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REPRESENTATIONS OF NORMALIZER SUBGROUPS OF MAXIMAL TORI OF THE CLASSICAL GROUP OF TYPE C

By

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Abstract. We study representations of the normalizer subgroup N of a maximal torus of the classical group of type C, $Sp(n)$. We obtain a formula of the irreducible characters of N, and give the branching rule from $Sp(n)$ to N.

1. Introduction

The research of representations and characters of $Sp(n)$, the classical group of type C, has been developed and we have the characterization of the irreducible representations and formulae of the dimensions and characters (see [W]).

Restriction of an irreducible character of $Sp(n)$ to a maximal torus T is a polynomial invariant under the action of the Weyl group of type C. The Weyl group of a semisimple Lie group is obtained as the quotient of the normalizer subgroup of a maximal torus by the maximal torus itself. When we research representations of the semisimple Lie groups, it is important to decompose the representation space into the weight spaces of the maximal torus. The weight spaces are permuted by the action of the Weyl group. So, maximal tori and Weyl groups play a crucial role to investigate the representations of the semisimple Lie groups.

We consider the representations of the normalizer subgroup N of a maximal torus of $Sp(n)$. The group N has the properties of both the maximal torus and the Weyl group. Indeed, N includes the maximal torus T that gives the weight space decomposition and the Weyl group N/T permutes the weights. Each of the

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characters of $SP(n)$ is determined by its restriction to a maximal torus T, since any element of G is conjugate to an element of T. This restriction is a polynomial function on T which is invariant under the action of the Weyl group $W = N/T$. So, the research of difference between representations of N and representations of the whole group $Sp(n)$ is an interesting subject. To compare the representations of $Sp(n)$ and N, we consider the restriction of the representation of $Sp(n)$ to N and give a combinatorial formula for the multiplicities of the irreducible representations of N in the restriction of the irreducible representation of $Sp(n)$ to N.

The representation theory of N has been developed in the context of the zero-weight representation and so many interesting results are obtained (see [AMT], [Mat], [Na], [Ni], [MT]).

In this paper, we use the method given by Clifford [C] to determine irreducible characters of N. Each element of N is determined by $w \in N/T$ and $t \in T$. We write the corresponding element as $n_w t$. Then, we obtain the character value of $n_w t$ of irreducible representations of N.

In the remainder of this section, we summarize the contents of this paper.

In section 2, basic facts and notations are introduced to proceed the arguments, and we have a criterion given by Clifford of the irreducibility of representations of N.

In section 3, we determine the character value at $n_w t$ of irreducible representations of N.

In section 4, we write the value of elementary symmetric functions at eigenvalues of $n_w t$ in terms of w and t. Then, the character value of an irreducible representation of $Sp(n)$ at $n_w t$ is expressed by w and t.

In section 5, we obtain the branching rule between N and $Sp(n)$. We use an inner product on the space of characters of N given by normalized Haar measure on N.

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2. The Irreducible Representations of N

In this paper, define the classical group of type C, $Sp(n)$, as follows;

$$Sp(n) := \{g \in U(2n) \mid {}^t g J_n g = J_n\},$$

where $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in GL(2n, \mathbf{R})$ and I_n is the identity matrix of degree n.

We fix a maximal torus T of $Sp(n)$ as follows;

$$T := \{diag(t_1, t_2, \dots, t_n, \bar{t}_1, \bar{t}_2, \dots, \bar{t}_n) \mid t_i = e^{\sqrt{-1}\theta_i}, \theta_i \in \mathbf{R}\},$$

where $\bar{t}_i$ is the complex conjugate of t_i. Let N be the normalizer subgroup of T. Then, the following sequence becomes exact;

$$1 \to T \to N \to W \to 1 \ \ (exact),$$

where W is the Weyl group of type C, which is isomorphic to the semi-direct product $\mathfrak{S}_n \ltimes (\mathbf{Z}_2)^n$.

For $i > 0$, let $t_{-i} := \bar{t}_i$. Then, the elements t of T are expressed as follows;

$$t = diag(t_1, t_2, \dots, t_n, t_{-1}, t_{-2}, \dots, t_{-n}).$$

The group W consists of the permutations σ on the set

$$\{1, 2, \dots, n, -1, -2, \dots, -n\},$$

which satisfy the condition $\sigma(-i) = -\sigma(i)$. The group W can be regarded as a subgroup of $\mathfrak{S}_{2n}$. For $w \in W$, we use the same symbol w for the permutation matrix corresponding to w in $U(2n)$. Then, the matrix is of type $w = \begin{pmatrix} A & C \\ C & A \end{pmatrix}$, where the matrix w is a permutation matrix of size $2n \times 2n$, the size of block matrices A and C is $n \times n$, and the matrices A, C satisfy the conditions ${}^tAA + {}^tCC = I_n$ and ${}^tAC = A{}^tC = 0$.

Notation 2.1. *For each* $w \in W$, $w = \begin{pmatrix} A & C \\ C & A \end{pmatrix}$, *we set* $n_w = \begin{pmatrix} A & C \\ -C & A \end{pmatrix}$.

Then, $n_w \in Sp(n)$, and we obtain

$$n_w^{-1} t n_w = diag(t_{w(1)}, t_{w(2)}, \dots, t_{w(n)}, t_{w(-1)}, t_{w(-2)}, \dots, t_{w(-n)}).$$

Remark 2.2. *Let* $x_1, x_2, \dots, x_n$ *be the generators of* W *as Coxeter group, where* x_n *corresponds to the long root. For* x_i $(i = 1, 2, \dots, n-1)$, *we have the following expression*;

$$n_{x_i} = \begin{pmatrix} A_i & 0 \\ 0 & A_i \end{pmatrix},$$

where

$$A_i = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix},$$

and

$$n_{x_n} = \begin{pmatrix} I_{n-1} & & & \\ & 0 & & 1 \\ & & I_{n-1} & \\ & -1 & & 0 \end{pmatrix}.$$

In the matrices A_i *and* n_{x_n}, *the entries which are not written are* 0.

In fact, we can choose elements $\tilde{n}_{x_i}$ *of* N *corresponding to* x_i *as*

$$\tilde{n}_{x_i} = \begin{pmatrix} B_i & 0 \\ 0 & B_i \end{pmatrix},$$

where

$$B_i = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix}$$

for $i = 1, 2, \dots, n-1$, *and*

$$\tilde{n}_{x_n} = n_{x_n}.$$

In the matrix B_i, *the entries which are not written are* 0.

For each i, *where* $i = 1, 2, \dots, n$, *elements* n_{x_i} *and* $\tilde{n}_{x_i}$ *of* N *differ by an element of* T; $n_{x_i}^{-1}\tilde{n}_{x_i} \in T$. *In* [MT], $\tilde{n}_{x_i}$*'s are used to proceed the argument* (*see* [MT], Remark 5.2).

Here, we consider the irreducibility of representations of N.

Theorem 2.3 (Clifford [C]). *Let* (ρ, V) *be a finite dimensional continuous representation of* N. *Then, we obtain the weight space decomposition of* V *with respect to* T *as follows*;

$$V = V_{\mu_1} \oplus V_{\mu_2} \oplus \cdots \oplus V_{\mu_r},$$

where $\mu_i : T \to \mathbf{C}^\times$ *is a continuous homomorphism, and*

$$V_{\mu_i} = \{v \in V \mid \forall t \in T, \rho(t)v = \mu_i(t)v \ \ (\mu_i(t) \in \mathbf{C}^\times)\}.$$

Fix a weight μ. *Then,*

$$V_{\mu} = \{v \in V \mid \forall t \in T, \rho(t)v = \mu(t)v \ \ (\mu(t) \in \mathbf{C}^\times)\}.$$

Let N_μ be the maximum subgroup of N that stabilizes the weight space V_μ. Then, the representation (ρ, V) is irreducible if and only if the following two conditions hold;

(a) $(\rho|_{N_\mu}, V_\mu)$ *is an irreducible representation of* N_μ,

(b) $V = \rho(N)V_\mu$. $\square$

Weyl group W acts on the set of weights of the irreducible representation (ρ, V). The weights are permuted under the action of W and form a W-orbit. For a weight μ, we can define a subgroup N_μ of N as the stabilizer subgroup of μ by the action. Then, the stabilizer subgroup N_μ is the maximum subgroup of N that stabilizes the weight space V_μ. In the set of weights, we introduce the dominance order by which the following weight μ becomes the highest weight;

$$\mu(t) = t_1^{p_1} t_2^{p_2} \cdots t_n^{p_n}, \quad p_i \in \mathbf{Z}_{\geq 0}, \quad p_1 \geq p_2 \geq \cdots \geq p_n \geq 0. \tag{2.1}$$

For the highest weight μ, we obtain the weight space V_μ and the maximum stabilizer subgroup N_μ. Let $W_\mu := N_\mu/T$. Then, we can parameterize the irreducible representation ρ by the weight μ and an irreducible representation φ of W_μ in the context of [C].

Each element of W can be uniquely written in product of the following elements;

$$(i_1 i_2 \cdots i_k \; -i_1 - i_2 \cdots -i_k),$$

$$(i_1 i_2 \cdots i_k)(-i_1 - i_2 \cdots -i_k).$$

Namely, i and $-i$ appear in one cycle element simultaneously or not. Define a cycle element to be *self-contained* if i and $-i$ appear in the expression, and to be *separated* otherwise. For separated case, we have a pair of cycle elements. The self-contained cycle elements have even length. For $w \in W$, if w is decomposed into cycle elements all of which are separated, then we call the element w to be separated.

Let $W_\mu := N_\mu/T \subset W$. Then, W_μ is isomorphic to the direct product of Weyl groups.

