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Reliability of multi-state consecutive *k*-out-of-*n* series and parallel systems

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Abstract. In this paper, we focus on multi-state complex systems and their reliability. First, we provide a formula which computes the reliability of consecutive k-out-of-n: G systems. Then, we extend the used method to obtain the reliability of consecutive k-out-of-L_n: G series and consecutive k-out-of-L_n: G parallel systems. In the end, we illustrate all theoretical results by numerical examples.

Résumé. Dans ce papier, nous nous intéressons aux systèmes multi-états à configurations complexes et leur fiabilité. Nous commençons par établir une formule qui permet de calculer la fiabilité des systèmes k-consécutifs-sur-n : G. Ensuite, nous étendons la méthode utilisée pour obtenir la fiabilité des systèmes k-consécutifs-sur-L_n: G série et k-consécutifs-sur-L_n: G parallèle. Enfin, nous illustrons tous les résultats théoriques par des exemples numériques.

Key words: Reliability; Multi-state system; Multi-state complex system; Consecutive k-out-of-n: G system; Consecutive k-out-of-L_n: G series system; Consecutive k-out-of-L_n: G parallel system.
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1. Introduction

A consecutive k-out-of-n: G system is a system compound of n linearly components. This system works if and only if at least k consecutive components are working. The consecutive k-out-of-n: G systems have attracted the attention of many researchers because of their wide applications: telecommunication, pipelines, distribution of water...etc. From their

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appearance in Kontoleon (1980), the reliability and several opened problems related to the binary consecutive k-out-of-n: G systems had been resolved El-Neweihi *et al.* (1978),Ross (1979), Chiang and Niu (1981), Derman *et al.* (1982), Wu and Chen (1994), Shanthikumar (1982), Kuo *et al.* (1990). But seen that the multi-state systems are more flexible and reflect many real situations, a generalization of the binary consecutive k-out-of-n to the multi-state consecutive k-out-of-n has been done in Shanthikumar (1982); Hwang and Yao (1989). In Zuo *et al.* (2003); Kossov and Preuss (1995); Milinowski and Preuss (1995), the definition of a binary consecutive k-out-of-n system was extended to the multi-state context where the system is binary but their components are multi-state.

Recently, Huang *et al.* (2003) proposed more general definitions for the multi-state consecutive k-out-of-n: F, G systems where the number of functioning components depends on the state of the system. They established a recursive algorithm to evaluate the state distribution of decreasing consecutive k-out-of-n: G systems and another algorithm used to bound the state distribution of increasing consecutive k-out-of-n: F systems. Zuo *et al.* (2003) presented a recursive formula used to evaluate the state distribution of consecutive k-out-of-n: G systems when $M = 3$, and also an algorithm when $M \geq 4$. Yamamoto *et al.* (2006) proposed a recursive formula for the reliability of a consecutive k-out-of-n: G system without conditions on k or M.

Belaloui and Ksir (2007) proposed a non-recursive formula to evaluate directly the reliability of a consecutive k-out-of-n: G system without any condition on k or M.
In the literature, there are other approaches to compute the reliability of multi-state systems among them the U.M.G.F method (Universal Generating function Method) Ushakov (1986, 1997); Levitin (2004). Usually, most of the studies assume independence among components, because computation of reliability characteristics of a system that consists of dependent components Levitin (2004) is difficult especially when a specific type of dependence is not known.

In this work, we provide an explicit formula for evaluating the reliability of consecutive k-out-of-n: G systems. Then, we extend the used method in order to obtain the reliability of two important complex systems which are consecutive k-out-of-L_n: G series and consecutive k-out-of-L_n: G parallel systems.

The following work will be organized as follows: in section 2, we present some essential notions and definitions used in the whole paper, in section 3, we give a formula which evaluates the reliability of a consecutive k-out-of-n: G system, extensions are done to establish explicit formulas for the reliability of consecutive k-out-of-L_n: G series and consecutive k-out-of-L_n: G parallel systems. To conclude, we treat numerical examples in section 4 to show the application of the obtained results.

