Afrika Statistika

Afrika Statistika
Vol. 13 (3), 2018, pages 1795 –1822.
DOI: http://dx.doi.org/10.16929/as/1795.134

ISSN 2316-090X

A Central limit Theorem of dependent sums of standard exponential functionals motivated by extreme value theory

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Received: Juin 1, 2018; Accepted: October 29, 2018

Abstract. Consider the following demimartingale

$$\sum_{j=1}^{k-1} f(j)(\exp(-\gamma \sum_{h=j+1}^{k-1} E_h/h) - \exp(-\gamma \sum_{h=j}^{k-1} E_h/h)),$$

where $E_1, E_2, ...$ are independent standard exponential random variables, $\gamma > 0$, k is a positive integer and $f(j)$ is an increasing function of the integer $j \geq 1$. We find general conditions under which the central limit theorem (CLT) holds and next apply the results to find the asymptotic behavior of the functional Hill within the Extreme Value Theory (EVT) field. This results show a new trend for the central limit theorem issue for non-stationary sequences of associated variables.

Key words: Extreme value theory; Associated random variables; demimartingales; asymptotic laws; functional Hill processes; extreme value theory; statistical tests.
AMS 2010 Mathematics Subject Classification Objects : Primary 62E20; 62F12; 60F05. Secondary 60B10; 60F17.

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Résumé. Consider the following demimartingale

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1. Introduction

Let $n \geq 1$, $1 < k(n) < n$, $\gamma > 0$, let $E_{1,n}, E_{2,n}, ..., E_{n,n}$ be an array of independent standard exponential random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and finally let $f(j)$ be an increasing function of $j \in \mathbb{N}\setminus\{0\}$. The following stochastic process

$$W_{k(n),n}(f) = \sum_{j=1}^{k(n)-1} f(j)(\exp(F_{j+1,n}) - \exp(F_{j,n})), \tag{1}$$

where

$$F_{j,n} = -\gamma \sum_{h=j}^{k(n)-1} E_{h,n}/h, \ 1 \leq j < k, \ F_{k(n),n} = 0,$$

plays an important role in extreme value theory where it guides the asymptotic expansion of the so-called functional Hill process when the underlying distribution is in the Weibull extremal domain of attraction. For example for $f(j) = j^\tau$, the asymptotic behavior of the function Hill process remained unsolved for $0 < \tau \leq 1/2$ and this behavior was found by Dème *et al.* (2012) and Dème *et al.* (2012), for the Frechet and Gumbel extremal domain. But as for the Weibull case, the methods to be used radically differ. While sums of independent real random vectors theory is used for the Frechet and Gumbel domains, dependent variables methods should be used for the Weibul case. Indeed, it will be shown later that the behavior of the functional Hill process, in the Weibull domain case, is guided by (1). It happens that $W_{k(n),n}(f)$, when centered at its expectation, is a demi-martingale and is closely related to the theory of associated random variables. Our guess is then that a correct handling of such processes should use methods of this modern theory of dependent random variables.

We then may begin by introducing to the associated random variables concept which goes back to Lehman (1966) in the bivariate case. Notice that we will lessen the notation by putting $k(n) = k$ in the sequel.

The concept of association for random variables generalizes that of independence and seems to model a great variety of stochastic models. *We point out now that we will equally say that a sequence of random variables is associated or that the elements of the sequence are associated.* As immediate examples, Gaussian random vectors with nonnegatively correlated components (Pitt (1982)) are associated. As well the order statistics associated with a finite set of independent are associated and a homogenous Markov chain is also associated (Daley (1968)). Such a property also arises in Physics, and is quoted under the name of FKG property (Fortuin *et al.* (1971)), in percolation theory and even in Finance (see Jiazhu (2002)). The final definition is given by Esary *et al.* (1967) in

Definition 1. A finite sequence of rv's $(X_1, ..., X_n)$ are associated when for any couple of real and coordinate-wise non-decreasing functions h et g defined on $\mathbb{R}^n$, we have

$$Cov(h(X_1, ..., X_n), \quad g(X_1, ..., X_n)) \geq 0 \tag{2}$$

An infinite sequence of rv's are associated whenever all its finite subsequences are associated.

We have a few number of interesting properties to be found in (Rao (2012)) : **(P1)** A sequence of independent rv's is associated. **(P2)** Partial sums of associated rv's are associated. **(P3)** Non-decreasing functions of associated variables are associated. **(P4)** Let the sequence $Z_1, Z_2, ..., Z_n$ be associated and let $(a_i)_{1 \leq i \leq n}$ be positive numbers and $(b_i)_{1 \leq i \leq n}$ real numbers. Then the *rv*'s $a_i(Z_i - b_i)$ are associated.

Demimartingales are set from associated centered variables exactly as martingales are derived from partial sums of centered independent random variables. We have

Definition 2. A sequence of rv's $\{S_n, n \geq 1\}$ in $L^1(\Omega, \mathcal{A}, \mathbb{P})$ is a demimartingale when for any $j \geq 1$,for any coordinatewise nondecreasing function g defined on $\mathbb{R}^j$, we have

$$\mathbb{E}\big((S_{j+1} - S_j)\ g(S_1, ..., S_j)\big) \geq 0, \quad j \geq 1. \tag{3}$$

Two particular cases should be highlighted. First any martingale is a demimartingale. Secondly, partial sums $S_0 = 0$, $S_n = X_1 + ... + X_n$, $n \geq 1$, of associated and centered random variables $X_1, X_2, ...$ form a demimartingale for, in this case, (3) becomes :

$$\mathbb{E}\left\{(S_{j+1} - S_j)\ g(S_1, ..., S_j)\right\} = \mathbb{E}\left\{X_{j+1}\ g(S_1, ..., S_j)\right\} = Cov\left\{X_{j+1}, g(S_1, ..., S_j)\right\},$$

since $\mathbb{E}X_{j+1} = 0$. Since $(x_1, ..., x_{j+1}) \longmapsto x_{j+1}$ and $(x_1, ..., x_{j+1}) \longmapsto g(x_1, ..., x_j)$ are coordinate-wise nondecreasing functions and since the $X_1, X_2, ..$ are associated, we get

$$\mathbb{E}\left\{(S_{j+1} - S_j)\ g(S_1, ..., S_j)\right\} = Cov\left\{X_{j+1}\ g(S_1, ..., S_j)\right\} \geq 0.$$

We may easily show that (1) forms a demi-martingale. Indeed we have

$$W_{k,n}^* = E(W_{k,n}) - W_{k,n} = \sum_{j=1}^{j=k-1} \overline{f}(j)(S_j^* - s_j^*),$$

where $\overline{f}(j) = (f(j) - f(j-1))$, $S_j^* = \exp(F_{j,n})$ and $s_j^* = \mathbb{E}S_j^*$. Hence it is natural to try to apply demi-martingale methods. In this spirit, Fall *et al.* (2012) used Theorem 2.4.2 of (Rao (2012)) to show that weakly converges *a.s.* to a finite random variable W with finite expectation

$$\sum_{j=1}^{+\infty}(f(j) - f(j-1))j^{-1/2} < \infty. \tag{K0}$$

It is easy to check that for the class $f(j) = j^\tau$, $(K0)$ holds for $\tau < 1/2$. In this latter paper, the law of W has been described and used for statistical tests purposes.

