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LOCAL HARDY–LITTLEWOOD MAXIMAL OPERATOR IN VARIABLE LEBESGUE SPACES

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ABSTRACT. We investigate the class $\mathcal{B}^{loc}(\mathbb{R}^n)$ of exponents $p(\cdot)$ for which the local Hardy–Littlewood maximal operator is bounded in variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Littlewood–Paley square function characterization of $L^{p(\cdot)}(\mathbb{R}^n)$ spaces with the above class of exponent are also obtained.

1. INTRODUCTION

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and the corresponding variable exponent Sobolev spaces $W^{k,p(\cdot)}$ are of the interest for their applications to the problems in fluid dynamics [26, 27], partial differential equations with non-standard growth conditions and calculus of variations [1, 2, 11, 12], image processing [3, 14, 22].

The boundedness of Hardy–Littlewood maximal operator is very important tool to get boundedness of more complicated operators such as singular integral operators, commutators of singular integrals, Riesz potential and many other operators. Conditions for the boundedness of the Hardy–Littlewood maximal operator on variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ have been studied in [8, 9, 6, 25, 17, 15, 21]. For an overview we refer to the monographs [10] and [4].

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Let $p : \mathbb{R}^n \longrightarrow [1, \infty)$ be a measurable function. Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the space of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$

$$\int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx < \infty,$$

with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Given a locally integrable function f on $\mathbb{R}^n$, the Hardy–Littlewood maximal operator M is defined as follows

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q |f(y)| dy,$$

where the supremum is taken over all cubes Q containing x. Throughout the paper, all cubes are assumed to have their sides parallel to the coordinate axes.

Let f be locally integrable function f on $\mathbb{R}^n$. We consider the local variant of the Hardy–Littlewood maximal operator given by

$$M^{loc} f(x) = \sup_{Q \ni x, |Q| \leq 1} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Denote by $\mathcal{B}(\mathbb{R}^n)$ ($\mathcal{B}^{loc}(\mathbb{R}^n)$) the class of all measurable functions $p : \mathbb{R}^n \longrightarrow [1, \infty)$ for which the operator M (operator M^{loc}) is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Given any measurable function $p : \mathbb{R}^n \longrightarrow [1, \infty)$, let $p_- = \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x)$ and $p_+ = \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x)$. Below we assume that $1 < p_- \leq p_+ < \infty$.

It has been proved by Diening [8] that if $p(\cdot)$ satisfies the following uniform continuity condition

$$|p(x) - p(y)| \leq \frac{c}{\log(1/|x-y|)}, \quad |x - y| < 1/2, \tag{1.1}$$

and if $p(\cdot)$ is a constant outside some large ball, then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. After that the second condition on $p(\cdot)$ has been improved independently by Cruz–Uribe, Fiorenza, and Neugebauer [6] and Nekvinda [25]. It is shown in [6] that if $p(\cdot)$ satisfies (1.1) and

$$|p(x) - p_\infty| \leq \frac{c}{\log(e + |x|)} \tag{1.2}$$

for some $p_\infty > 1$, then $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. In [25], the boundedness of M is deduced from (1.1) and the integral condition more general than (1.2) condition: there exist constants c, p_∞, such that $0 < c < 1, p_\infty > 1$, and

$$\int_{\mathbb{R}^n} c^{\frac{1}{|p(x) - p_\infty|}} dx < \infty.$$

The condition (1.1) is named the local log-Hölder continuity condition and the condition (1.2) the log-Hölder decay condition (at infinity). The conditions (1.1) and (1.2) together are named global log-Hölder continuity condition. These conditions are connected to the geometry of the space $L^{p(\cdot)}(\mathbb{R}^n)$.

By $\mathcal{X}^n$ we denote the set of all open cubes in $\mathbb{R}^n$ and by $\mathcal{Y}^n$ ($\mathcal{Y}^n_{loc}$) we denote the set of all families $\mathcal{Q} = \{Q_i\}$ of disjoint, open cubes in $\mathbb{R}^n$ (with measure less than 1) such that $\bigcup Q_i = \mathbb{R}^n$ (we ignore the difference in notation caused by a null set).

Everywhere below by $l_\mathcal{Q}$ we denote a Banach sequential space (BSS) (see for the definition in [23]). Let $\{e_Q\}$ be standard unit vectors in $l_\mathcal{Q}$.