2. Essential Notions and Definitions

Before stating the main results, let us introduce useful notations, assumptions and definitions.

Notation.

N the number of blocks in the system.
k the minimum number of consecutive components required for the system to work.
L_n the number of components in block n, $n \in \{1, 2, \ldots, N\}$.
$M + 1$ the number of states of components, blocks and the system.
M: perfect functioning, 0: complete failure.
$S = \{0, 1, \ldots, M\}$.
X_i state of component i, $X_i \in S$.
$X_n : (X_1, X_2, \ldots, X_{l_n})$ vector of states of components in block n.
$p_i\,(h) = P\,\{X_i = h\}$
Φ_c structure function of a consecutive k-out-of-n: G system, $\Phi_c \in S$.
Φ_n structure function of block n, $\Phi_n \in S$.
Φ_{cs} structure function of a consecutive k-out-of-L_n: G series system, $\Phi_{cs} \in S$.
Φ_{cp} structure function of a consecutive k-out-of-L_n: G parallel system, $\Phi_{cp} \in S$.

Assumptions

The system is multi-state monotone(blocks and global system).
The X_i are mutually independent.
The blocks(res. the components)are independent.
Each component, each block and the system can assume $M + 1$ states.
The components states distributions in each block are known.

We also need the following definitions.

Definition 1. A multi-state consecutive k-out-of-n: G system is a system with n multi-state components, linearly arranged. This system works if and only if at least k consecutive components work ($k \leq n$). Its structure function is given by

$$\Phi_c\,(x) = \max_{1 \leq i \leq n-k+1} \left(\min_{i \leq j \leq i+k-1} X_j \right) \tag{1}$$

Definition 2. A multi-state consecutive k-out-of-L_n: G series system is a system with N blocks, linearly arranged, labeled B_1,B_2,...,B_N. Each block has the configuration multi-state consecutive k-out-of-L_n: G system. The system works if all the blocks work and each block works if and only if at least k consecutive components work($k \leq L_n$). Its structure function is given by

$$\Phi_{cs}\,(x) = \min_{1 \leq n \leq N} \left[\max_{(n-1)L_n+1 \leq s \leq nL_n-k+1} \left(\min_{s \leq l \leq s+k-1} X_l \right) \right] \tag{2}$$

Definition 3. A multi-state consecutive k-out-of-L_n: G parallel system is a system with N blocks, arranged in parallel, labeled B_1,B_2,...,B_N. Each block has the configuration multi-state consecutive k-out-of-L_n: G system. The system works if at least one block works and each block works if and only if at least k consecutive components work($k \le L_n$). Its structure function is given by

$$\Phi_{cp}(x) = \max_{1 \le n \le N} \left[\max_{(n-1)L_n+1 \le s \le nL_n-k+1} \left(\min_{s \le l \le s+k-1} X_l \right) \right] \tag{3}$$

Finally, we have the following convention.

Convention : In our further considerations, we assume that for $l = 1, 2, \ldots, M$

$$\begin{cases} X_i \ge l \Longrightarrow & \text{the component } i \text{ works} \\ X_i < l \Longrightarrow & \text{the component } i \text{ fails} \end{cases}$$

$$\begin{cases} \Phi_n(X) \ge l \Longrightarrow & \text{the block } n \text{ works} \\ \Phi_n(X) < l \Longrightarrow & \text{the block } n \text{ fails} \end{cases}$$

3. Main Results

According to the total probability theorem, the reliability of a system mentioned in Definition 1 is given as follows.