It would also be natural to see how to apply the available central limit theorems for associated random variables. Most of them are established for stationary sequences (see the following citations in Rao (2012) : Newman (1980, 1984), p.14; Wood (1983), p.15; Dewan and Prakasa Rao (1997b), p.16). A few number of central limits theorems proposed for non stationary case. We may cite Cox and Grimmett (1984), Oliveira (2012), (see also Rao (2012) in page 15). The one in Cox and Grimmett (1984), seems to be the most general one. Unfortunately, this requires that for some constant $c_1 > 0$, for large values of n

$$\mathbb{V}ar\left(\overline{f}(j)(S_j^* - s_j^*)\right) \geq c_1 > 0. \tag{C1}$$

But we have, by (20), for large values of ,

$$\mathbb{V}ar(\overline{f}(j)(S_j^* - s_j^*)) \leq \gamma^2 \overline{f}(j)^2/j.$$

In particular, for $f(j) = j$, it is known since Diop and Lo (2009) that $W_{k,n}^*$ is asymptotically central and yet $(C1)$ does not hold. This highlights once a again that the general central theorem limit is yet to be done.

The present study is then an important case of asymptotic normality of demi-martingales or sums of associated centered random variables that are not stationary.

The rest of the paper is organized as follows : The main result is stated in Section 2 where is it described and commented. An application in Extreme Value Theory is given. The full and detailed proof is given in Section 3 while all technical computations are postponed in Section 4 as an appendix. The paper is concluded in Section 5.

2. Statement of the Main Result, Comments and Applications

To begin with, we define the class of functions for which our results. Define for $1 \leq j \leq k-1$, $B(1,j,k) = \exp(-\gamma \sum_{h=j}^{k-1} 1/h)$, $1 \leq L \leq j$,

$$f^*(L,j) = \sum_{h=L}^{j} \overline{f}(h) B(1,h,k) \tag{4}$$

and next for $L \geq 1$

$$\sigma_n^2(L,f) = \sum_{j=L}^{k-1} f^*(j)^2 j^{-2}.$$

We shall require the following conditions

$$\forall (L \geq 1),\ \sigma_n^2(L,f) \to \infty, \tag{K1}$$

$$\forall (L \geq 1),\ B(n,f) = \max\{f^*(j)j^{-1}, 1 \leq j \leq k\}/\sigma_n(L,f) \to 0 \tag{K2}$$

$$\forall (L \geq 1) \lim_{n \to +\infty} \sigma_n(L,f)/\sigma_n(1,f) = 1 \tag{K3}$$

$$\forall (L_1 \geq 1), (\forall L_2 \geq 1), \lim_{n \to +\infty} \sigma_n^{-1}(L_1,f) \sum_{j=L_2}^{k-1} \overline{f}(j) j^{-1} B(1,j,k) = 0 \tag{K4}$$

For short, we will write $\sigma_n(1,f) = \sigma_n$.We are ready to state our result.

We are going to state the main theorem. Next, we will make a number of comments and applications.

2.1. A central limit theorem

Theorem 1. *Let (K1), (K2), (K3) and (K4) hold, then*

$$\sigma_n^{-1}(E(W_{k,n}) - W_{k,n}) \to_d \mathcal{N}(0,1).$$

Let us see what happens for the :

2.2. The Diop-Lo class

For this class, that is $f(j) = j^{\tau}$, $\tau > 0$, we prove that $(K1) - (K4)$ hold for $\tau > 1/2$ in the Appendix II. But for $\tau = 1/2$, we get non asymptotic normality as in Fall *et al.* (2012) with $\tau < 1/2$. Putting this with the results in Fall *et al.* (2012), we conclude that we have asymptotic normality for $\tau > 1/2$ and not for $\tau \leq 1/2$. We have the interesting remark that the boundary case $\tau = 1/2$ joins the normality case for the Gumbel and Frechet case and the non-normality case for the Weibull case.

2.3. Application to Extreme value Theory

2.3.1. Asymptotic results in the Weibull case

The following stochastic process

$$T_n(f) = \sum_{j=1}^{j=k(n)} f(j) \left(\log X_{n-j+1,n} - \log X_{n-j,n}\right) \tag{5}$$

was introduced by Dème *et al.* (2012) as a generalization of the Diop and Lo (1994) Diop and Lo (2009) continuous generalization of the Hill statistic for $f(j) = j^{\tau}, \tau > 1$. The Hill statistic corresponds to $\tau = 1$ and is denoted here by $T_n(1)$. This latter plays a key role in Univariate Extreme Value Theorem (UEVT).

This theory has its foundations in finding the asymptotic law of the maximum observation $X_{n,n} = \max(X_1, ..., X_n)$. It is said that the underlying distribution function F of the observations is attracted to some *df* H if for some sequences $(a_n > 0)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, we have for continuity point x of H,

$$\lim_{n\to\infty} P(\frac{X_{n,n} - b_n}{a_n} \leq x) = \lim_{n\to\infty} F^n(a_n x + b_n) = H(x).$$

It is known that, when it is non-degenerated, H can be parametrized as $G_\gamma(x) = \exp(-(1+\gamma x)^{-1/\gamma}), 1 + \gamma x > 0,\ \gamma \in \mathbb{R}$ named as the Generalized Extreme Value (GEV) distribution. It is said that F is in the domain of attraction of G_γ, hereby denoted as : $F \in D(G_\gamma)$. The reader is referred to *de* Haan and Feireira (2006), Resnick (1987), Galambos (1985), Beirlant *et al.* (2004) and Lo et al. (2018) for a modern account of UEVT.

Although the parameter γ of the GDP is continuous, the three cases $(\gamma < 0)$, $\gamma = 0$ and $\gamma > 0$, respectively named as Weibull, Gumbel and Frechet cases, may behave radically differently. But in all the cases, the Hill statistic is used to estimate what is called the extremal index in the following sense : For $\gamma \geq 0$, $k^{-1}T_n(1) \to \gamma$ as $n \to +\infty$ and $k/n \to 0$; for $\gamma < 0$, then the upper end-point of $G(x) = F(e^x)$ defined by $y_0 = \log \sup\{x \in R, F(x) < 1\}$, is finite and $k^{-1}T_n(1) \,/ (x_0 - G^{-1}(1-k/n)) \to (1-\gamma)^{-1}$ $n \to +\infty$ and $k/n \to 0$ and G^{-1} stands for the generalized inverse function of G.

The Diop and Lo generalization of Hill estimator

$$D_n(\tau) = \sum_{j=1}^{j=k(n)} j^{\tau} \left(\log X_{n-j+1,n} - \log X_{n-j,n}\right), \tau > 0,$$

has been studied in (Diop and Lo (2009)) where its asymptotic normality was proved for any $\tau > 1/2$. Recently, the functional form $T_n(f)$, which generalizes $D_n(\tau) = T_n(f_\tau) = D_n(\tau)$, has been extensively studied in the Frechet and Gumbel cases by Dème *et al.* (2012), who proved this : $T_n(f)$ has a Gaussian limiting process when $A(2, f) = \sum_{j=1}^{+\infty} f(j)^2/j^2 = +\infty$ and

$$B_n(f) =: \max\{f(j)/j, 1 \leq j \leq k\}/(\sum_{j=1}^{k} f(j)^2/j^2)^{1/2} \rightarrow 0,$$

as $n \rightarrow +\infty$. It has a non Gaussian limiting process when

$$A(2, f) < +\infty.$$

When particularized for f_τ, we get that asymptotic normality holds for $\tau \geq 1/2$ and not for $0 < \tau < 1/2$. Their results are based on sums of independent random variables, and then on Kolmogorov type theorems (see Loève (1977)). When put together, for the class of functions f_τ, we remark that the behavior of $T_n(f_\tau)$ is known for any γ in the whole extremal domain except for the Weibull domain and for $0 < \tau < 1/2$, that if for small parameters $\tau's$. In a recent paper, Fall *et al.* (2012), found thus behavior, that is for $0 < \tau < 1/2$.