Definition 2.4. *For the highest weight* μ, $\mu(t) = t_1^{p_1} t_2^{p_2} \cdots t_n^{p_n}$, $p_i \in \mathbf{Z}_{\geq 0}$, $p_1 \geq p_2 \geq \cdots \geq p_n \geq 0$, *define the number of* p_i*'s which are equal to* 0 *to be* n_0, *and the number of distinct elements which are not equal to* 0 *in the set* $\{p_1, \ldots, p_n\}$ *to be* q. *We define the numbers* $n_1, n_2, \ldots, n_q$ *as follows*;

$$\begin{aligned} p_1 = p_2 = \cdots = p_{n_1} \\ > p_{n_1+1} = p_{n_1+2} = \cdots = p_{n_1+n_2} \\ > \cdots > p_{n_1+\cdots+n_{q-1}+1} = p_{n_1+\cdots+n_{q-1}+2} = \cdots = p_{n_1+\cdots+n_q} > 0. \end{aligned} \qquad \square$$

DEFINITION 2.5. *For* $0 \le i \le q$, *define the sets* I_i, I_i' *as follows*;

$$I_i := \{n_1 + \cdots + n_{i-1} + 1, n_1 + \cdots + n_{i-1} + 2, \dots, n_1 + \cdots + n_{i-1} + n_i\}$$
$$(i = 1, 2, \dots),$$

$$I_0 := \{n_1 + \cdots + n_q + 1, n_1 + \cdots + n_q + 2, \dots, n_1 + \cdots + n_q + n_0\},$$

$$I_i' := \{-k \mid k \in I_i\}. \qquad \square$$

DEFINITION 2.6. *Define* $W(A_{n_i-1})$ *to be the group which consists of all the separated permutations* σ *on the set* $I_i \cup I_i'$ *with the conditions*

$$\sigma(I_i) \subset I_i, \quad \sigma(I_i') \subset I_i',$$

and $W(C_{n_0})$ *is the Weyl group of type* C *on the set* $I_0 \cup I_0'$. $\square$

Then, we have the following equation;

$$W_\mu = W(A_{n_1-1}) \times W(A_{n_2-1}) \times \cdots \times W(A_{n_q-1}) \times W(C_{n_0}). \tag{2.2}$$

As in the notation 2.1, let $n_w = \begin{pmatrix} A & C \\ -C & A \end{pmatrix}$, where $w = \begin{pmatrix} A & C \\ C & A \end{pmatrix}$. Then, each element of N can be written as $n_w t$ uniquely for $w \in W$, $t \in T$, and we obtain the following proposition.

PROPOSITION 2.7. *For the highest weight* μ, *we define a map* $\tilde{\mu} : N_\mu \to \mathbf{C}^\times$ *as follows*;

$$\tilde{\mu}(n_w t) := \mu(t) \quad (\forall t \in T). \tag{2.3}$$

Then, the map $\tilde{\mu}$ *becomes a character of* N_μ *and we have* $\tilde{\mu}|_T = \mu$.

PROOF. It is clear that $\tilde{\mu}$ is a well-defined map. Immediately, we have $\tilde{\mu}|_T = \mu$. We show that $\tilde{\mu}$ is a group homomorphism from N_μ to $\mathbf{C}^\times$.

For elements $n_w t, n_{w'} t' \in N_\mu$, we have

$$\begin{aligned} (n_w t)(n_{w'} t') &= n_w n_{w'} (n_{w'}^{-1} t n_{w'}) t' \\ &= n_{ww'} (n_{ww'}^{-1} n_w n_{w'}) (n_{w'}^{-1} t n_{w'}) t'. \end{aligned}$$

Since $n_{ww'}^{-1}n_w n_{w'} \in T$, we have

$$\begin{aligned}\tilde{\mu}((n_w t)(n_{w'}t')) &= \tilde{\mu}(n_{ww'}(n_{ww'}^{-1}n_w n_{w'})(n_{w'}^{-1}tn_{w'})t') \\ &= \mu((n_{ww'}^{-1}n_w n_{w'})(n_{w'}^{-1}tn_{w'})t') \\ &= \mu(n_{ww'}^{-1}n_w n_{w'})\mu(n_{w'}^{-1}tn_{w'})\mu(t').\end{aligned}$$

Then, we have

$$\tilde{\mu}((n_w t)(n_{w'}t')) = \mu(n_{ww'}^{-1}n_w n_{w'})\mu(t)\mu(t') \tag{2.4}$$

from the condition $n_{w'} \in N_\mu$.

Here, we determine the value of $\mu(n_{ww'}^{-1}n_w n_{w'})$. Let $n_0' = n_1 + \cdots + n_q$. Then, the matrix $n_{ww'}^{-1}n_w n_{w'}$ is expressed as follows;

$$n_{ww'}^{-1}n_w n_{w'} = \begin{pmatrix} I_{n_0'} & & & \\ & D_{n_0} & & \\ & & I_{n_0'} & \\ & & & D_{n_0} \end{pmatrix},$$

where D_{n_0} is a diagonal matrix of size $n_0 \times n_0$, and the entries of the matrix $n_{ww'}^{-1}n_w n_{w'}$ which are not written are 0.

Since $p_i = 0$ for $i = n_0' + 1, n_0' + 2, \dots, n_0' + n_0$, we have

$$\mu(n_{ww'}^{-1}n_w n_{w'}) = 1.$$

Then, from (2.4), we have the following equations;

$$\begin{aligned}\tilde{\mu}((n_w t)(n_{w'}t')) &= 1 \cdot \mu(t)\mu(t') \\ &= \tilde{\mu}(n_w t)\tilde{\mu}(n_{w'}t'). \end{aligned} \tag{2.5}$$

The equation (2.5) shows that the map $\tilde{\mu} : N_\mu \to \mathbf{C}^\times$ is a character of N_μ. □

From the proposition 2.7, we obtain the fact that for any irreducible representation μ of T, we have a representation $\tilde{\mu}$ of N_μ which satisfies $\tilde{\mu}|_T = \mu$.

Next, we consider representations of N_μ given by representations of W_μ.

Definition 2.8. *Let $\pi : N_\mu \to W_\mu$ be the quotient map and φ a representation of W_μ. Then, we define a representation $\tilde{\varphi}$ of N_μ as follows;*

$$\tilde{\varphi} := \varphi \circ \pi. \tag{2.6}$$

□

Then, we have

$$\tilde{\varphi}(n_w t) = \varphi(w). \tag{2.7}$$

LEMMA 2.9 (Clifford [C]). *Let* μ *be an irreducible representation of* T. *Then, we have a stabilizer subgroup* N_μ *and a group representation* $\tilde{\mu}$ *of* N_μ *as* (2.3). *For an irreducible representation* φ *of* W_μ, *we have a representation* $\tilde{\varphi}$ *of* N_μ *as* (2.6). *Then, the representation* $\tilde{\mu} \otimes \tilde{\varphi}$ *becomes an irreducible representation of* N_μ. ☐

From (2.3) and (2.7), we have

$$(\tilde{\mu} \otimes \tilde{\varphi})(n_w t) = \mu(t) \otimes \varphi(w). \tag{2.8}$$

THEOREM 2.10 (Clifford [C]). *Let* (ρ, V) *be an irreducible representation of* N, μ *the highest weight of* (ρ, V). *Define a representation* φ *of* W_μ *as follows*;

$$\varphi(w) := \rho(n_w) \quad (w \in W_\mu). \tag{2.9}$$

Then, (φ, V_μ) *is an irreducible representation of* W_μ *and the following condition holds*;
for the representation $\tau(\mu, \varphi)$ *of* N_μ *defined as*

$$\tau(\mu, \varphi)(n_w t) := (\tilde{\mu} \otimes \tilde{\varphi})(n_w t) = \mu(t) \otimes \varphi(w), \tag{2.10}$$

we have

$$\rho \cong \tau(\mu, \varphi) \uparrow_{N_\mu}^{N}. \tag{2.11}$$

☐

From the theorem 2.3, lemma 2.9 and theorem 2.10, we obtain the following theorem.

THEOREM 2.11 (Clifford [C]). *The irreducible representation* (ρ, V) *of* N *is parameterized uniquely by the highest weight* μ *and an irreducible representation* φ *of* W_μ *up to equivalence. Moreover, let* (ρ, V), (ρ', V') *be irreducible representations of* N, μ, μ' *the weights of them and* φ, φ' *irreducible representations of* W_μ, $W_{\mu'}$ *respectively. Let* $\rho \cong \tau(\mu, \varphi) \uparrow_{N_\mu}^{N}$, $\rho' \cong \tau(\mu', \varphi') \uparrow_{N_{\mu'}}^{N}$. *Then,* (ρ, V) *and* (ρ', V') *are equivalent if and only if there exists an element* $w \in W$ *by which* $\mu' = w \cdot \mu$ *(in which case we have* $W_\mu = W_{\mu'}$*) and* $\varphi' = w \cdot \varphi$ *hold, where* $(w \cdot \mu)(t) = \mu(n_w^{-1} t n_w)$, $(w \cdot \varphi)(x) = \varphi(w^{-1} x w)$ *for* $t \in T$ *and* $x \in W_\mu$. ☐

NOTATION 2.12. *Let* (ρ, V) *be an irreducible representation of* N. *From the equation* (2.11), *we have*

$$\rho \cong \tau(\mu, \varphi) \uparrow_{N_\mu}^{N}.$$

Then, we write $\tau(\mu, \varphi) \uparrow_{N_\mu}^{N}$ *as* $\theta_{\mu, \tau(\mu, \varphi)}$;

$$\rho \cong \theta_{\mu, \tau(\mu, \varphi)}. \tag{2.12}$$

3. The Irreducible Characters of N

Each element of N can be written as $n_w t$ where n_w is given in the notation 2.1 and $t \in T$. Fix an element $w \in W$ and $t \in T$. The system of representatives of $N/N_\mu \cong W/W_\mu$ forms a finite set. Let

$$R = \{w_1, w_2, \dots w_\beta\} \tag{3.1}$$

be one of the complete sets of representatives. For each $w_i \in R$, we have $n_{w_i} \in N$ as in the notation 2.1. Then,

$$V = \bigoplus V_i, \tag{3.2}$$

where $V_i = \rho(n_{w_i}) V_\mu$. Then, $\rho(n_w t)$ permutes the summands V_i. Hence, we have

$$tr\, \rho(n_w t) = \sum_{\substack{i \text{ s.t.} \\ \rho(n_w t) V_i = V_i}} tr\, \rho(n_w t)|_{V_i}. \tag{3.3}$$

For $v \in V_\mu$, we obtain the following equations;

$$\begin{aligned} \rho(n_w t)\rho(n_{w_i})v &= \rho(n_w t n_{w_i})v \\ &= \rho(n_{w_i} \cdot n_{w_i}^{-1} n_w n_{w_i} \cdot n_{w_i}^{-1} t n_{w_i})v \\ &= \mu(n_{w_i}^{-1} t n_{w_i}) \rho(n_{w_i}) \rho(n_{w_i}^{-1} n_w n_{w_i})v. \end{aligned} \tag{3.4}$$

So, from (3.4), if $n_{w_i}^{-1} n_w n_{w_i} \notin N_\mu$, then

$$\rho(n_w t) V_i \neq V_i$$

and the summand V_i gives no contribution to the value of $tr\, \rho(n_w t)$.