Theorem 1. *If $k \ge n-k$, then*

$$\mathbb{P}(\Phi_c \ge l) = \sum_{i=1}^{n-k+1} \left(\prod_{j=i}^{k+i-1} \sum_{h=l}^{M} p_j(h) \right) - \sum_{i=1}^{n-k} \left(\prod_{j=i}^{k+i} \sum_{h=l}^{M} p_j(h) \right). \tag{4}$$

Proof. By setting

$$Y_i = \min_{i \le j \le i+k-1} X_j,$$

in Definition 1, we have

$$P(\Phi_c \ge l) = P\left(\max_{1 \le i \le n-k+1} Y_i \ge l \right) = 1 - P(Y_1 < l, Y_2 < l, \ldots, Y_{n-k+1} < l). \tag{5}$$

But, we also have

$$\begin{aligned} P(Y_1 < l, Y_2 < l, \ldots, Y_{n-k+1} < l) &= P(Y_1 < l) \times P(Y_2 < l / Y_1 < l) \times \ldots \\ &\times P(Y_{n-k+1} < l / Y_1 < l, \ldots, Y_{n-k} < l). \end{aligned}$$

with

$$\begin{aligned} P(Y_1 < l) &= P\left(\min_{1 \le j \le k} X_j < l \right) \\ &= 1 - \prod_{j=1}^{k} \sum_{h=l}^{M} p_j(h). \end{aligned}$$

and

$$\begin{aligned} P(Y_2 < l/Y_1 < l) &= \frac{P(Y_2 < l, Y_1 < l)}{P(Y_1 < l)} \\ &= \frac{P((\min_{1\le j\le k} X_j < l) \cap (\min_{2\le j\le k+1} X_j < l))}{P(\min_{1\le j\le k} X_j < l)} \\ &= \frac{1 - \left(\prod_{j=1}^{k}\sum_{h=l}^{M} p_j(h) + \prod_{j=2}^{k+1}\sum_{h=l}^{M} p_j(h) - \prod_{j=1}^{k+1}\sum_{h=l}^{M} p_j(h)\right)}{1 - \prod_{j=1}^{k}\sum_{h=l}^{M} p_j(h)}. \end{aligned}$$

Then, by induction, we get the following general formula

$$P(Y_{n-k+1} < l/Y_1 < l, \ldots, Y_{n-k} < l) = \frac{1 - \left(\sum_{i=1}^{n-k+1}\left(\prod_{j=i}^{k+i-1}\sum_{h=l}^{M} p_j(h)\right) - \sum_{i=1}^{n-k}\left(\prod_{j=i}^{k+i}\sum_{h=l}^{M} p_j(h)\right)\right)}{1 - \left(\sum_{i=1}^{n-k}\left(\prod_{j=i}^{k+i-1}\sum_{h=l}^{M} p_j(h)\right) - \sum_{i=1}^{n-k-1}\left(\prod_{j=i}^{k+i}\sum_{h=l}^{M} p_j(h)\right)\right)}$$

As a result, we get

$$P(Y_1 < l, Y_2 < l, \ldots, Y_{n-k+1} < l) = 1 - \left[\sum_{i=1}^{n-k+1}\left(\prod_{j=i}^{k+i-1}\sum_{h=l}^{M} p_j(h)\right) - \sum_{i=1}^{n-k}\left(\prod_{j=i}^{k+i}\sum_{h=l}^{M} p_j(h)\right)\right]$$

By substitution in Formula (5) we obtain

$$P(\Phi_c \ge l) = \sum_{i=1}^{n-k+1}\left(\prod_{j=i}^{k+i-1}\sum_{h=l}^{M} p_j(h)\right) - \sum_{i=1}^{n-k}\left(\prod_{j=i}^{k+i}\sum_{h=l}^{M} p_j(h)\right)$$

The condition $k \ge n - k$ is necessary for our result and means that the number k of functioning components must be equal at least to the number of possible runs in the system excepting one. This condition allows having either connection or conflict between different runs of k consecutive components.

According to Theorem 1, the following special cases are clear. The first is the following.