Now, let us show how to apply our results in extreme value theory. In this paper, let consider the very simple case of

$$x_0 - G^{-1}(1-u) = u^{1/\gamma}, 0 \leq u \leq 1.$$

Here, we use the index $-\gamma < 0$ instead of $\gamma < 0$. We remark that $G(x) = F(e^x) \in D(G_{-1/\gamma})$ if and only if $F \in D(G_{-1/\gamma})$. We use the classical representation of the $Y_j = \log X_j$ associated with the distribution function $G(x) = F(e^x)$ through a sequence of independent standard uniform random variables U_1, U_2, ... , that is

$$\{Y_j, j \geq 1\} =_d \{G^{-1}(1-U_j), j \geq 1\}$$

and then

$$\{\{Y_{n-j+1,,n}, 1 \leq j \leq n\}, n \geq 1\} =_d \left\{\{G^{-1}(1-U_{j,n}), 1 \leq j \leq n\}, n \geq 1\right\}.$$

This gives

$$\begin{aligned} T_n(f)/(y_0 - Y_{n-k+1,n}) &= \sum_{j=1}^{j=k} f(j)\left(\log X_{n-j+1,n} - \log X_{n-j,n}\right) \\ &= \sum_{j=1}^{j=k} f(j) \frac{((y_0 - \log X_{n-j,n}) - (y_0 - \log X_{n-j+1,n}))}{(y_0 - Y_{n-k+1,n})} \\ &=_d \sum_{j=1}^{j=k} f(j) U_{k,n}((U_{j+1,n}/U_{k,n})^\gamma - (U_{j,n}/U_{k,n})^\gamma). \end{aligned}$$

We have for $1 \leq j \leq k-1$,

$$\begin{aligned}(U_{j,n}/U_{k,n})^{\gamma} &= \prod_{h=j}^{k-1} (U_{h,n}/U_{h+1,n})^{\gamma} \\ &= \exp\left(-\gamma \sum_{h=j}^{k-1} \frac{1}{h} \log(U_{h+1,n}/U_{h,n})^{h}\right) \\ &\equiv \exp\left(-\gamma \sum_{h=j}^{k-1} E_h^{(n)}/h\right).\end{aligned}$$

By the Malmquist's representation (see (Shorack and Shorack (1986)), p. 336 or Lo *et al.* (2016), Proposition 32, page 126), the random variables $E_h^{(n)}, 1 \leq h \leq n$, are independent and standard exponential random variables. We arrive at

$$\begin{aligned}&T_n(f)/(y_0 - Y_{n-k+1,n}) \\ &= \sum_{j=1}^{k} f(j) \left\{ \exp(-\gamma \sum_{h=j+2}^{k-1} E_h^{(n)}/h) - \exp\left(-\gamma \sum_{h=j}^{k-1} E_h^{(n)}/h\right) \right\}\end{aligned}$$

and we get

$$D_{k,n}(f) - (T_n(f)/(y_0 - Y_{n-k+1,n})) = W_{k,n}^{*}(f),$$

where $D_{k,n}(f) = f(k-1) - \sum_{j=1}^{k-1} \overline{f}(j) s_{j,k}^{*}$. At this step, we apply Theorem 1 to get the final result.

Proposition 1. *Let $X_1, X_2, ...$ be positive random variables with a finite endpoint x_0 such that $\log F^{-1}(1-u) = \log x_0 + u^{\gamma}, 0 \leq u \leq 1$ and $\gamma > 0$. Let f be an increasing function over the positive integers satisfying $(K1)-(K4)$ and put*

$$D_{k,n}(f) = f(k-1) - \sum_{j=1}^{k-1} \overline{f}(j) s_{j,k}^{*}.$$

Then

$$T_n^{*}(f) = D_{k,n}(f) - \Big(T_n(f)/(\log x_0 - \log X_{n-k+1,n})\Big)$$

converges in distribution to a $\mathcal{N}(0,1)$ random variable. As well, for $\tau > 1/2$, $T_n^{*}(f_\tau)$ converges in distribution to a $\mathcal{N}(0,1)$ random variable.

We indeed remark that for this simple case in the Weibull case, the law of the functional Hill process is determined for $1/2 < \tau$. For the general case, we have the following Karamata representation when F is in the Weibull case of parameter $\gamma > 0 : x_0(F) < \infty$ and

$$\log x_0 - F^{-1}(1-u) = (1+p(u))u^{\gamma} \exp(\int_u^1 b(t)t^{-1}dt), \tag{6}$$

where $(p(u), b(u)) \to (0,0)$ as $u \to 0$. In a coming paper, we will determine general conditions on f, b and p under which $T_n^*(f)$ behaves as $W_{k,n}^*$ as in the present case.

3. Proof of The Main Result

Let $\varepsilon > 0$ fixed and let $A(\varepsilon) = [-\varepsilon - 1/2, \varepsilon - 1/2] \equiv [a_1(\varepsilon), a_2(\varepsilon)]$. By Fact 1 in Section 4, there exists $u_0 > 0$ so that

$$u \leq u_0, \log(1+u) = u + \theta u^2 \text{ with } \theta \in A(\varepsilon). \tag{7}$$

Now we fix L so that

$$2L \geq \gamma/u_0;\ 1 + \gamma^2 L^{-1} \exp(2\gamma^2 L^{-1}) \leq \varepsilon. \tag{8}$$

By (22)-(25), we may fix such that for $j \geq L$:

$$\sum_{h=j}^{\infty} h^{-2} \leq 2j^{-1}; \sum_{h=j}^{\infty} h^{-4} \leq 2j^{-3}/3. \tag{9}$$

Now, we expand

$$\begin{aligned} T_n &= \sigma_n^{-1} \sum_{j=1}^{k-1} \overline{f}(j)(S_{j,k,n}^* - s_{j,k,n}^*). \\ &= \sigma_n^{-1} \sum_{j=1}^{L-1} \overline{f}(j)(S_{j,k,n}^* - s_{j,k,n}^*) + \sigma_n^{-1} \sum_{j=L}^{k-1} \overline{f}(j)(S_{j,k,n}^* - s_{j,k,n}^*) \equiv T_n(1) + T_n(2). \end{aligned}$$

We have

$$\psi_{T_n}(v) = \mathbb{E} \exp(ivT_n) = \mathbb{E} \{\exp(ivT_n(1)) \exp(ivT_n(2))\}$$

and next

$$\psi_{T_n}(v) - \mathbb{E} \exp(ivT_n(2)) = \mathbb{E} \{\exp(ivT_n(1)) - 1\} \exp(ivT_n(2)).$$

But

$$|\psi_{T_n}(v) - \mathbb{E} \exp(ivT_n(2))| \leq \mathbb{E} |\exp(ivT_n(1)) - 1|.$$

Since L is fixed and $\sigma_n \to +\infty$ as $n \to \infty$, $\exp(ivT_n(1)) - 1$, which is bounded by the constant 2, tends to zero in probability and by the dominated convergence Theorem, $E\left|\exp(ivT_n(1)) - 1\right| \to 0$ and

$$|\psi_{T_n}(v) - \mathbb{E}\exp(ivT_n(2))| \to 0. \tag{10}$$

Now we turn to

$$\psi_{T_n}(v,2) = \mathbb{E}\exp(ivT_n(2))$$

and transform it as

$$\psi_{T_n}(v,2) = \exp\left(-iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)s^*_{j,k,n}\right) \tag{11}$$

$$\times\mathbb{E}\exp(iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)S^*_{j,k,n}) \equiv A_n(1)\times A_n(2)$$