Assume that for some $g \in R$, the summand $\rho(n_g) V_\mu$ is fixed by the action of $\rho(n_w t)$. Then, $n_g^{-1} n_w n_g$ is an element of N_μ. Here, we have

$$n_g^{-1} n_w n_g = n_{g^{-1}wg}(n_{g^{-1}wg}^{-1} n_g^{-1} n_w n_g),$$

and $n_{g^{-1}wg}^{-1} n_g^{-1} n_w n_g$ is an element of T. So, we obtain $g^{-1}wg \in N_\mu/T = W_\mu$.

For each w, consider

$$U^{w} = \{u \in W \mid u^{-1}wu \in W_{\mu}\}. \tag{3.5}$$

For $\delta, \eta \in U^{w}$ which satisfy that $\delta^{-1}w\delta$ and $\eta^{-1}w\eta$ are in the same conjugacy class of W_{μ}, we have $\delta \in Z_{W}(w)\eta W_{\mu}$, where $Z_{W}(w)$ is the centralizer subgroup of w in W.

NOTATION 3.1. *Let* $\{\eta_1, \eta_2, \dots, \eta_l\}$ *be the complete set of representatives of the following quotient*;

$$Z_{W}(w)\backslash U^{w}/W_{\mu}.$$

Then, we have a decomposition of U^{w} into the equivalence classes;

$$U^{w} = Z_{W}(w)\eta_1 W_{\mu} \sqcup Z_{W}(w)\eta_2 W_{\mu} \sqcup \cdots \sqcup Z_{W}(w)\eta_l W_{\mu}. \tag{3.6}$$

On the other hand, let

$$U_r^{w} := (Z_{W}(w)\eta_r W_{\mu}) \cap R. \tag{3.7}$$

Then, from (3.6), we have

$$Z_{W}(w)\eta_r W_{\mu} = \bigsqcup_{w_i \in U_r^{w}} w_i W_{\mu}, \tag{3.8}$$

$$U^{w} = \bigsqcup_{r=1}^{l} \left(\bigsqcup_{w_i \in U_r^{w}} w_i W_{\mu} \right). \tag{3.9}$$

THEOREM 3.2. *Let* (ρ, V) *be an irreducible representation of* N, μ *the highest weight of* ρ, φ *the representation of* W_{μ} *given in* (2.9), ξ *the character of* φ. *Then, we can write*

$$\rho \cong \theta_{\mu, \tau(\mu,\varphi)} = (\tilde{\mu} \otimes \tilde{\varphi}) \uparrow_{N_{\mu}}^{N}$$

as in section 2, (2.10), (2.12). *Let* $n_w t$ *be an element of* N *given by the notation* 2.1, *and* $\{\eta_1, \eta_2, \dots, \eta_l\}$ *be the set given in the notation* 3.1. *Then, the character value determined by the element* $n_w t$ *on space* V *with representation* ρ *is written as follows*;

$$tr\ \rho(n_w t) = \sum_{r=1}^{l} \xi(\eta_r^{-1} w \eta_r) \sum_{w_i \in U_r^{w}} \mu(n_{w_i^{-1}ww_i}^{-1} n_{w_i}^{-1} n_w n_{w_i}) t_{w_i(1)}^{p_1} t_{w_i(2)}^{p_2} \cdots t_{w_i(n)}^{p_n}. \tag{3.10}$$

Proof. From (2.10), (2.11), (3.3), (3.4), (3.8) and (3.9), we obtain the following equations;

$$\begin{aligned}
tr\ \rho(n_w t) &= \sum_{\substack{w_i \in R,\\ w_i^{-1} w w_i \in W_\mu}} \mu(n_{w_i}^{-1} t n_{w_i})\ tr\ \rho(n_{w_i}^{-1} n_w n_{w_i})|_{V_\mu} \\
&= \sum_{r=1}^{l} \sum_{w_i \in U_r^w} \mu(n_{w_i}^{-1} t n_{w_i})\ tr\ \rho(n_{w_i}^{-1} n_w n_{w_i})|_{V_\mu} \\
&= \sum_{r=1}^{l} \sum_{w_i \in U_r^w} tr\ \tau(\mu, \varphi)(n_{w_i}^{-1} n_w n_{w_i}) \mu(n_{w_i}^{-1} t n_{w_i}) \\
&= \sum_{r=1}^{l} \sum_{w_i \in U_r^w} tr\ \tau(\mu, \varphi)(n_{w_i^{-1} w w_i} \cdot n_{w_i^{-1} w w_i}^{-1} n_{w_i}^{-1} n_w n_{w_i}) \mu(n_{w_i}^{-1} t n_{w_i}) \\
&= \sum_{r=1}^{l} \sum_{w_i \in U_r^w} tr\ \varphi(w_i^{-1} w w_i) \mu(n_{w_i^{-1} w w_i}^{-1} n_{w_i}^{-1} n_w n_{w_i}) \mu(n_{w_i}^{-1} t n_{w_i}).
\end{aligned}$$

Let ξ be the character of the irreducible representation φ of W_μ. Then, we obtain the following equation;

$$tr\ \rho(n_w t) = \sum_{r=1}^{l} \sum_{w_i \in U_r^w} \xi(w_i^{-1} w w_i) \mu(n_{w_i^{-1} w w_i}^{-1} n_{w_i}^{-1} n_w n_{w_i}) \mu(n_{w_i}^{-1} t n_{w_i}). \tag{3.11}$$

On the other hand, we have

$$\xi(w_i^{-1} w w_i) = \xi(\eta_r^{-1} w \eta_r), \tag{3.12}$$

$$\mu(n_{w_i}^{-1} t n_{w_i}) = t_{w_i(1)}^{p_1} t_{w_i(2)}^{p_2} \cdots t_{w_i(n)}^{p_n}, \tag{3.13}$$

for the element $w_i \in U_r^w$. Then, from (3.11), (3.12) and (3.13), we obtain the following equation;

$$tr\ \rho(n_w t) = \sum_{r=1}^{l} \xi(\eta_r^{-1} w \eta_r) \sum_{w_i \in U_r^w} \mu(n_{w_i^{-1} w w_i}^{-1} n_{w_i}^{-1} n_w n_{w_i}) t_{w_i(1)}^{p_1} t_{w_i(2)}^{p_2} \cdots t_{w_i(n)}^{p_n},$$

which gives the same value as (3.10). □

Remark 3.3. *For* $w_i \in U_r^w \subset Z_W(w) \eta_r W_\mu$, *we can write* $w_i = z_i^r \eta_r h_i^r$, *where* $z_i^r \in Z_W(w)$ *and* $h_i^r \in W_\mu$. *For each* w_i, *fix* z_i^r *and* h_i^r *which satisfy* $w_i = z_i^r \eta_r h_i^r$. *Then,*

$$\{z_i^r \mid w_i = z_i^r \eta_r h_i^r\} \tag{3.14}$$

is the representatives of the following quotient set;

$$Z_W(w)/(Z_W(w) \cap \eta_r W_\mu \eta_r^{-1}). \tag{3.15}$$

4. The Value of Symmetric Functions at Eigenvalues of $n_w t$

In this section, we express the value of elementary symmetric functions at the eigenvalues of the element $n_w t$ in N.

As in the notation 2.1, let

$$n_w = \begin{pmatrix} A & C \\ -C & A \end{pmatrix}, \quad t = \begin{pmatrix} s & 0 \\ 0 & \bar{s} \end{pmatrix},$$

where t is a diagonal matrix of $Sp(n)$. Then, the characteristic polynomial of $n_w t$ is written as follows;

$$det(xI_{2n} - n_w t) = det\begin{pmatrix} xI_n - As & -C\bar{s} \\ Cs & xI_n - A\bar{s} \end{pmatrix}.$$

Let e_k be the k-*th* elementary symmetric function, and let $\varepsilon_k(n_w t)$ be the value of the function e_k at the eigenvalues of $n_w t$. Then, we obtain the characteristic polynomial as the polynomial of x with coefficients $\pm\varepsilon_k(n_w t)$;

$$det(xI_{2n} - n_w t) = x^{2n} - \varepsilon_1(n_w t)x^{2n-1} + \varepsilon_2(n_w t)x^{2n-2} - \cdots + (-1)^{2n}\varepsilon_{2n}(n_w t).$$

Fix an element w in W. Let $f_k^w(t)$ be the function on T whose value at t is given as $\varepsilon_k(n_w t)$. Here, we determine the form of the function $f_k^w(t)$ on T.