Corollary 1. *If all the components are identical, then we have*

$$P(\Phi_c \ge l) = (n-k+1)\left(\sum_{h=l}^{M} p_j(h)\right)^{k} - (n-k)\left(\sum_{h=l}^{M} p_j(h)\right)^{k+1}. \quad (6)$$

The second is

Corollary 2. *If $k = n$, then*

$$P(\Phi_c \ge l) = \prod_{j=1}^{n}\sum_{h=l}^{M} p_j(h). \quad (7)$$

which is the reliability of a multi-state series system.

Now, taking into account Definitions 2 and 3, and by using Theorem 1, we can derive the reliability of multi-state consecutive k-out-of-L_n: G series and parallel systems respectively given in the next result

Theorem 2. *If $k \geq L_n - k$ for $n = \overline{1, N}$, then*

$$\mathbb{P}(\Phi_{cs} \geq l) = \prod_{n=1}^{N} \left[\sum_{i=1}^{L_n-k+1} \left(\prod_{j=i}^{k+i-1} \sum_{h=l}^{M} p_j(h) \right) - \sum_{i=1}^{L_n-k} \left(\prod_{j=i}^{k+i} \sum_{h=l}^{M} p_j(h) \right) \right], \tag{8}$$

and

$$\mathbb{P}(\Phi_{cp} \geq l) = 1 - \prod_{n=1}^{N} \left[1 - \left(\sum_{i=1}^{L_n-k+1} \left(\prod_{j=i}^{k+i-1} \sum_{h=l}^{M} p_j(h) \right) - \sum_{i=1}^{L_n-k} \left(\prod_{j=i}^{k+i} \sum_{h=l}^{M} p_j(h) \right) \right) \right]. \tag{9}$$

Corollary 3. *In Theorem 2, if all the blocks are identical with $L_1 = L_2 = \ldots = L_N = L$, then*

$$\mathbb{P}(\Phi_{cs} \geq l) = \left[\sum_{i=1}^{L-k+1} \left(\prod_{j=i}^{k+i-1} \sum_{h=l}^{M} p_j(h) \right) - \sum_{i=1}^{L-k} \left(\prod_{j=i}^{k+i} \sum_{h=l}^{M} p_j(h) \right) \right]^N, \tag{10}$$

and

$$\mathbb{P}(\Phi_{cp} \geq l) = 1 - \left[1 - \left(\sum_{i=1}^{L-k+1} \left(\prod_{j=i}^{k+i-1} \sum_{h=l}^{M} p_j(h) \right) - \sum_{i=1}^{L-k} \left(\prod_{j=i}^{k+i} \sum_{h=l}^{M} p_j(h) \right) \right) \right]^N. \tag{11}$$

4. Numerical examples

Example 1.

Let consider a consecutive k-out-of-L_n: G series system with 3 blocks ($N = 3$) where $L_1 = 3$, $L_2 = 4$ and $L_3 = 5$. The state distributions of components in the blocks are given in Tables 1 to 3.

$p_{1,0} = 0.2$	$p_{1,1} = 0.1$	$p_{1,2} = 0.7$
$p_{2,0} = 0.1$	$p_{2,1} = 0.3$	$p_{2,2} = 0.6$
$p_{3,0} = 0.2$	$p_{3,1} = 0.3$	$p_{3,2} = 0.5$

Table 1. State Distributions of Components in the First Block

We suppose that the first and the second blocks work if and only if at least 2 consecutive components work in each one of them, however, at least three consecutive working

$p_{1,0} = 0.2$	$p_{1,1} = 0.1$	$p_{1,2} = 0.4$	$p_{1,3} = 0.3$
$p_{2,0} = 0.1$	$p_{2,1} = 0.1$	$p_{2,2} = 0.3$	$p_{2,3} = 0.5$
$p_{3,0} = 0.2$	$p_{3,1} = 0.1$	$p_{3,2} = 0.2$	$p_{3,3} = 0.5$
$p_{4,0} = 0.1$	$p_{4,1} = 0.1$	$p_{4,2} = 0.4$	$p_{4,3} = 0.4$