Let us handle the quantities involved in (11). Now by (7) and (8), we have

$$s^*_{j,k,n} = \exp(-\sum_{h=j}^{k-1}\log(1+\gamma/h)) = \exp(-\gamma\sum_{h=j}^{k-1}1/h)\exp(-\gamma^2\sum_{h=j}^{k-1}\theta_h(\varepsilon)h^{-2}),$$

where each θ_h is in $A(\varepsilon)$. This yields, by recalling that

$$B(1,j,k) = \exp(-\gamma\sum_{h=j}^{k-1}1/h)$$

and by denoting

$$B(2,j,k) = \exp(-\gamma^2\sum_{h=j}^{k-1}\theta_h(\varepsilon)h^{-2})$$

$$A_n(1) = \exp(-iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)B(2,j,k)).$$

But

$$\begin{aligned} A_n(1) &= \exp(-iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)) \\ &\times \exp(-iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k))(B(2,j,k)-1)). \end{aligned}$$

Now by (25),

$$\sum_{h=j}^{\infty} h^{-2} \leq 2j^{-1}$$

and then

$$\left| -\gamma^2 \sum_{h=j}^{k-1} \theta_h(\varepsilon) h^{-2} \right| \leq \gamma^2 \sum_{h=j}^{\infty} h^{-2} \leq 2\gamma^2 j^{-1}.$$

* Further by Lemma 1 in Section 4, we have for

$$x = -\gamma^2 \sum_{h=j}^{k-1} \theta_h(\varepsilon) h^{-2},$$

$$|B(2,j,k) - 1| \leq |x| \left(1 + |x| / 2 \exp(|x|)\right).$$

Hence, for $j \geq L$, we have

$$|B(2,j,k) - 1| \leq 2\gamma^2 j^{-1} (1 + \gamma^2 L^{-1} \exp(2\gamma^2 L^{-1})).$$

This and (8) together yield

$$\left| \sigma_n^{-1} \sum_{j=L}^{k-1} \overline{f}(j) B(1,j,k))(B(2,j,k) - 1)) \right| \leq 2\gamma^2 (1+\varepsilon) \times \sigma_n^{-1} \sum_{j=L}^{k-1} \overline{f}(j) B(1,j,k)) j^{-1}.$$

*The right-member of this inequality tends to zero by $(K4)$ and then

$$A_n(1) = \exp(-iv\sigma_n^{-1} \sum_{j=L}^{k-1} \overline{f}(j) B(1,j,k))(1 + o(1)). \tag{12}$$

We focus now on $A_n(2)$. We remind that

$$S_{j,k,n}^{*} = \exp(-\gamma \sum_{h=j}^{k-1} E_h/h) = \left(\exp(-\gamma \sum_{h=j}^{k-1} 1/h) \times \exp(-\gamma \sum_{h=j}^{k-1} (E_h - 1)/h) \right)$$

$$= \exp\left(B(1,j,k) \times \exp(-\gamma \sum_{h=j}^{k-1} (E_h - 1)/h) \right).$$

Put $R(j,k) = -\gamma \sum_{h=j}^{k-1} (E_h - 1)/h$ and $R^{*}(j,k) = \exp(-\gamma \sum_{h=j}^{k-1} (E_h - 1)/h) - (1 + R(j,k))$. Remark that by Lemma 1,

$$|R^*(j,k)| \leq \frac{1}{2}R(j,k)^2 \exp(R(j,k)).$$

Hence

$$\begin{aligned}
A_n(2) &= \mathbb{E}\exp(iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)(1+R(j,k)+R^*(j,k))B(1,j,k) \\
&= \exp(iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)) \times \mathbb{E}\exp(iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)R(j,k)) \\
&\times \exp(iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)R^*(j,k)B(1,j,k)) \\
&\equiv A_n(2,1,v) \times A_n(2,2,v) \times A_n(2,3,v). \qquad (13)
\end{aligned}$$

We have to show that $A_n(2,3,v)$ tends to 1. We use Cauchy-Scharwz inequality and get

$$\mathbb{E}R^*(j,k) \leq \frac{1}{2}\mathbb{E}R(j,k)^2\exp(R(j,k)) \leq \frac{1}{2}(\mathbb{E}R(j,k)^4)^{1/2} \times (\mathbb{E}\exp(2R(j,k)))^{1/2}.$$

We easily get by using the moment function of standard exponential law

$$\mathbb{E}\exp(2R(j,k)) = \exp(+2\gamma\sum_{h=j}^{k-1}1/h) \times \exp(-\sum_{h=j}^{k-1}\log(1+2\gamma/j)).$$

By using (7) and (8), we get

$$\mathbb{E}\exp(2R(j,k)) = \exp(-4\gamma^2\sum_{h=j}^{k-1}\theta_h h^{-2}),$$

where for each h, $\theta_h \in A(\varepsilon)$. Since for $j \geq L$,

$$\left|\sum_{h=j}^{k-1}\theta_h h^{-2}\right| \leq \left|\sum_{h=j}^{k-1}h^{-2}\right| \leq 2j^{-1} \to 0,$$

we have

$$\mathbb{E}\exp(2R(j,k)) \to 1.$$

Next, we may get, as demonstrated in the Appendix (see (27)) that for $j \geq L$,

$$\mathbb{E}R(j,k)^4 \leq 78j^{-2}.$$

We arrive at

$$\mathbb{E}\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)R^*(j,k)B(1,j,k)) \leq (1+o(1))\sqrt{78}\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)j^{-1}B(1,j,k)),$$

with tends to zero by (K4) and then

$$A_n(2,3,v) = \mathbb{E}\exp(iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)R(j,k)) \to 1 \tag{14}$$

by the dominated convergence theorem. We now get by putting together (12), (13) and (14)

$$\psi_{T_n}(v) = (1+o(1))A_n(2,2,v). \tag{15}$$

Now

$$\begin{aligned} A_n(2,2,v) &= \mathbb{E}\exp(iv\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)R(j,k)) \\ &= \mathbb{E}\exp\left(-i\gamma v\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)\sum_{h=j}^{k-1}\frac{E_h-1}{h}\right) \\ &= \mathbb{E}\exp\left(-i\gamma v\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)\left(\sum_{h=j}^{k-1}\frac{E_h}{h}-\sum_{h=j}^{k-1}\frac{1}{h}\right)\right) \\ &= \mathbb{E}\exp\left(i\gamma v\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)\left(\sum_{h=j}^{k-1}\frac{1}{h}\right)\right) \\ &\times \mathbb{E}\exp\left(-i\gamma v\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)\sum_{h=j}^{k-1}\frac{E_h}{h}\right). \end{aligned} \tag{16}$$

Finally by letting

$$A_n(2,4,v) = \mathbb{E}\exp(-i\gamma v\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)\sum_{h=j}^{k-1}\frac{E_h}{h}),$$

we use Lemma 2 and remind the notation (4) to get

$$A_n(2,4,v) = \mathbb{E}\exp(-i\gamma v\sigma_n^{-1}\sum_{j=L}^{k-1}f^*(j)E_j/j).$$

By using the characteristic function of the exponential law, we have

$$A_n(2,4,v) = \exp(-\sum_{j=L}^{k-1}\log(1+i\gamma v\sigma_n^{-1}(L)f^*(j)/j)).$$

By $(K2)$, we are entitled to use Fact 1 to have for n large enough (say $n \geq N_0$) to ensure $\gamma v\sigma_n^{-1}(L)B_n(L,f^*) \leq u_0$, and get

$$A_n(2,4,v) = \exp(-i\gamma v\sigma_n^{-1}(L)\sum_{j=L}^{k-1} f^*(j)/j)) \times \exp(+\gamma^2 v^2\sigma_n^{-2}\sum_{j=L}^{k-1}\theta_j(\varepsilon)f^*(j)^2/j^2). \quad (17)$$