Definition 4.1. *For each cycle element* $\gamma = (i_1 i_2 \cdots i_s)$, *define* $t(\gamma)$ *to be a monomial* $t_{i_1} t_{i_2} \cdots t_{i_s}$, *and* $|\gamma|$ *to be the length of* γ. *For a cycle element* $\gamma = (i_1 i_2 \cdots i_s)$ *in the cycle expression of* w, *define a matrix* $n_\gamma = (c_{ij})_{1 \le i,j \le 2n}$ *of size* $2n \times 2n$ *to be as follows*;
for $n_w = (n_{ij})_{1 \le i,j \le 2n}$,

$$c_{ij} = \begin{cases} n_{ij} & (i = \gamma(j), j \in \{i_1, \ldots, i_s\}) \\ 1 & (i = j, j \notin \{i_1, \ldots, i_s\}) \\ 0 & \textit{otherwise}. \end{cases}$$

Then, define the value $det(\gamma)$ *to be as follows*;

$$det(\gamma) = det(n_\gamma). \tag{4.1}$$

□

Let the cycle expression of w be as follows;

$$w = \gamma_1\gamma_2\cdots\gamma_j. \tag{4.2}$$

We set $k_i = |\gamma_i|$ and let $\zeta_{i,1}, \zeta_{i,2}, \dots, \zeta_{i,k_i}$ be the roots of the equation $x^{k_i} + (-1)^{k_i}\, det(\gamma_i)t(\gamma_i) = 0$. Then, the eigenvalues of $n_w t$ are given as follows;

$$(\zeta_{1,1}, \zeta_{1,2}, \dots, \zeta_{1,k_1}, \dots \zeta_{j,k_j}). \tag{4.3}$$

On the other hand, for $\boldsymbol{x}_1 = (x_1, x_2, \dots, x_{k_1}), \boldsymbol{x}_2 = (x_{k_1+1}, x_{k_1+2}, \dots, x_{k_1+k_2}), \dots,$ $\boldsymbol{x}_j = (x_{k_1+\cdots+k_{j-1}+1}, \dots, x_{k_1+\cdots+k_j})$, where $k_1 + k_2 + \cdots + k_j = n$, we have

$$e_k(\boldsymbol{x}_1, \dots, \boldsymbol{x}_j) = \sum_{l_1+\cdots+l_j=k} e_{l_1}(\boldsymbol{x}_1)\cdots e_{l_j}(\boldsymbol{x}_j). \tag{4.4}$$

By substituting the eigenvalues of $n_w t$ in $(x_1, x_2, \dots, x_n)$ of the equation (4.4), we have the following equation;

$$e_k(\zeta_{1,1}, \dots, \zeta_{j,k_j}) = \sum_{l_1+\cdots+l_j=k} e_{l_1}(\zeta_{1,1}, \dots, \zeta_{1,k_1})\cdots e_{l_j}(\zeta_{j,1}, \dots, \zeta_{j,k_j}), \tag{4.5}$$

and we have

$$e_{l_i}(\zeta_{i,1}, \dots, \zeta_{i,k_i}) = \begin{cases} det(\gamma_i)t(\gamma_i) & (l_i = k_i) \\ 1 & (l_i = 0) \\ 0 & otherwise. \end{cases} \tag{4.6}$$

So, we have the following lemma.

Lemma 4.2. *We can express the function value of $f_k^w(t)$ at t as follows;*

$$f_k^w(t) = \sum_{\{\gamma_{j_1},\dots,\gamma_{j_l}\}} det(\gamma_{j_1})t(\gamma_{j_1})\; det(\gamma_{j_2})t(\gamma_{j_2})\cdots det(\gamma_{j_l})t(\gamma_{j_l}), \tag{4.7}$$

where $\gamma_{j_1}, \dots, \gamma_{j_l}$ run over distinct cycle elements appearing in the cycle expression of w, and satisfy the condition

$$|\gamma_{j_1}| + |\gamma_{j_2}| + \cdots + |\gamma_{j_l}| = k, \tag{4.8}$$

and $det(\gamma_{j_k})$ is the value defined in definition 4.1 *corresponding to the cycle element γ_{j_k}. The set $\{\gamma_{j_1}, \dots, \gamma_{j_l}\}$ appears exactly once in the sum.*

Proof. The value $f_k^w(t)$ is obtained by substituting the eigenvalues of $n_w t$ to the symmetric function e_k. So, $e_k(\zeta_{1,1}, \dots, \zeta_{j,k_j})$, the left hand side of (4.5), is the value $f_k^w(t)$. From (4.5) and (4.6), we obtain the equation (4.7). □

REMARK 4.3. *In case γ is self-contained, we obtain $det(\gamma) = +1$. The reason of this is explained as follows. Since the length of γ is even as the element of $\mathfrak{S}_{2n}$, we obtain $sgn(\gamma) = -1$. Furthermore, in the matrix n_γ defined in definition* 4.1, *there are odd number of (-1)'s. So, we have $det(\gamma) = (-1) \cdot (-1) = +1$. With the condition $t(\gamma) = 1$, we obtain $det(\gamma)t(\gamma) = 1$.*

Here, in case separated γ_1 and γ_2 are expressed as $\gamma_1 = (i_1 i_2 \cdots i_m)$, $\gamma_2 = (-i_1 -i_2 \cdots -i_m)$ respectively, we obtain

$$det(\gamma_1) = det(\gamma_2) = +1 \ \ or \ \ -1$$

and $t(\gamma_1) = \bar{t}(\gamma_2)$, so we obtain $t(\gamma_1) \cdot t(\gamma_2) = 1$, $det(\gamma_1) \cdot det(\gamma_2) = +1$.

5. The Branching Rule from $S_p(n)$ to N

In this section, we calculate the multiplicity of the irreducible representation of N in the restriction of the irreducible representation of $Sp(n)$ to N.

Let $\rho = \theta_{\mu,\tau(\mu,\varphi)}$ as (2.12), where μ is the highest weight of ρ given in (2.1), N_μ is the stabilizer of μ, $W_\mu = N_\mu/T$, $R = \{w_1, w_2, \dots, w_\beta\}$ is a complete system of representatives of $N/N_\mu \cong W/W_\mu$ and φ is an irreducible representation of W_μ (see theorem 2.3, (2.2), (2.9), (3.1)).

Let dn be the normalized Haar measure on N with $\int_N dn = 1$. For characters ψ, ψ' of N, define an inner product $\langle \psi, \psi' \rangle$ as follows;

$$\langle \psi, \psi' \rangle = \int_N \psi \bar{\psi}' \, dn \tag{5.1}$$

Then, the value is the same as the following integration value;

$$\frac{1}{|W|} \sum_{w \in W} \int_T \psi(n_w t) \overline{\psi'(n_w t)} \, dt, \tag{5.2}$$

where we define the measure dt on T as follows;

$$dt = \frac{1}{(2\pi)^n} d\theta_1 \cdots d\theta_n, \quad t_i = e^{\sqrt{-1}\theta_i}, \quad t_{-i} = e^{-\sqrt{-1}\theta_i}. \tag{5.3}$$

Then, the irreducible characters of N form orthonormal basis under the inner product (5.1).

LEMMA 5.1. *For the measure dt on T, we have the following equation;*

$$\int_T (t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}) \overline{(t_1^{b_1} t_2^{b_2} \cdots t_n^{b_n})} \, dt = \begin{cases} 1 & (a_i = b_i, i = 1, 2, \dots, n) \\ 0 & (otherwise) \end{cases} \tag{5.4}$$

□

Fix an element $w \in W$. From (3.5), notation 3.1 and (3.7), we have U^w, $\{\eta_1, \dots, \eta_l\}$, U_r^w.

Let $n_0, n_1, \dots, n_q$ be the numbers defined in the definition 2.4. For $t \in T$, we can write as follows;

$$\mu(n_{w_i}^{-1} t n_{w_i}) = (t_{w_i(1)} \cdots t_{w_i(n_1)})^{p_1'} \cdots (\cdots t_{w_i(n_1+\cdots+n_q)})^{p_q'} \tag{5.5}$$

where $p_1', p_2', \dots, p_q'$ are all the distinct non-zero numbers in $\{p_1, p_2, \dots, p_n\}$, $\mu(t) = t_1^{p_1} t_2^{p_2} \cdots t_n^{p_n}$, with the condition $p_1' > p_2' > \cdots > p_q' > 0$.