Table 2. State Distributions of Components in the Second Block

$p_{1,0} = 0.2$	$p_{1,1} = 0.2$	$p_{1,2} = 0.3$	$p_{1,3} = 0.3$
$p_{2,0} = 0.3$	$p_{2,1} = 0.2$	$p_{2,2} = 0.2$	$p_{2,3} = 0.3$
$p_{3,0} = 0.1$	$p_{3,1} = 0.4$	$p_{3,2} = 0.2$	$p_{3,3} = 0.3$
$p_{4,0} = 0.3$	$p_{4,1} = 0.5$	$p_{4,2} = 0.1$	$p_{4,3} = 0.1$
$p_{5,0} = 0.2$	$p_{5,1} = 0.6$	$p_{5,2} = 0.1$	$p_{5,3} = 0.1$

Table 3. State Distributions of Components in the Third Block

components are needed for the third block to work.

Using (4), one computes the reliability of each block in the system according to the number k of working components and the results are given in Tables 4 to 6.

k \ l	2	3
0	1	1
1	0.936	0.72
2	0.51	0.42

Table 4. Reliability of the First Block

k \ l	2	3	4
0	1	1	1
1	0.936	0.7056	0.5184
2	0.84	0.5264	0.3136
3	0.425	0.145	0.03

Table 5. Reliability of the Second Block

The reliability of the system depends mainly on the number k of working components in each block. For two components in the first block, two in the second and three in the third one, we have

$$\mathbb{P}(\Phi_{cs} \geq 1) = 0,6512897664.$$

It seems not practical to give all values for the reliability of the system, but it is more significant to show them in a graphical form. For this, we use the histogram in Figure 1

k \ l	3	4	5
0	1	1	1
1	0.7434	0.42336	0.28224
2	0.18	0.034	0.006
3	0.0354	0.00333	0.00027

Table 6. Reliability of the Third Block

where the reliability of the system for $l = 1$ i.e. $\mathbb{P}(\Phi_{cs} \geq 1)$ and for $l = 2$ i.e. $\mathbb{P}(\Phi_{cs} \geq 2)$ taking all possible combinations of working components in the system is represented using the blue and the red color respectively.

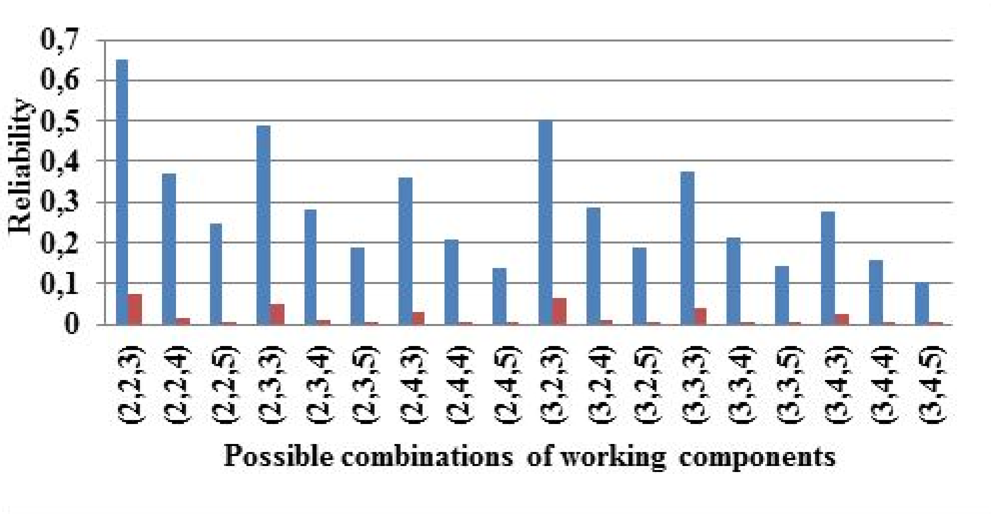

Fig. 1. Histogram of System Reliability

In this figure, we can see that the reliability of the system diminishes when the number of working components arises, because the structure of the system in this case is closer to a series configuration than to a parallel one.