But by Lemma 2, we have the identity

$$\exp(-i\gamma v\sigma_n^{-1}\sum_{j=L}^{k-1} f^*(j)/j)) = \exp\left(-i\gamma v\sigma_n^{-1}\sum_{j=L}^{k-1}\overline{f}(j)B(1,j,k)\left(\sum_{h=j}^{k-1}\frac{1}{h}\right)\right). \quad (18)$$

Now by combined (15), (16), (17) and (18), we finally get

$$\psi_{T_n}(v) = (1+o(1))\exp(+\gamma^2v^2\sigma_n^{-2}\sum_{j=L}^{k-1}\theta_j(\varepsilon)f^*(j)^2/j^2).$$

where each $\theta_j(\varepsilon)$ is in $A(\varepsilon)$.Then for each $\varepsilon > 0, n \geq N_0$, by using the full notation of σ_n, we obtain

$$\gamma^2v^2(-\frac{1}{2}-\varepsilon)\sigma_n^{-2}(1,f)/\sigma_n^{-2}(L,f) \leq \sigma_n^{-2}\sum_{j=L}^{k-1}\theta_j(\varepsilon)f^*(j)^2/j^2)$$

$$\leq \gamma^2v^2(-\frac{1}{2}+\varepsilon)\sigma_n^{-2}(1,f)/\sigma_n^{-2}(L,f)$$

and then for each $\varepsilon > 0$ fixed,

$$\exp(\gamma^2(-\frac{1}{2}-\varepsilon)v^2) \leq \lim\inf_{n\to\infty}\psi_{T_n}(v) \leq \lim\sup_{n\to\infty}\exp(\gamma^2(-\frac{1}{2}+\varepsilon)v^2).$$

By letting $n \to \infty$ and by using

$$\psi_{T_n}(v) \to \exp(-\gamma^2v^2/2).$$

This achieves the proof of our theorem.

4. Appendix I

4.1. Some useful facts

Lemma 1. *For any* $x \in \mathbb{R}$

$$|\exp(x) - 1 - x| \leq x^2 e^{|x|}/2.$$

Proof. We have for any $x \in \mathbb{R}$,

$$\exp(x) - 1 - x = \sum_{s=2}^{\infty}\frac{x^s}{s!}.$$

Then

$$\begin{aligned}
|\exp(x) - 1 - x| &\leq \sum_{s=2}^{\infty} \frac{|x|^s}{s!} \\
&= (x^2/2)(1 + \sum_{s=3}^{\infty} \frac{2\,|x|^{s-2}}{s!}) \\
&= (x^2/2)(1 + \sum_{s=3}^{\infty} \frac{|x|^{s-2}}{(s-2)!} \times \frac{2}{s(s-1)}) \\
&\leq (x^2/2)(1 + \sum_{s=3}^{\infty} \frac{|x|^{s-2}}{(s-2)!}) \\
&= x^2 e^{|x|}/2,
\end{aligned}$$

where we used

$$\frac{1}{s!} = \frac{1}{(s-2)! \times (s-1)s}.$$

Lemma 2. *Let $r \geq 2$ be an integer and $(x_1, y_1), ..., (x_r, y_r)$ be r couples of real numbers and let b an integer such that $1 \leq s < r$. Then we have*

$$\sum_{j=s}^{r} x_j \sum_{h=j}^{r} y_h = \sum_{j=s}^{r} x_j^*(s) y_j,$$

where for $s \leq j \leq r$

$$x_j^*(s) = \sum_{h=s}^{j} x_j$$

4.2. Moment estimation

This subsection is devoted to the computations of the moments of

$$S_j^* = exp(-\gamma \sum_{h=j}^{k-1} E_h/h)$$

where the $E_h's$ are independent standard exponential random variables, and their approximations for large values of j. We begin to give a particular and useful for the expansion of the logarithm function.

Fact 1. Let $\varepsilon > 0$ be fixed for once. There exists $0 < u_0$ such that

$$\forall (u \in Z), 0 < |u| < u_0, \ \log(1+u) = u + \theta(\varepsilon, u)u^2,$$

where $\theta(\varepsilon, u) \in [-\varepsilon - 1/2, \varepsilon - 1/2] \equiv A(\varepsilon) = [a_1(\varepsilon), a_2(\varepsilon)]$. For any integer $m \geq 1$, let $J_0(m)$ such that $J_0(m) \geq \gamma m/u_0$ so that

$$j \geq J_0(m) \Longrightarrow \log(1+\gamma m/j) = u + \theta_j u^2 \text{ with } \theta_j \in A(\varepsilon), \ u = \gamma m/j.$$

In the remainder, we concentrate on the moment computations.

4.2.1. Exact values

We have for any integer $m \geq 1$

$$\mathbb{E}\left((S_j^*)^m\right) = E(\exp(-m\gamma \sum_{h=j}^{k-1} E_h/h) = \prod_{h=j}^{k-1} E(-m\gamma E_h/h) = \prod_{h=j}^{k-1} (1+m\gamma/h)^{-1}$$

$$\mathbb{E}\left((S_j^*)^m\right) = \exp\left(-\sum_{h=j}^{k-1} \log(1+m\gamma/h)\right).$$

Now for $j \geq J_0(m)$,

$$\mathbb{E}\left((S_j^*)^m\right) = \exp\left(-m\gamma \sum_{h=j}^{k-1}(1/h) - m^2\gamma^2 \sum_{h=j}^{k-1} \theta_h/h^2\right).$$

4.2.2. Approximated values for moments

We have by

$$\left| m^2\gamma^2 \sum_{h=j}^{k-1} \theta_h/h^2 \right| \leq |a_1(\varepsilon)|\, m^2\gamma(\frac{1}{j} - \frac{1}{k-1} - \frac{1}{(k-1)^2}) \leq \frac{|a_1(\varepsilon)|\, m^2\gamma}{j}.$$

For

$$\frac{|a_1(\varepsilon)|\, m^2\gamma}{J_1(\varepsilon, m)} \leq \varepsilon,$$

we have

$$j \geq J_1(\varepsilon, m) \vee J_0(m) \Longrightarrow \exp\left(-m^2\gamma^2 \sum_{h=j}^{k-1} \theta_h/h^2\right) \leq e^{\varepsilon}.$$

Next by (24),

$$\exp(-m\gamma \sum_{h=j}^{k-1}(1/h)) = \left(\frac{j}{k-1}\right)^{m\gamma} \exp\left(-m\gamma \left\{\sum_{h=j}^{k-1} \frac{1}{h} - \log((k-1)/j)\right\}\right)$$

with *

$$\exp(-1/j) \leq \exp\left(-m\gamma\left\{\sum_{h=j}^{k-1}\frac{1}{h} - \log((k-1)/j)\right\}\right) \leq \exp(-1/(k-1)).$$

We finally have for $j \geq J_1(\varepsilon, m) \vee J_0(m)$

$$\mathbb{E}\left((S_j^*)^m\right) = \left(\frac{j}{k-1}\right)^{m\gamma} B(1,m,j)B(2,m,j), \tag{19}$$

with

$$0 \leq B(1,j) = 1 + O\left(\frac{|a_1(\varepsilon)|\, m^2\gamma}{j}\right) \text{ and } B(2,j) = 1 + O(j^{-1})).$$

4.2.3. Approximated values for variances

We have for $j > J_0(2)$

$$\mathbb{E}\left((S_j^*)^2\right) = \exp\left(-2\gamma\sum_{h=j}^{k-1}(1/h) - 4^2\gamma^2\sum_{h=j}^{k-1}\theta_h(1)/h^2\right).$$

and for $j > J_0(1)$

$$\mathbb{E}\left(S_j^{*2}\right) = \left(-\gamma\sum_{h=j}^{k-1}(1/h) - \gamma^2\sum_{h=j}^{k-1}\theta_h(2)/h^2\right)^2$$

$$= \exp\left(-2\gamma\sum_{h=j}^{k-1}(1/h) - 2\gamma^2\sum_{h=j}^{k-1}\theta_h(2)/h^2\right).$$