Let $w_0 = w_i^{-1} w w_i \in W_\mu$. Then, from the definition 2.6, w_0 is written in product of elements of $W(A_{n_k-1})$, $k = 1, \dots, q$, and $W(C_{n_0})$;

$$w_0 = \delta_1 \delta_2 \cdots \delta_q \delta_0, \tag{5.6}$$

where

$$\delta_k \in W(A_{n_k-1}) \quad (k = 1, \dots, q), \quad \delta_0 \in W(C_{n_0}).$$

For $k = 1, \dots, q$, let

$$\delta_k = \delta_{k,1} \delta'_{k,1} \delta_{k,2} \delta'_{k,2} \cdots \delta_{k,s_k} \delta'_{k,s_k}, \tag{5.7}$$

$$\delta_0 = \delta_{0,1} \cdots \delta_{0,s_0} \tag{5.8}$$

be the cycle expression of δ_k and δ_0 in W, where $\delta_{k,l}$'s are permutations on I_k and $\delta'_{k,l}$'s are permutations on I_k' respectively with $t(\delta_{k,1}) = \overline{t(\delta'_{k,1})}$. Then, we obtain the cycle expression of w as follows;

$$w = \gamma_{1,1} \gamma'_{1,1} \cdots \gamma_{1,s_1} \gamma'_{1,s_1} \gamma_{2,1} \gamma'_{2,1} \cdots \gamma_{0,s_0}, \tag{5.9}$$

where

$$\gamma_{k,l} = w_i \delta_{k,l} w_i^{-1}, \quad \gamma'_{k,l} = w_i \delta'_{k,l} w_i^{-1}. \tag{5.10}$$

Lemma 5.2. *Let* $\gamma_{k,l}$ *be given as* (5.10). *Then, we obtain the following equation*;

$$\mu(n_{w_i}^{-1} t n_{w_i}) = (t(\gamma_{1,1}) \cdots t(\gamma_{1,s_1}))^{p_1'} \cdots (t(\gamma_{q,1}) \cdots t(\gamma_{q,s_q}))^{p_q'}. \tag{5.11}$$

Proof. Let $n_1' = 0$, $n_k' = n_1 + \cdots + n_{k-1}$, $k = 2, 3, \dots, q$. Then, we have the following equation;

$$t_{n_k'+1} t_{n_k'+2} \cdots t_{n_k'+n_k} = t(\delta_{k,1}) \cdots t(\delta_{k,s_k}). \tag{5.12}$$

Then, we obtain the following equation;

$$t_{w_i(n_k'+1)} t_{w_i(n_k'+2)} \cdots t_{w_i(n_k'+n_k)} = t(\gamma_{k,1}) \cdots t(\gamma_{k,s_k}). \tag{5.13}$$

From (5.5) and (5.13), we obtain the following equation;

$$\mu(n_{w_i}^{-1} t n_{w_i}) = (t(\gamma_{1,1}) \cdots t(\gamma_{1,s_1}))^{p_1'} \cdots (t(\gamma_{q,1}) \cdots t(\gamma_{q,s_q}))^{p_q'},$$

by which the result follows. □

Let ψ be the character of ρ. Then, from the theorem 3.2, (3.10), we have

$$\psi(n_w t) = \sum_{r=1}^{l} \xi(\eta_r^{-1} w \eta_r) \sum_{w_i \in U_r^w} \mu(n_{w_i^{-1} w w_i}^{-1} n_{w_i}^{-1} n_w n_{w_i}) t_{w_i(1)}^{p_1} t_{w_i(2)}^{p_2} \cdots t_{w_i(n)}^{p_n}. \tag{5.14}$$

Let χ_λ be an irreducible character of $Sp(n)$ with $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, and $\chi_\lambda^w(t)$ the value of χ_λ at the element $n_w t$;

$$\chi_\lambda^w(t) = \chi_\lambda(n_w t). \tag{5.15}$$

Let ψ be the irreducible character of N. Then, the multiplicity of ψ in $\chi_\lambda \downarrow_N$, $\langle \psi, \chi_\lambda \downarrow_N \rangle$, is given as follows;

$$\langle \psi, \chi_\lambda \downarrow_N \rangle = \int_N \psi \cdot \overline{\chi_\lambda \downarrow_N}\, dn. \tag{5.16}$$

Here, we express the function χ_λ by the elementary symmetric functions.

THEOREM 5.3 (Koike-Terada [KT1]). *Let χ_λ be the irreducible character of $Sp(n)$. Then, we have*

$$\chi_\lambda = |e_{({}^t\lambda)^*} - e_{({}^t\lambda)^* - 2(1^l)}, e_{({}^t\lambda)^* + (1^l)} - e_{({}^t\lambda)^* - 3(1^l)}, \dots, e_{({}^t\lambda)^* + (l-1)(1^l)} - e_{({}^t\lambda)^* - (l+1)(1^l)}|, \tag{5.17}$$

where $l = \lambda_1$ and for a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, we define

$$\lambda^* = (\lambda_1, \lambda_2 - 1, \dots, \lambda_n - (n-1)) \in \mathbf{Z}^n.$$

□

Let ${}^t\lambda = (\lambda_1', \lambda_2', \dots, \lambda_l')$ be the transposed partition of λ with $l = \lambda_1$. Expanding the right hand side of (5.17), we obtain the following equation;

$$\chi_\lambda = \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\mathrm{sgn}(\sigma)) \sum_{J \subset \{1,2,\dots,\lambda_1\}} (-1)^{|J|} e_{m_1} e_{m_2} \cdots e_{m_{\lambda_1}}, \tag{5.18}$$

where we define

$$m_k = \begin{cases} \lambda_{\sigma(k)}' - (\sigma(k) - 1) - (k+1) & (k \in J) \\ \lambda_{\sigma(k)}' - (\sigma(k) - 1) + (k-1) & (k \notin J). \end{cases} \tag{5.19}$$

From the equation (4.3), we have the eigenvalues of $n_w t$ as follows;

$$\zeta = (\zeta_{1,1}, \zeta_{1,2}, \dots, \zeta_{1,k_1}, \dots \zeta_{j,k_j}).$$

Proposition 5.4. *The value* $\chi_\lambda(n_w t) = \chi_\lambda^w(t)$ *is expressed as follows;*

$$\chi_\lambda^w(t) = \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\operatorname{sgn}(\sigma)) \sum_{J \subset \{1,2,\dots,\lambda_1\}} (-1)^{|J|} f_{m_1}^w(t) f_{m_2}^w(t) \cdots f_{m_{\lambda_1}}^w(t), \tag{5.20}$$

where m_k, $k = 1, 2, \dots, \lambda_1$ *are given in* (5.19).

Proof. Substituting ζ in the equation (5.17), from (5.18), (4.5), (4.6), we obtain the following equations;

$$\begin{aligned}\chi_\lambda^w(t) &= \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\operatorname{sgn}(\sigma)) \sum_{J \subset \{1,2,\dots,\lambda_1\}} (-1)^{|J|} e_{m_1}(\zeta) e_{m_2}(\zeta) \cdots e_{m_{\lambda_1}}(\zeta) \\ &= \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\operatorname{sgn}(\sigma)) \sum_{J \subset \{1,2,\dots,\lambda_1\}} (-1)^{|J|} f_{m_1}^w(t) f_{m_2}^w(t) \cdots f_{m_{\lambda_1}}^w(t)\end{aligned}$$

which gives the equation (5.20). □

Fix an element $u \in U_r^w$ and $J \subset \{1, 2, \dots, \lambda_1\}$. Then, we have m_k, $k = 1, 2, \dots, \lambda_1$ as in (5.19). Here, we determine the coefficient of the term $\mu(u^{-1}tu)$ in the function value $f_{m_1}^w(t) \cdots f_{m_{\lambda_1}}^w(t)$.

For $V = \mathbf{C}^{2n}$, let $E^k = V \wedge V \wedge \cdots \wedge V$ (k *multiple of* V) be the k-th alternative tensor space. Then, $f_{m_1}^w(t) \cdots f_{m_{\lambda_1}}^w(t)$ is the character value at $n_w t$ on the representation space $E^{m_1} \otimes E^{m_2} \otimes \cdots \otimes E^{m_{\lambda_1}}$.

Let $v_1, v_2, \dots, v_n, v_{-1}, \dots, v_{-n}$ be the basis of V consisting of the weight vectors of t. Then, we have the basis of $E^{m_1} \otimes E^{m_2} \otimes \cdots \otimes E^{m_{\lambda_1}}$ as follows;

$$v_1^{a_1^1} \wedge v_2^{a_2^1} \wedge \cdots \wedge v_n^{a_n^1} \wedge v_{-1}^{a_{-1}^1} \wedge \cdots \wedge v_{-n}^{a_{-n}^1} \otimes \cdots \otimes v_1^{a_1^{\lambda_1}} \wedge \cdots \wedge v_n^{a_n^{\lambda_1}} \wedge v_{-1}^{a_{-1}^{\lambda_1}} \wedge \cdots \wedge v_{-n}^{a_{-n}^{\lambda_1}}, \tag{5.21}$$

where

$$a_l^k \in \{0, 1\}, \quad k = 1, \dots, \lambda_1,\ l = 1, \dots, n, -1, \dots, -n, \tag{5.22}$$

with

$$\sum_l a_l^k = m_k. \tag{5.23}$$

The basis of $E^{m_1} \otimes \cdots \otimes E^{m_{\lambda_1}}$ as (5.21) which give contribution to the character value at $n_w t$ are eigenvectors of n_w. So, we obtain the condition for v to be an eigenvector of n_w as follows;

for any cycle $\gamma = (i_1, i_2, \dots, i_s)$ which appears in the cycle expression of w, we have

$$a_{i_1}^k = a_{i_2}^k = \cdots = a_{i_s}^k, \quad k = 1, 2, \dots, \lambda_1. \tag{5.24}$$

Then, for self-contained $\gamma = (i_1, \dots, i_s, -i_1, \dots, -i_s)$, we have

$$a_{i_1}^k = a_{-i_1}^k = a_{i_2}^k = a_{-i_2}^k = \cdots = a_{i_s}^k = a_{-i_s}^k. \tag{5.25}$$

LEMMA 5.5. *Let v be an eigenvector of $n_w t$ in $E^{m_1} \otimes \cdots \otimes E^{m_{\lambda_1}}$ that satisfies* (5.21), (5.22), (5.23), (5.24) *with m_k, $k = 1, 2, \dots, \lambda_1$ given in* (5.19). *For $\mu(t) = t_1^{p_1} t_2^{p_2} \cdots t_n^{p_n}$, we define $p_{-l} = -p_l$ for $l > 0$. If the eigenvalue of v is expressed as scalar multiple of the term $\mu(u^{-1}tu)$, then we have the following condition*;

for $d_l = \sum_{k=1}^{\lambda_1} a_l^k$, we have

$$d_l - d_{-l} = p_{u^{-1}(l)}, \quad l = 1, 2, \dots, n. \tag{5.26}$$

PROOF. Since $\mu(u^{-1}tu) = t_1^{p_{u^{-1}(1)}} \cdots t_n^{p_{u^{-1}(n)}}$, the power of t_l in the eigenvalue of the weight vector (5.21) is given as $p_{u^{-1}(l)}$. There appear $t_l^{d_l}$ and $t_{-l}^{d_{-l}}$ in the eigenvalue and $t_{-l} = t_l^{-1}$. Then, we have the following equation;

$$d_l - d_{-l} = p_{u^{-1}(l)},$$

by which the equation (5.26) follows. □

Then, we obtain a matrix (a_l^k), $k = 1, 2, \dots, \lambda_1$, $l = 1, 2, \dots, n, -1, \dots, -n$ which satisfies the conditions (5.22), (5.23), (5.24), (5.26).

