+\gamma_+)/2} \times h_m(1+k)^{\gamma_+} \exp(-(1+k)^2/2c_1) > A$$

for m large enough. Thus, $(F_m(t), G_m(t))$ is not global in time by Proposition 3.1.

Next, we consider the case (ii). Since $u(x,t) \geq S(t)u_0(x)$, $u \in H_{a_1}$, $I_\nu(z) \geq Cz^\nu$ and $\int_\Omega \psi_1(\phi)^2\, d\phi$ is constant, it follows that

$$\begin{aligned} u(x,t) &\geq \int_0^\infty \int_\Omega G(r,\theta,\rho,\phi,t) u_0(\rho,\phi)\rho^{N-1}\, d\phi d\rho \\ &\geq C\int_0^\infty \frac{r^{\gamma_+}\rho^{N+\gamma_+-1}}{t^{\gamma_+ + N/2}} \exp\left(-\frac{r^2+\rho^2}{4t}\right)(1+\rho^2)^{-a_1/2}\psi_1(\theta)\, d\rho. \end{aligned}$$

Then, since $\gamma_+ = \nu_1 - (N-2)/2$, we obtain

$$u(x, Y_m^2) \geq C\left(\frac{r}{Y_m}\right)^{\gamma_+} \exp\left(-\frac{r^2}{4Y_m^2}\right) \int_0^\infty \frac{1}{Y_m}\left(\frac{\rho}{Y_m}\right)^{N/2+\nu_1} \times \exp\left(-\frac{\rho^2}{4Y_m^2}\right)(1+\rho^2)^{-a_1/2}\psi_1(\theta)\, d\rho.$$

Since $Y_m = \sqrt{c_1|x_m|^2}$, we have for $x \in B(x_m, k|x_m|)$

$$u(x, Y_m^2) \geq C(1+k)^{-N/2}(1-k)^{1+\nu_1} c_1^{-\gamma_+/2} \exp\{-(1+k)^2/c_1\} \times \psi_1(\theta)\int_0^\infty \frac{1}{Y_m}\left(\frac{\rho}{Y_m}\right)^{N/2+\nu_1} \exp\left(-\frac{\rho^2}{4Y_m^2}\right)(1+\rho^2)^{-a_1/2}\, d\rho.$$

Putting $\chi = \rho/Y_m$, we have

$$u(x, Y_m^2) \geq C\psi_1(\theta)\int_0^\infty \chi^{N/2+\nu_1} \exp\left(-\frac{\chi^2}{4}\right)(1+\chi^2 Y_m^2)^{-a_1/2}\, d\chi.$$

Note that $1 + \chi^2 Y_m^2 \leq Y_m^2(1+\chi^2)$ if m is large enough. Then, we obtain

$$\begin{aligned} u(x, Y_m^2) &\geq C\psi_1(\theta)|Y_m|^{-a_1}\int_0^\infty \chi^{N/2+\nu_1} \exp\left(-\frac{\chi^2}{4}\right)(1+\chi^2)^{-a_1/2}\, d\chi \\ &\geq C\psi_1(x/|x|)|x_m|^{-a_1} \end{aligned}$$

for sufficiently large m. Since $\int_{B(x_m,k|x_m|)} \psi_1(x/|x|)\zeta_m(x)\,dx$ is constant, we have

$$F_m(Y_m^2) \geq \int_{B(x_m,k|x_m|)} u(x, Y_m^2)\zeta_m(x)\,dx$$

$$\geq C|x_m|^{-a_1} \int_{B(x_m,k|x_m|)} \psi_1(x/|x|)\zeta_m(x)\,dx \geq C|x_m|^{-a_1}.$$

Since $u_0 \in H_{a_1}$ with $a_1 < \alpha_1$, we have

$$|x_m|^{\alpha_1} F_m(Y_m^2) \geq C|x_m|^{\alpha_1 - a_1} > A$$

for sufficiently large m. If $v_0 \in H_{a_2}$ with $a_2 < \alpha_2$, we similarly have

$$|x_m|^{\alpha_2} G_m(Y_m^2) \geq C|x_m|^{\alpha_2 - a_2} > B$$

for m large enough. Thus, $(F_m(t), G_m(t))$ is not global in time by Proposition 3.1. □

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