Thus

$$Var(S_j^*) = \exp(2\gamma\sum_{h=j}^{k-1}1/h)$$

$$\times\left\{\exp(-4^2\gamma^2\sum_{h=j}^{k-1}\theta_h(1)/h^2) - \exp(-2\gamma^2\sum_{h=j}^{k-1}(2\theta_h(1)-\theta_h(2))/h^2)\right\}.$$

Since $x = 4^2\gamma^2\sum_{h=j}^{k-1}\theta_h(1)/h^2$ and $y = 2\gamma^2\sum_{h=j}^{k-1}\theta_h(2)/h^2$ are both nonnegative, we have $|e^x - e^y| \leq |x-y|$. Thus

$$0 \leq \exp(-4^2\gamma^2\sum_{h=j}^{k-1}\theta_h(1)/h^2) - \exp(-2\gamma^2\sum_{h=j}^{k-1}\theta_h(2))/h^2)$$

$$\leq 2\gamma^2\sum_{h=j}^{k-1}|2\theta_h(1)-\theta_h(2)|/h^2 \leq \frac{2\gamma^2\,|a_1(\varepsilon)|}{j},$$

by (25). Hence

$$Var(S_j^*) = \left(\frac{j}{k-1}\right)^{2\gamma} V(1,j)V(2,j) \tag{20}$$

with

$$|V(1,j)| = 1 + O(j^{-1}) \text{ and } 0 \leq V(2,j) \leq \frac{2\gamma^2 |a_1(\varepsilon)|}{j}.$$

4.2.4. Covariance approximate values

Let $\ell > 1$ and consider $\sigma_{j,j+\ell} = cov(S_{j+\ell}^*, S_j^*)$. We have

$$\mathbb{E}\left(S_j^*\right) = \exp(\sum_{h=j}^{k-1} - \log(1+\gamma/h))$$

$$= \exp(\sum_{h=j}^{j+\ell-1} - \log(1+\gamma/h)) \exp(\sum_{h=j+\ell}^{k-1} - \log(1+\gamma/h))$$

$$= \mathbb{E}\left(S_{j+\ell}^*\right) \exp(\sum_{h=j}^{j+\ell-1} - \log(1+\gamma/h)).$$

Also

$$S_j^* S_{j+\ell}^* = \exp(-\gamma \sum_{h=j}^{k-1} E_h/h) \exp\left(-\gamma \sum_{h=j+\ell}^{k-1} E_h/h\right)$$

$$= \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h - \gamma \sum_{h=j+\ell}^{k-1} E_h/h) \exp(-\gamma \sum_{h=j+\ell}^{k-1} E_h/h)$$

$$= \exp(-2\gamma \sum_{h=j+\ell}^{k-1} E_h/h) \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h) = \left(S_{j+\ell}^*\right)^2 \exp(-\gamma \sum_{h=j}^{j+\ell-1} E_h/h).$$

Hence

$$\mathbb{E}(S_j^* S_{j+\ell}^*) = E\left(S_{j+\ell}^*\right)^2 \exp(\sum_{h=j}^{j+\ell-1} - \log(1+\gamma/h)).$$

For $j \geq J_0(1) \vee J_0(2)$,

$$cov(S_j^*, S_{j+\ell}^*) = Var(S_{j+\ell}^*) \exp(\sum_{h=j}^{j+\ell-1} - \log(1+\gamma/h))$$

$$cov(S_j^*, S_{j+\ell}^*) = Var(S_{j+\ell}^*) \exp(-\gamma \sum_{h=j}^{j+\ell-1} 1/h - \gamma^2 \sum_{h=j}^{j+\ell-1} \theta_h/h^2)$$

$$= Var(S_{j+\ell}^*) \left(\frac{j}{j+\ell-1}\right)^{\gamma} (1 + O(j^{-1})). \tag{21}$$

4.3. Integral computations

Let $b \geq 1$, we get by comparing the area under the curve $x \longmapsto x^{-b}$ from j to $k-1$ and those of the rectangles based on the intervals $[h, h+1]$, $j = 1, .., k-2$, we get

$$\sum_{h=j}^{k-2} h^{-b} \leq \int_j^{k-1} x^{-b} dx \leq \sum_{h=j+1}^{k-1} h^{-b},$$

that is

$$\int_j^{k-1} x^{-b} dx + j^{-b} \leq \sum_{h=j}^{k-1} h^{-b} \leq \int_j^{k-1} x^{-b} dx + (k-1)^{-b}. \tag{22}$$

As well, by comparing the area under the curve $x \longmapsto x^b$ from j to $k-1$ and those of the rectangles based on the intervals $[h, h+1]$, $j = 1, .., k-2$, we also get

$$\int_j^{k-1} x^b dx + (k-1)^b \leq \sum_{h=j}^{k-1} h^{-b} \leq \int_j^{k-1} x^b dx + j^b. \tag{23}$$

For $b = 1$, on has

$$\frac{1}{k-1} \leq (\sum_{h=j}^{k-1} \frac{1}{h}) - \log((k-1)/j) \leq \frac{1}{j}. \tag{24}$$

For $b = 2$, on has

$$\frac{1}{j} - \frac{1}{k-1} + \frac{1}{(k-1)^2} \leq \sum_{h=j}^{k-1} h^{-2} \leq \frac{1}{j} - \frac{1}{k-1} + \frac{1}{j^2}, \tag{25}$$

that is

$$\frac{1}{(k-1)^2} \leq \sum_{h=j}^{k-1} h^{-2} - \frac{1}{j}(1 - \frac{j}{k-1}) \leq \frac{1}{j^2}.$$

As well, we have for $b > 0$,

$$\sum_{h=j}^{k-2} h^b \leq \int_j^{k-1} x^b dx \leq \sum_{h=j+1}^{k-1} h^b$$

and then

$$\frac{1}{b+1}((k-1)^{b+1} - j^{b+1}) + j^b \leq \sum_{h=j}^{k-1} h^b \leq \frac{1}{b+1}((k-1)^{b+1} - j^{b+1}) + (k-1)^b. \tag{26}$$

Hence for j fixed and $k \to \infty$, we get $\sum_{h=j}^{k-1} h^b \sim (k-1)^{b+1}/(b+1)$.