Fix the space $E^{m_1} \otimes \cdots \otimes E^{m_{\lambda_1}}$ with m_k, $k = 1, 2, \dots, \lambda_1$ given by (5.19), and let μ be as in lemma 5.5. Then, for the fixed elements $w \in W$ and $u \in U_r^w$, we define M to be the set of all the matrices (a_l^k) that satisfy the conditions (5.22), (5.23), (5.24), (5.26). Then, we obtain the following proposition.

PROPOSITION 5.6. *Let the set X consist of all the weight vectors in the space $E^{m_1} \otimes \cdots \otimes E^{m_{\lambda_1}}$ given as* (5.21) *that become eigenvectors of $n_w t$ and the eigenvalue is scalar multiple of the term $\mu(u^{-1}tu)$. Then, there exists one-to-one correspondence between the set M and the set X.*

Proof. For each vector v in X, v is written as (5.21) and we obtain one and only one matrix (a_l^k) which belongs to M. This correspondence is bijective. Indeed, for each matrix $(a_l^k) \in M$, we have the vector v defined as

$$v = v_1^{a_1^1} \wedge \cdots \wedge v_{-n}^{a_{-n}^1} \otimes \cdots \otimes v_1^{a_1^{\lambda_1}} \wedge \cdots \wedge v_{-n}^{a_{-n}^{\lambda_1}}.$$

Then, from (5.22) and (5.23), the vector v is an element of the alternative tensor space $E^{m_1} \otimes \cdots \otimes E^{m_{\lambda_1}}$. From (5.24), the vector v is an eigenvector of $n_w t$. Furthermore, from (5.26), the eigenvalue of the vector v is written as $\varepsilon \cdot \mu(u^{-1}tu)$. So, v is an element of X, and gives the matrix (a_l^k). Hence the correspondence is one-to-one between M and X. □

As in (5.9), we have the cycle expression of w as follows;

$$w = \gamma_{1,1}\gamma'_{1,1} \cdots \gamma_{1,s_1}\gamma'_{1,s_1}\gamma_{2,1}\gamma'_{2,1} \cdots \gamma_{0,s_0}.$$

Definition 5.7. *In the space $E^{m_1} \otimes \cdots \otimes E^{m_{\lambda_1}}$, let $v \in X$. Then, we have the matrix $(a_l^k) \in M$ corresponding to v. For each $i = 0, 1, \ldots, q$, define a matrix $A_i = (\alpha_{k,j}^i)$, where $k = 1, 2, \ldots, \lambda_1$, $j = 1, 2, \ldots, s_i$ as follows;*

for separated $\gamma_{i,j} = (h_1, h_2, \ldots, h_s)$,

$$\alpha_{k,j}^i = a_{h_1}^k (= a_{h_2}^k = \cdots = a_{h_s}^k). \tag{5.27}$$

Similarly, define a matrix $B_i = (\beta_{k,j}^i)$, where $k = 1, 2, \ldots, \lambda_1$, $j = 1, 2, \ldots, s_i$, as follows;

for separated $\gamma'_{i,j} = (-h_1, -h_2, \ldots, -h_s)$,

$$\beta_{k,j}^i = a_{-h_1}^k (= a_{-h_2}^k = \cdots = a_{-h_s}^k). \tag{5.28}$$

For self-contained $\gamma_{0,j} = (h_1, \ldots, h_s, -h_1, \ldots, -h_s)$, we use the same symbol $\gamma_{0,j}$ to express the cycle element and we define $\alpha_{k,j}^0 = a_{h_1}^k$ and $\beta_{k,j}^0 = 0$. □

Then, we obtain a pair of sequences of matrices

$$[(A_1, A_2, \ldots, A_q, A_0), (B_1, B_2, \ldots, B_q, B_0)], \tag{5.29}$$

which satisfies the following conditions;

$$\alpha_{k,j}^i, \beta_{k,j}^i \in \{0, 1\}, \quad i = 1, \ldots, q, k = 1, \ldots, \lambda_1, j = 1, \ldots, s_i, \tag{5.30}$$

$$\sum_{i=0}^{q} \sum_{j=1}^{s_i} \alpha_{k,j}^i |\gamma_{i,j}| + \beta_{k,j}^i |\gamma'_{i,j}| = m_k. \tag{5.31}$$

LEMMA 5.8. *Notations are as in definition* 5.7. *Let* $i = 0, 1, \dots, q$. *For a pair of separated cycle elements* $\gamma_{i,j}$, $\gamma'_{i,j}$, *we define numbers* $d_{i,j}$, $d'_{i,j}$ *as follows*;

$$d_{i,j} = \sum_{k=1}^{s_i} \alpha^i_{k,j}, \quad d'_{i,j} = \sum_{k=1}^{s_i} \beta^i_{k,j}, \quad j = 1, \dots, s_i. \tag{5.32}$$

Then, we have the following equation;

$$d_{i,j} - d'_{i,j} = p'_i, \tag{5.33}$$

where p'_i*'s are as in* (5.5) *with* $p'_0 = 0$. *For self-contained* $\gamma_{0,j}$, $d'_{0,j} = 0$ *and* $d_{0,j}$ *has no restriction.*

PROOF. Let $\gamma_{i,j} = (h_1, \dots, h_s)$, $\gamma'_{i,j} = (-h_1, \dots, -h_s)$. Then, $d_{i,j} = d_{h_1}$, $d'_{i,j} = d_{-h_1}$ and we have

$$d_{i,j} - d'_{i,j} = d_{h_1} - d'_{-h_1} = p_{u^{-1}(h_1)}.$$

Since $u^{-1}(h_1) \in I_i$ (see definition 2.5), we obtain the following equation;

$$d_{i,j} - d'_{i,j} = p'_i.$$

For self-contained $\gamma_{0,j} = (h_1, h_2, \dots, h_s)$, we have $\beta^0_{k,j} = 0$ and $d'_{0,j} = 0$. Since $t_{h_1} t_{h_2} \cdots t_{h_s} = 1$, we have $(t_{h_1} t_{h_2} \cdots t_{h_s})^{d_{0,j}} = 1$ and the number $d_{0,j}$ gives no contribution to the eigenvalue. Hence, the result follows. □

DEFINITION 5.9. *Notations are as in definition* 5.7. *Fix* $J \subset \{1, 2, \dots, \lambda_1\}$. *Then, we have a sequence* $(m_1, m_2, \dots, m_{\lambda_1})$ *where* m_k*'s are given in* (5.19). *Let* $\boldsymbol{v}$ *be the sequence defined as* $(m_1, m_2, \dots, m_{\lambda_1})$. *Define*

$$Mat(w, u, \boldsymbol{v}, \mu)$$

to be the set of the pair of sequences of matrices as (5.29) *that satisfies* (5.30), (5.31), (5.33). □

Then, we have the following proposition.

PROPOSITION 5.10. *Notations are as in lemma* 5.5, *definition* 5.7, *definition* 5.9. *Then, there exists one-to-one correspondence between the set* M *and the set* $Mat(w, u, \boldsymbol{v}, \mu)$.

Proof. Given the matrix $(a_l^k) \in M$, from the equation (5.27) and (5.28), there exists a unique pair of sequences of matrices as (5.29). Then, this correspondence is bijective. Indeed, for the pair

$$[(A_1, A_2, \dots, A_q, A_0), (B_1, B_2, \dots, B_q, B_0)],$$

set $a_l^k = \alpha_{k,j}^i$ when $\gamma_{i,j} = (h_1, \dots, h_s)$ and $l = h_t$ for a certain $t = 1, \dots, s$, or $a_l^k = \beta_{k,j}^i$ when $\gamma'_{i,j} = (-h_1, \dots, -h_s)$ and $l = -h_t$ for a certain $t = 1, \dots, s$. Then, from the conditions (5.30), (5.31), (5.33), the matrix (a_l^k), $k = 1, \dots, \lambda_1$, $l = 1, \dots, -n$ satisfies the conditions (5.22), (5.23), (5.24), (5.26). Hence, we have $(a_l^k) \in M$, and the pair given as (5.29) by the (a_l^k) coincides with the given pair

$$[(A_1, A_2, \dots, A_q, A_0), (B_1, B_2, \dots, B_q, B_0)].$$

Hence, the result follows. □

Proposition 5.11. *Notations are as in proposition 5.6 and proposition 5.10. Then, there exists one-to-one correspondence between the set X and the set $Mat(w, u, \mathbf{v}, \mu)$.*

Proof. From proposition 5.6 and proposition 5.10, the result follows. □

Definition 5.12. *Notations are as in proposition 5.11. Define $m(w, u, \mathbf{v}, \mu)$ to be the number of the elements in the set $Mat(w, u, \mathbf{v}, \mu)$.* □

Here, we investigate the eigenvalue of $v \in X$ for n_w.