Example 2.
In this example, we consider a consecutive k-out-of-L_n: G parallel system having three blocks ($N = 3$) identical to those defined in example 1. For the reliability of each block, we obtain the same results presented in Tables 4, 5 and 6.

However, for the whole system, the reliability depends-as in the previous example- on the number k of functioning components in each block. As an illustration, the reliability of the system corresponding to two functioning components in the first block, two in the second and three in the third block is

$$\mathbb{P}(\Phi_{cp} \geq 1) = 0,9989489664.$$

Contrary to the case of the example 1 where the system has only three states (0, 1 and 2), this system can hold four states (0, 1, 2 and 3). As in the first example and for the same possible combinations of working components, the reliability of the system is represented by the following histogram

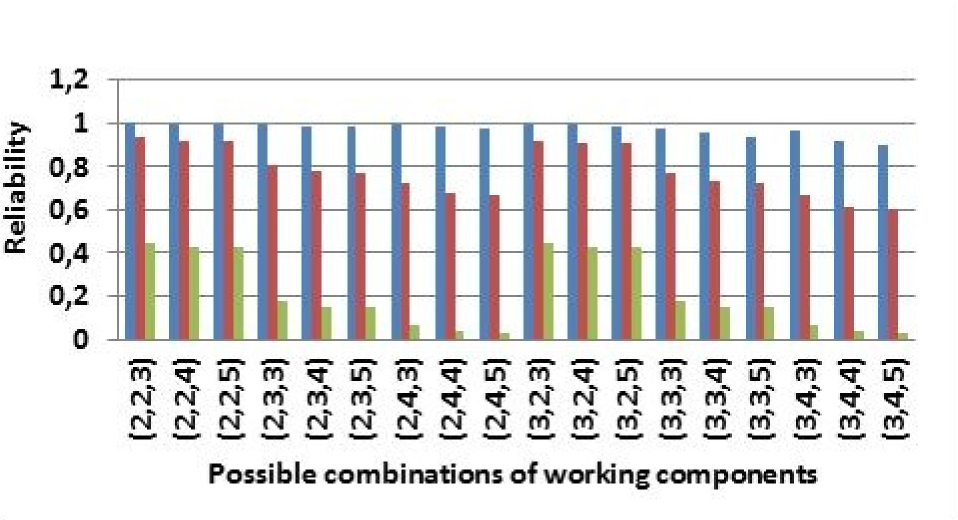

Fig. 2. Histogram of System Reliability

In Figure 2, the blue color refers to the reliability of the system for $l = 1$ i.e. $\mathbb{P}(\Phi_{cp} \geq 1)$, the red color refers to the reliability of the system for $l = 2$ i.e. $\mathbb{P}(\Phi_{cp} \geq 2)$ and the green color refers to the reliability of the system for $l = 3$ i.e. $\mathbb{P}(\Phi_{cp} \geq 3)$.

From this histogram, we can conclude that the reliability of the system for each combination is better than such in the previous example.

Conclusion.
We have provided a formula for the reliability of a multi-state consecutive k-out-of-n: G system with independent components when $k \geq n - k$ i.e. $2k \geq n$, more than, expressions for the reliability of consecutive k-out-of-L_n: G series and consecutive k-out-of-L_n: G parallel systems are derived. For the same condition ($2k \geq n$), and for arbitrarily two-state dependent components, general expressions for the reliability of consecutive k-out-of-n: G and consecutive k-out-of-n : F systems are devoted in Eryilmaz (2009). As it is mentioned in Eryilmaz (2009), when $2k < n$ and for the multi-state case, it is so difficult to derive a common expression because of the non-static behaviors of the obtained formulas when the number k of functioning components changes. Similar formulas for other multi-state complex systems will be the matter of future paper.

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