4.4. Expectation computing

We have to compute $\mathbb{E}(R^{*4}_{j,k})$. We begin to remark that for $I = \{i, i+1, ..., k-1\}^4$.

$$\left(\sum (E_j - 1)/j\right)^4 = \sum_{(i_1,\dots,i_4)\in I} \prod_{i=1}^{4} (E_{j_i} - 1)/j_i).$$

Consider the subset I_1 of I for which two components are equal and so are the two others. Let us split I into I$_1$, $I_2 = \{(i_1, ..., i_4) \in I, i_1 = ... = i_4\}$ and I_3. In I_3, one of the four component is different form the others and then because of the independence of the centered $rv's$ $(E_{j_i} - 1)/j_i)$, and we get

$$\mathbb{E} \sum_{(i_1,\dots,i_4)\in I_3} \prod_{i=1}^{4} (E_{j_i} - 1)/j_i) = 0.$$

Now we have

$$\mathbb{E} \sum_{(i_1,\dots,i_4)\in I_1} \prod_{i=1}^{4} (E_{j_i} - 1)/j_i) = 36\ \mathbb{E} \sum_{j\leq s<r\leq k-1} \{(E_s - 1)/s)\}^2 \{(E_r - 1)/r)\}^2$$

$$= 36 \sum_{j\leq s<r\leq k-1} s^{-2}r^{-2} = 18 \left\{ \left(\sum_{s=j}^{k-1} s^{-2}\right)^2 - (\sum_{s=j}^{k-1} s^{-4}) \right\}.$$

Thus since

$$\mathbb{E} \sum_{(i_1,\dots,i_4)\in I_2} \prod_{i=1}^{4} (E_{j_i} - 1)/j_i) = \mathbb{E} \sum_{s=j}^{k-1} (E_s - 1)/s)^4 = 9 \sum_{s=j}^{k-1} s^{-4},$$

we finally get by (9), for $j \geq L$,

$$\mathbb{E}(R^{*4}_{j,k}) = 9 \sum_{s=j}^{k-1} s^{-4} + 18 \left\{ \left(\sum_{s=j}^{k-1} j_s^{-2}\right)^2 - \left(\sum_{s=j}^{k-1} j_s^{-4}\right) \right\} = 18 \left(\sum_{s=j}^{k-1} j_s^{-2}\right)^2 - 9 \sum_{s=j}^{k-1} j_s^{-4}$$

$$\leq 72j^{-2} + 6j^{-3} \leq 78j^{-2}. \tag{27}$$

5. Conclusion

A coming paper will focus on the announcement concerning the extension the result of Proposition **??** for an arbitrary *cdf* F in the Weibull extremal domain represented by Formula (6). Besides, we intend to use the demi-martingale whose asymptotic law is given here to compare different central limit laws provided by different authors for associated sequences.

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6. Annexe II

This section is devoted to a detailed application of our results for the class of Diop-Lo, that is for functions $f_\tau(j) = j^\tau$, $\tau > 0$. In XXX, the class of $\tau < 1/2$ has been determined and showed to be non-gaussian. We focus on values of τ greater or equal to $1/2$. We need to show how that the results we get do not apply for $\tau \leq 1/2$.

We show here in details that the conditions $(K1)$, $(K2)$ and $(K3)$ hold for $\tau \geq 1/2$ and not for $\tau < 1/2$. We begin to remark that $\overline{f_\tau} = f_\tau(j) - f_\tau(j-1) \rightsquigarrow \tau j^{\tau-1}$ as $j \to +\infty$. Next, by (24) and (26), we have that $B(1,j,k) \rightsquigarrow \left(\frac{j}{k}\right)^\gamma$for $j \leq k$ large. We then see that for $\varepsilon > 0$, there exists $J > 1$ that for any $j \leq J$,

$$\overline{f}(j)B(1,j,k) = (1+\theta_j(\varepsilon))\, k^{-\gamma} j^{\tau+\gamma-1}$$

where $\theta_j(\varepsilon) \in [-\epsilon, \epsilon]$. Let $L > 1, k > J$.

6.1. Evaluation of $f^*(j)$

Depending on $L \leq j \leq J$ or $J \leq j \leq k-1$,we have either

$$f^*(j) = \sum_L^j \overline{f}(h)\, B(1,h,k) \leq \sum_L^J \overline{f}(h)\, B(1,h,k)$$

or

$$f^*(j) = \sum_L^J \overline{f}(h)\, B(1,h,k) + \sum_J^j \overline{f}(h)\, B(1,h,k)\,.$$

By focusing on large values of J,we then get

$$B_J(j) = \sum_J^j \overline{f}(h)\, B(1,h,k) = \tau(1+\theta(\varepsilon))\, k^{-\gamma} \sum_J^j h^{\tau+\gamma-1}$$

Since J is large enough, then by (24) – (26), we have

$$B_J(j) = \tau(1+\theta(\varepsilon))\, \frac{k^{-\gamma}}{\tau+\gamma} \left[j^{\tau+\gamma} - J^{\tau+\gamma}\right].$$

6.2. Evaluation of $\sigma_n^2(L, f_\tau)$

Recall

$$\sigma_n^2(L, f_\tau) = \sum_L^{k-1} f^*(j)^2\, j^{-2}$$

which gives

$$\sigma_n^2(L, f_\tau) = \sum_{L}^{J-1} f^*(j)^2 j^{-2} + \sum_{J}^{k-1} f^*(j)^2 j^{-2}$$

$$= \sum_{L}^{J-1} f^*(j)^2 j^{-2} + \sum_{J}^{k-1} \left[\sum_{h=L}^{J} \overline{f}(h) B(1,h,k)\right] j^{-2} + \sum_{J}^{k-1} B_{J,\gamma}^2 j^{-2}$$

$$= \sum_{L}^{J-1} f^*(j)^2 j^{-2} + \left[\sum_{h=L}^{J} \overline{f}(h) B(1,h,k)\right] \sum_{J}^{k-1} j^{-2}$$

$$+\tau(1+\theta(\varepsilon))^2 \sum_{J}^{k-1} \frac{1}{j^2(\tau+\gamma)^2}\left[j^{2\tau+2\gamma} - 2J^{\tau+\gamma} j^{\tau+\gamma} - J^{2\tau+2\gamma}\right] k. \tag{28}$$

We put

$$C(L,J,k) = \sum_{L}^{J-1} f^*(j)^2 j^{-2} + \left[\sum_{h=L}^{J} \overline{f}(h) B(1,h,k)\right] \sum_{J}^{k-1} j^{-2} \tag{29}$$

$$\to C(L,J,\infty) = C(L,J) < \infty.$$

The last term in (28) is a multiple of

$$D(J,k) = \sum_{j}^{k-1} \left[j^{2\tau+2\gamma-2} - 2J^{\tau+\gamma} j^{\tau+\gamma-2} - j^{-2} J^{2\tau+2\gamma}\right] k^{-2\gamma}.$$

and the coefficient of multiplicity is

$$\tau(1+\theta(\varepsilon))^2 (\tau+\gamma)^{-2} \to \tau(\tau+\gamma)^{-2}, \tag{30}$$

the limit being independent of L and of J. At this step, we distinguish three cases.

6.2.1. case $\tau + \gamma = 1$

We get

$$D(J,k) = \sum_{J}^{k-1} \left[1 - 2Jj^{-1} - j^{-2}J^2\right] k^{-2\gamma} =$$

$$\left[(k-1-j) - 2J\log\left(\frac{k-1}{J}\right)(1+o(1)) + \sum_{J}^{k-1} j^{-2}J^2\right] k^{-2\gamma} \tag{31}$$

$$= k^{1-2\gamma}(1+o(1)).$$

6.2.2. Case $2\tau + 2\gamma = 1$

We have

$$D(J,k) = \sum_{J}^{k-1} \left[j^{-1} - 2J^{1/2}j^{-3/2} - j^{-2}J\right] k^{-2\gamma}$$

$$= \left[\log(\frac{k-1}{J})(1+o(1)) + 4J^{1/2}(k-1)^{-1/2}(1+o(1)) + J\sum_{J}^{k-1} j^{-2}\right] k^{-2\gamma}$$

$$= k^{-2\gamma}(\log k)(1+o(1)) \tag{32}$$

6.2.3. Case $2\tau + 2\gamma \neq 1, \tau + \gamma \neq 1$

We have

$$D(J,L) = (k-1)^{2\tau+2\gamma-1}\left[\frac{1}{2\tau+2\gamma-1}\left(1 - \frac{J^{2\tau+2\gamma-1}}{(k-1)^{2\tau+2\gamma-1}}\right)(1+o(1))\right.$$

$$\left. + \frac{J^{\tau+\gamma}}{\tau+\gamma-1}\left(\frac{1}{(k-1)^{\tau+\gamma}} - \frac{J^{\tau+\gamma-1}}{(k-1)^{2\tau+2\gamma-1}}\right)(1+o(1)) + J^{\tau+\gamma}\sum_{J}^{k-1} j^{-2}\right] k^{-2\gamma}$$

$$= \frac{k^{2\tau+2\gamma-1}}{(2\tau+2\gamma-1)}(1+o(1))\, k^{-2\gamma} = (2\tau+2\gamma-1)^{-1} k^{2\tau-1} \tag{33}$$