Lemma 5.13. *Let m_k, $k = 1, 2, \dots, \lambda_1$ be given as in (5.19). For each $v \in X$ in the space $E^{m_1} \otimes \cdots \otimes E^{m_{\lambda_1}}$, we have the following equation;*

$$\begin{aligned} n_w v = {} & (det(\gamma_{1,1})\, det(\gamma_{1,2}) \cdots det(\gamma_{1,s_1}))^{p'_1} \\ & \cdot (det(\gamma_{2,1}) \cdots)^{p'_2} \cdots (\cdots det(\gamma_{q,s_q}))^{p'_q} v. \end{aligned} \tag{5.34}$$

Proof. Let $n_{\gamma_{i,j}}$, $n_{\gamma'_{i,j}}$ be the matrices given as in definition 4.1 for $\gamma_{i,j}$ and $\gamma'_{i,j}$. Then, we have the following equation;

$$n_w = n_{\gamma_{1,1}} n_{\gamma'_{1,1}} \cdots n_{\gamma'_{1,s_1}} n_{\gamma_{2,1}} \cdots n_{\gamma_{0,s_0}}. \tag{5.35}$$

Then, for $n_{\gamma_{i,j}}$ and $n_{\gamma'_{i,j}}$, we have

$$(n_{\gamma_{i,j}} n_{\gamma'_{i,j}}) v = (det(n_{\gamma_{i,j}}))^{d_{i,j}} \cdot (det(n_{\gamma'_{i,j}}))^{d'_{i,j}} \cdot v \tag{5.36}$$

Using the facts $det(n_{\gamma_{i,j}})\, det(n_{\gamma'_{i,j}}) = 1$, $d_{i,j} - d_{i,j} = p'_i$ and the notation in definition 4.1, we have the following equation;

$$n_w v = (det(\gamma_{1,1})\, det(\gamma_{1,2}) \cdots det(\gamma_{1,s_1}))^{p'_1}$$
$$\cdot\, (det(\gamma_{2,1}) \cdots)^{p'_2} \cdots (\cdots det(\gamma_{q,s_q}))^{p'_q} v,$$

by which the result follows. □

Definition 5.14. *Let* $\mu(t) = t^{p_1} \cdots t^{p_n}$ *and fix the element* $u \in U_r^w$. *For the fixed element* w *where*

$$w = \gamma_{1,1}\gamma'_{1,1} \cdots \gamma_{1,s_1}\gamma'_{1,s_1}\gamma_{2,1}\gamma'_{2,1} \cdots \gamma_{0,s_0},$$

we define the number $sgn(w,u,\mu)$ *as follows;*

$$sgn(w,u,\mu) = (det(\gamma_{1,1})\, det(\gamma_{1,2}) \cdots det(\gamma_{1,s_1}))^{p'_1}$$
$$\cdot\, (det(\gamma_{2,1}) \cdots)^{p'_2} \cdots (\cdots det(\gamma_{q,s_q}))^{p'_q}. \tag{5.37}$$

□

Then we obtain the following equation;

$$n_w v = sgn(w,u,\mu) v. \tag{5.38}$$

Proposition 5.15. *Notations are as in definition* 5.12 *and definition* 5.14. *Let* $\boldsymbol{v} = (m_1, m_2, \ldots, m_{\lambda_1})$ *be the sequence given in definition* 5.9. *Then, we have the following equation*;

$$\int_T \mu(u^{-1}tu) \cdot \overline{f^w_{m_1}(t) \cdots f^w_{m_{\lambda_1}}(t)}\, dt = sgn(w,u,\mu) \cdot m(w,u,\boldsymbol{v},\mu) \tag{5.39}$$

Proof. Since the number of eigenvectors v which gives eigenvalue

$$sgn(w,u,\mu)\mu(u^{-1}tu)$$

is given as $m(w,u,\boldsymbol{v},\mu)$, the coefficient of $\mu(u^{-1}tu)$ in the character value $f^w_{m_1}(t) \cdots f^w_{m_{\lambda_1}}(t)$ is given as $sgn(w,u,\mu)m(w,u,\boldsymbol{v},\mu)$. Hence, the result follows. □

Example. Let $w = (123)(-1,-2,-3)(456)(-4,-5,-6) \in W_6$. We calculate the coefficient of the term $t(123) = t_1 t_2 t_3$ of the polynomial $f_6^w(t) f_3^w(t)$ at $n_w t$. Here,

$$\begin{aligned} f_6^w(t) &= 2 + t_1t_2t_3t_4t_5t_6 + t_1t_2t_3t_{-4}t_{-5}t_{-6} \\ &\qquad + t_{-1}t_{-2}t_{-3}t_4t_5t_6 + t_{-1}t_{-2}t_{-3}t_{-4}t_{-5}t_{-6} \\ f_3^w(t) &= t_1t_2t_3 + t_{-1}t_{-2}t_{-3} + t_4t_5t_6 + t_{-4}t_{-5}t_{-6}, \end{aligned}$$

so we obtain the following equations;

$$\begin{aligned} f_6^w(t)f_3^w(t) &= 4t_1t_2t_3 + 4t_{-1}t_{-2}t_{-3} + 4t_4t_5t_6 + 4t_{-4}t_{-5}t_{-6} \\ &\quad + (t_1t_2t_3)^2t_4t_5t_6 + (t_1t_2t_3)^2t_{-4}t_{-5}t_{-6} \\ &\quad + (t_{-1}t_{-2}t_{-3})^2t_4t_5t_6 + (t_{-1}t_{-2}t_{-3})^2t_{-4}t_{-5}t_{-6} \\ &\quad + t_1t_2t_3(t_4t_5t_6)^2 + t_{-1}t_{-2}t_{-3}(t_4t_5t_6)^2 \\ &\quad + t_1t_2t_3(t_{-4}t_{-5}t_{-6})^2 + t_{-1}t_{-2}t_{-3}(t_{-4}t_{-5}t_{-6})^2, \end{aligned}$$

and we obtain the coefficient of the term $t_1t_2t_3$ as 4.

Next, we consider the matrices. At first, we obtain the following table;

	(123)	(−1−2−3)	(456)	(−4−5−6)
$f_6^w(t)$	1	1	0	0
$f_3^w(t)$	1	0	0	0

In the $(1,1)$-entry of the table, we have the number 1. This means that we use $t(123)$ appearing in a monomial of $f_6^w(t)$ to construct a monomial $t(123)$ in $f_6^w(t)f_3^w(t)$. So, this table means we choose monomials $t(123)t(-1-2-3)$ in $f_6^w(t)$ and $t(123)$ in $f_3^w(t)$ to construct a monomial $t(123)$ in $f_6^w(t)f_3^w(t)$.

From the table, we obtain the following matrix;

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In the same manner, we obtain the further three matrices;

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

So, the number of matrices which satisfy the conditions is 4, which coincides with the coefficient of $t_1t_2t_3$ in the function value of $f_6^w(t)f_3^w(t)$ at n_wt. □

For $u, \tilde{u} \in U_r^w$, we compare $m(w, u, \boldsymbol{v}, \mu)$ with $m(w, \tilde{u}, \boldsymbol{v}, \mu)$. From remark 3.3, we have $u = z_u^r\eta_rh_u^r$, $\tilde{u} = z_{\tilde{u}}^r\eta_rh_{\tilde{u}}^r$ with $z_u^r, z_{\tilde{u}}^r \in Z_W(w)$, $h_u^r, h_{\tilde{u}}^r \in W_\mu$.

Let

$$w = \gamma_{1,1}\gamma'_{1,1}\cdots\gamma_{1,s_1}\gamma'_{1,s_1}\gamma_{2,1}\gamma'_{2,1}\cdots\gamma_{0,s_0}$$

be a cycle expression of w given by w_0 as in (5.9), (5.10). Similarly, we write

$$w = \tilde{\gamma}_{1,1}\tilde{\gamma}'_{1,1}\cdots\tilde{\gamma}_{1,s_1}\tilde{\gamma}'_{1,s_1}\tilde{\gamma}_{2,1}\tilde{\gamma}'_{2,1}\cdots\tilde{\gamma}_{0,s_0}, \tag{5.40}$$

where

$$\tilde{\gamma}_{i,j} = \tilde{u}\tilde{\delta}_{i,j}\tilde{u}^{-1}, \quad \tilde{\gamma}'_{i,j} = \tilde{u}\tilde{\delta}'_{i,j}\tilde{u}^{-1}, \tag{5.41}$$

for

$$\begin{aligned}\tilde{w}_0 &= \tilde{u}^{-1}w\tilde{u}\\ &= \tilde{\delta}_{1,1}\tilde{\delta}'_{1,1}\cdots\tilde{\delta}_{1,\tilde{s}_1}\tilde{\delta}'_{1,\tilde{s}_1}\tilde{\delta}_{2,1}\tilde{\delta}'_{2,1}\cdots\tilde{\delta}_{0,\tilde{s}_0}.\end{aligned} \tag{5.42}$$

Then, there exists an element $z \in Z_W(w)$ by which the following conditions hold;

(1) $\tilde{u}(I_i) = zu(I_i)$, $\tilde{u}(I'_i) = zu(I'_i)$, where I_i, I'_i are given in the definition 2.5.

(2) For each pair of cycle elements $\tilde{\gamma}_{i',j'}$ and $\tilde{\gamma}'_{i',j'}$, there exists a unique pair of cycle elements $\gamma_{i,j}$ and $\gamma'_{i,j}$ which satisfies $\tilde{\gamma}_{i',j'} = z\gamma_{i,j}z^{-1}$ and $\tilde{\gamma}'_{i',j'} = z\gamma'_{i,j}z^{-1}$. Furthermore, we have $i' = i$, $|\tilde{\gamma}_{i',j'}| = |\gamma_{i,j}|$, $|\tilde{\gamma}'_{i',j'}| = |\gamma'_{i,j}|$.

(3) For each $i = 0, 1, \ldots, q$, we have $s_i = \tilde{s}_i$.

(4) The set $\{\gamma_{1,1}, \gamma'_{1,1}, \ldots, \gamma_{0,s_0}\}$ coincides with the set $\{\tilde{\gamma}_{1,1}, \tilde{\gamma}'_{1,1}, \ldots, \tilde{\gamma}_{0,s_0}\}$.