6.3. Evaluation of $S(L,k)$

$$S(L,k) = \sum_{L}^{k-1} \overline{f}(j)\, j^{-1} B(1,j) = \sum_{L}^{j} \overline{f}(j)\, j^{-1} B(1,j) \leq \sum_{L}^{J} \overline{f}(j)\, j^{-1} B(1,j)$$

Or

$$\sum_{L}^{k-1} \overline{f}(j)\, j^{-1} B(1,j) = \sum_{L}^{J} \overline{f}(j)\, j^{-1} B(1,j) + \sum_{J}^{k-1} \overline{f}(j)\, j^{-1} B(1,j)$$

But

$$\sum_{J}^{k-1} \overline{f}(j)\, j^{-1} B(1,j) = (1+\theta(\epsilon)) \sum_{J}^{k-1} \tau j^{\tau-1} j^{-1} \left(\frac{j}{k}\right)^{\gamma}$$

$$= \tau(1+\theta(\epsilon))\, k^{-\gamma} \sum_{J}^{k-1} j^{\tau+\gamma-2}$$

6.3.1. Case $\gamma + \tau \neq 1$

The sum is

$$\sum_{J}^{k-1} \overline{f}(j)\, j^{-1} B(1,j) = \frac{\tau\,(1+\theta\,(\epsilon))\,k^{-\gamma}}{(\tau+\gamma-1)}\left[(k-1)^{\tau+\gamma-1} - J^{\tau+\gamma-1}\right].$$

Then

$$S(L,k) = S(L,J+1) + \frac{\tau\,(1+\theta\,(\epsilon))}{(\tau+\gamma-1)}\left[k^{\tau-1}\,(1+o\,(1)) - \frac{J^{\tau+\gamma-1}}{k^{\gamma}}\right] \tag{34}$$

6.3.2. Case $\gamma + \tau = 1$

The expression is

$$S(L,k) = S(L,J+1) + \tau\,(1+\theta\,(\epsilon))\,k^{-\gamma}\sum_{J}^{k-1} j^{-1} = \tau k^{-\gamma}(\log k)(1+o(1)) \tag{35}$$

6.4. Check the conditions for $\tau > 1/2$.

First remark that $2\tau + 2\gamma > 1$. Secondly, $\tau + \gamma > 1/2$. We have to consider the two subcases.

6.4.1. subcase : $\tau + \gamma = 1$.

By (28)-(31), we see that $\sigma_n^2\,(L, f_\tau) = C(J,L,k) + \tau(\tau+\gamma)^{-2}k^{1-2\gamma}$. Here, $2\gamma < 1$ surely and for any $L_1 > 0$ and $L_2 > 0$

$$\sigma_n^2\,(L_i, f_\tau) \to \infty, \sigma_n^2\,(L_1, f_\tau)\,/\sigma_n^2\,(L_2, f_\tau)\,, \tag{36}$$

since $C(J, L_i, k)$ are bounded by $C(J, L_i, \infty)$. As well, since $S(L_1, J+1)$ is bounded, we have by (35),

$$S(L_1,k)/\sigma_n\,(L_2, f_\tau) = O((\log k)k^{-1/2}) \to 0.$$

Finally, to check that $\max\{f^*(j)j^{-1}, 1 \leq j \leq k\}/\sigma_n\,(L, f_\tau) \to \infty$, we have to check that $\max\{B_J(j)(j)j^{-1}, J \leq j \leq k\}/\sigma_n\,(L, f_\tau) \to 0$. But

$$B_J(j)j^{-1}/\sigma_n\,(L, f_\tau) = \tau\,(1+\theta\,(\varepsilon))\,\frac{1}{\tau+\gamma}\,[1 - J/j]\,/k^{1/2} \leq \frac{\tau\,(1+\theta\,(\varepsilon))}{(\tau+\gamma)k^{1/2}} \to 0.$$

Hence the conditions $(K1) - (K4)$ hold.

6.4.2. subcase $\tau+\gamma \neq 1$.

By (33), we see that $\sigma_n^2(L, f_\tau) = C(J,L,k) + \tau(\tau+\gamma)^{-2}(2\tau+2\gamma-1)^{-1}k^{2\tau-1}$. We easily obtain (36) . By using (34), we get

$$S(L_1,k)/\sigma_n(L_2,f_\tau) = o(1) + \frac{\tau(1+\theta(\epsilon))}{(\tau+\gamma-1)}\left[k^{-1/2}(1+o(1)) - \frac{J^{\tau+\gamma-1}}{k^{\gamma+2\tau-1}}\right] \to 0.$$

Finally

$$\begin{aligned} B_J(j)j^{-1}/\sigma_n(L,f_\tau) &= \frac{\tau(1+o(1))}{\tau+\gamma} \\ \times \left(\frac{\tau}{(\tau+\gamma)^{-2}(2\tau+2\gamma-1)^{-1}(\tau+\gamma-1)}\right)^{-1/2} &\frac{[j^{\tau+\gamma} - J^{\tau+\gamma}]}{k^{\tau+\gamma+1/2}} \\ &= O(1)k^{-1/2}\left[(j/k)^{\tau+\gamma} - k^{-(\tau+\gamma)}J^{\tau+\gamma}\right] \\ &\leq O(1)k^{-1/2}\left[1 + k^{-(\tau+\gamma)}J^{\tau+\gamma}\right] \to 0, \end{aligned}$$

uniformly in $j \in [J,k]$. Here again, all the conditions $(K1)-(K5)$ hold.

6.5. Case $\tau < 1/2$

We also have the three cases. For $2\tau+2\gamma-1=0$, $\sigma_n^2(L,\tau)$ does not converges to $+\infty$ by (32). For $\tau+\gamma-1=0$, then $2\gamma-1>0$ and by (31), $\sigma_n^2(L,\tau)$ is bounded. Finally for $2\tau+2\gamma-1\neq 0$ and $\tau+\gamma-1\neq 0$, by (33) $\sigma_n^2(L,\tau)$ bounded. In summary, our results does not hold for $\tau<1/2$.

6.6. Case $\tau = 1/2$

Here we surely have $2\tau+2\gamma \neq 1$. Hence depending on $\tau+\gamma = (1/2)+\gamma = 1$ or not, we use (31) or (33) to get that $(K1)$ does not hold. In this case, we use (20) to see that

$$Var(\overline{f(j)}\ (S_j^* - s_j^*)) \sim C(\gamma,\tau)j^{2\tau+2\gamma-3}k^{2\gamma}.$$

where $C(\gamma,\tau)$ is some constant depending on γ and τ.This implies that for L large enough,

$$\sum_{j=L}^{k-1}\left\|\overline{f(j)}\ (S_j^* - s_j^*)\right\|_2 = C(\gamma,\tau)^{1/2}k^{2\tau-1}$$

and then, for $\tau = 1/2$,

$$\overline{\lim} EW_{k,n}^* \leq \overline{\lim}\sum_{j=1}^{k-1}\left\|\overline{f(j)}\ (S_j^* - s_j^*)\right\|_2 < \infty.$$

By applying Theorem 2.4.2 of (Rao (2012)) , we conclude that $EW^*_{k,n}$ converges to a random variable W with finite expectation.