PROPOSITION 5.16. *Let $u, \tilde{u} \in U_r^w$. Then, we have the following equation;*

$$m(w, u, \boldsymbol{v}, \mu) = m(w, \tilde{u}, \boldsymbol{v}, \mu). \tag{5.43}$$

PROOF. We compare the set $Mat(w, u, \boldsymbol{v}, \mu)$ with the set $Mat(w, \tilde{u}, \boldsymbol{v}, \mu)$.

For each pair of sequences of matrices

$$[(A_1, A_2, \ldots, A_q, A_0), (B_1, B_2, \ldots, B_q, B_0)],$$

we obtain a unique pair of sequences of matrices

$$[(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_q, \tilde{A}_0), (\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_q, \tilde{B}_0)] \tag{5.44}$$

defined as follows;
for $i' = 0, 1, \ldots, q$,

$$\tilde{A}_{i'} = (\tilde{\alpha}^{i'}_{k,j'}), \tag{5.45}$$

$$\tilde{\alpha}^{i'}_{k,j'} = \alpha^{i'}_{k,j}, \tag{5.46}$$

and

$$\tilde{B}_{i'} = (\tilde{\beta}^{i'}_{k,j'}), \tag{5.47}$$

$$\tilde{\beta}^{i'}_{k,j'} = \beta^{i'}_{k,j}, \tag{5.48}$$

for $\tilde{\gamma}_{i',j'} = z\gamma_{i',j}z^{-1}$, $k = 1, \dots, \lambda_1$, $j' = 1, \dots, s_{i'}$.

Then, the pair (5.44) satisfies the following conditions;

(1) $\tilde{\alpha}^{i'}_{k,j'}, \tilde{\beta}^{i'}_{k,j'} \in \{0, 1\}$, for $i' = 0, 1, \dots, q$, $k = 1, \dots, \lambda_1$, $j' = 1, \dots, s_{i'}$.

(2) Since we have $|\tilde{\gamma}_{i',j'}| = |\gamma_{i,j}|$, $|\tilde{\gamma}'_{i',j'}| = |\gamma'_{i,j}|$, we obtain the following equation;

$$\begin{aligned}
&\sum_{i'=0}^{q}\sum_{j'=1}^{s_{i'}} \tilde{\alpha}^{i'}_{k,j'}|\tilde{\gamma}_{i',j'}| + \tilde{\beta}^{i'}_{k,j'}|\tilde{\gamma}'_{i',j'}| \\
&\quad = \sum_{i'=0}^{q}\sum_{j=1}^{s_{i'}} \alpha^{i'}_{k,j}|\gamma_{i',j}| + \beta^{i'}_{k,j}|\gamma'_{i',j}| \\
&\quad = m_k.
\end{aligned}$$

(3) For $\tilde{d}_{i',j'}$ and $\tilde{d}'_{i',j'}$ given as

$$\tilde{d}_{i',j'} = \sum_{k=1}^{\lambda_1} \tilde{\alpha}^{i'}_{k,j'}, \quad \tilde{d}'_{i',j'} = \sum_{k=1}^{\lambda_1} \beta^{i'}_{k,j'}, \quad j' = 1, \dots, s_{i'},$$

we have the following equation;

$$\tilde{d}_{i',j'} - \tilde{d}'_{i',j'} = p'_{i'}.$$

From the conditions (1), (2), (3), the pair (5.44) belongs to the set $Mat(w, \tilde{u}, \boldsymbol{v}, \mu)$. This correspondence is bijective. So, the result follows. □

Definition 5.17. *For* U^w_r*, we have* η_r *given in notation* 3.1. *Then, we define* $m(w, \eta_r, \boldsymbol{v}, \mu)$ *as follows;*

$$m(w, \eta_r, \boldsymbol{v}, \mu) = m(w, w_i, \boldsymbol{v}, \mu) \tag{5.49}$$

for an element $w_i \in U^w_r$*.* □

Then, we obtain the following equation;

$$\int_T \mu(n_{w_i}^{-1} t n_{w_i})\overline{f^w_{m_1}(t) \cdots f^w_{m_{\lambda_1}}(t)}\, dt = sgn(w, w_i, \mu) m(w, \eta_r, \boldsymbol{v}, \mu). \tag{5.50}$$

PROPOSITION 5.18. *Let the notations be as in* (5.15), *theorem* 5.3, *proposition* 5.4, *definition* 5.14, *proposition* 5.15 *definition* 5.17, (5.50). *Then, we have the following equation*;

$$\int_T \mu(n_{w_i}^{-1} t n_{w_i}) \overline{\chi_\lambda^w(t)}\, dt = sgn(w, w_i, \mu) \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\mathrm{sgn}(\sigma)) \sum_J (-1)^{|J|} m(w, \eta_r, \boldsymbol{v}, \mu), \tag{5.51}$$

where J's are given as $J \subset \{1, 2, \dots, \lambda_1\}$ *and for each J,* $\boldsymbol{v}$ *is given in definition* 5.9.

PROOF. From (5.20), (5.50), we obtain the following equations;

$$\begin{aligned}
&\int_T \mu(n_{w_i}^{-1} t n_{w_i}) \overline{\chi_\lambda^w(t)}\, dt \\
&= \int_T \mu(n_{w_i}^{-1} t n_{w_i}) \overline{\sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\mathrm{sgn}(\sigma)) \sum_J (-1)^{|J|} f_{m_1}^w(t) f_{m_2}^w(t) \cdots f_{m_{\lambda_1}}^w(t)}\, dt \\
&= \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\mathrm{sgn}(\sigma)) \sum_J (-1)^{|J|} \int_T \mu(n_{w_i}^{-1} t n_{w_i}) \overline{f_{m_1}^w(t) f_{m_2}^w(t) \cdots f_{m_{\lambda_1}}^w(t)}\, dt \\
&= sgn(w, w_i, \mu) \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\mathrm{sgn}(\sigma)) \sum_J (-1)^{|J|} m(w, \eta_r, \boldsymbol{v}, \mu),
\end{aligned}$$

which is equal to the right hand side of (5.51). □

From (5.14), we have the following equation;

$$\psi(n_w t) = \sum_{r=1}^{l} \xi(\eta_r^{-1} w \eta_r) \sum_{w_i \in U_r^w} \mu(n_{w_i^{-1} w w_i}^{-1} n_{w_i}^{-1} n_w n_{w_i}) \mu(n_{w_i}^{-1} t n_{w_i}), \tag{5.52}$$

where ξ is given as in theorem 3.2.

THEOREM 5.19. *Under the situation of the proposition* 5.18 *and* (5.52), *we obtain the multiplicity of* ψ *in* $\chi_\lambda \downarrow_N$, $\langle \psi, \chi_\lambda \downarrow_N \rangle$, *as follows*;

$$\begin{aligned}
\langle \psi, \chi_\lambda \downarrow_N \rangle = \frac{1}{|W|} \sum_w \sum_{r=1}^{l} \xi(\eta_r^{-1} w \eta_r) \\
\cdot \sum_{w_i \in U_r^w} \mu(n_{w_i^{-1} w w_i}^{-1} n_{w_i}^{-1} n_w n_{w_i})\ sgn(w, w_i, \mu) \\
\cdot \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\mathrm{sgn}(\sigma)) \sum_J (-1)^{|J|} m(w, \eta_r, \boldsymbol{v}, \mu).
\end{aligned} \tag{5.53}$$

Proof. We obtain the following equations;

$$\begin{aligned}
\langle \psi, \chi_\lambda \downarrow_N \rangle &= \frac{1}{|W|}\sum_{w}\int_T \psi(n_w t)\overline{\chi_\lambda^w(t)}\, dt \\
&= \frac{1}{|W|}\sum_{w}\int_T \sum_{r=1}^{l} \xi(\eta_r^{-1} w \eta_r) \sum_{w_i \in U_r^w} \mu(n^{-1}_{w_i^{-1} w w_i} n^{-1}_{w_i} n_w n_{w_i}) \\
&\quad \cdot \mu(n^{-1}_{w_i} t n_{w_i}) \cdot \overline{\chi_\lambda^w(t)}\, dt \\
&= \frac{1}{|W|}\sum_{w}\sum_{r=1}^{l} \xi(\eta_r^{-1} w \eta_r) \sum_{w_i \in U_r^w} \mu(n^{-1}_{w_i^{-1} w w_i} n^{-1}_{w_i} n_w n_{w_i}) \\
&\quad \cdot \int_T \mu(n^{-1}_{w_i} t n_{w_i}) \cdot \overline{\chi_\lambda^w(t)}\, dt \\
&= \frac{1}{|W|}\sum_{w}\sum_{r=1}^{l} \xi(\eta_r^{-1} w \eta_r) \sum_{w_i \in U_r^w} \mu(n^{-1}_{w_i^{-1} w w_i} n^{-1}_{w_i} n_w n_{w_i}) \\
&\quad \cdot \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\mathrm{sgn}(\sigma)) \sum_{J} (-1)^{|J|}\, sgn(w, w_i, \mu) m(w, \eta_r, \mathbf{v}, \mu) \\
&= \frac{1}{W}\sum_{w}\sum_{r=1}^{l} \xi(\eta_r^{-1} w \eta_r) \\
&\quad \cdot \sum_{w_i \in U_r^w} \mu(n^{-1}_{w_i^{-1} w w_i} n^{-1}_{w_i} n_w n_{w_i})\, sgn(w, w_i, \mu) \\
&\quad \cdot \sum_{\sigma \in \mathfrak{S}_{\lambda_1}} (\mathrm{sgn}(\sigma)) \sum_{J} (-1)^{|J|} m(w, \eta_r, \mathbf{v}, \mu),
\end{aligned}$$

by which (5.53) holds. □

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