Definition 1.1. Let $l = \{l_\mathcal{Q}\}_{\mathcal{Q}\in\mathcal{Y}^n}$ ($l = \{l_\mathcal{Q}\}_{\mathcal{Q}\in\mathcal{Y}^n_{loc}}$) be a family of BSS. A space $L^{p(\cdot)}(\mathbb{R}^n)$ is said to satisfy a uniformly upper (lower) $l-$estimate ($l_{loc}-$estimate) if there exists a constant $C > 0$ such that for every $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\mathcal{Q} \in \mathcal{Y}^n$ ($\mathcal{Q} \in \mathcal{Y}^n_{loc}$) we have

$$\|f\|_{p(\cdot)} \le C\|\sum_{Q_i\in\mathcal{Q}} \|f\chi_{Q_i}\|_{p(\cdot)} \cdot e_{Q_i}\|_{l_\mathcal{Q}} \quad \left(\|\sum_{Q_i\in\mathcal{Q}} \|f\chi_{Q_i}\|_{p(\cdot)} \cdot e_{Q_i}\|_{l_\mathcal{Q}} \le C\|f\|_{p(\cdot)}\right).$$

Definition 1.1 was introduced by Kopaliani in [16]. The idea of Definition 1.1 is simply to generalize the following property of the Lebesgue-norm:

$$\|f\|^p_{L^p} = \sum_i \|f\chi_{\Omega_i}\|^p_{L^p}$$

for a partition of $\mathbb{R}^n$ into measurable sets Ω_i.

Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. For any $\mathcal{Q} \in \mathcal{Y}^n$ we define the space $l^{\mathcal{Q},p(\cdot)}$ by

$$l^{\mathcal{Q},p(\cdot)} := \left\{\bar{t} = \{t_Q\}_{Q\in\mathcal{Q}} : \sum_{Q\in\mathcal{Q}} |t_Q|^{p_Q} < \infty\right\},$$

equipped with the Luxemburg's norm, where the numbers p_Q are defined as $\frac{1}{p_Q} = \frac{1}{|Q|}\int_Q \frac{1}{p(x)}dx$. Analogously we define the space $l^{\mathcal{Q},p'(\cdot)}$ where $\frac{1}{p(t)} + \frac{1}{p'(t)} = 1$, $t \in \mathbb{R}^n$.

Note that if $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ then for simple functions we have uniformly lower and upper $l = \{l^{\mathcal{Q},p(\cdot)}\}_{\mathcal{Q}\in\mathcal{Y}^n}$ estimates.

Theorem 1.2. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ then uniformly*

$$\|\sum_{Q\in\mathcal{Q}} t_Q\chi_Q\|_{p(\cdot)} \asymp \|\sum_{Q\in\mathcal{Q}} t_Q\|\chi_Q\|_{p(\cdot)}e_Q\|_{l^{\mathcal{Q},p(\cdot)}} \tag{1.3}$$

and

$$\|\sum_{Q\in\mathcal{Q}} t_Q\chi_Q\|_{p'(\cdot)} \asymp \|\sum_{Q\in\mathcal{Q}} t_Q\|\chi_Q\|_{p'(\cdot)}e_Q\|_{l^{\mathcal{Q},p'(\cdot)}}. \tag{1.4}$$

Above theorem is another version of necessary part of Diening's Theorem 4.2 in [9] (proof may be found in [18]). Note that conditions (1.3) and (1.4) in general do not imply $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. The proof (see in [20]) relies on the example constructed by Lerner in [21]. We give the proof of this fact also here.

Let $E = \cup_{k\ge1}(e^{k^3}, e^{k^3 e^{1/k^2}})$ and

$$p_0(x) = \int_{|x|}^\infty \frac{1}{t\log t}\chi_E(t)dt. \tag{1.5}$$

There exist $\alpha > 1$ and $\beta_0\,(1/\alpha < \beta_0 < 1)$ such that $p_0(\cdot) + \alpha \in \mathcal{B}(\mathbb{R})$ and $\beta_0(p_0(\cdot)+\alpha) \notin \mathcal{B}(\mathbb{R})$ (see [21, Theorem 1.7]). Note that for a space $L^{p(\cdot)}(\mathbb{R})$, with $p(\cdot) = p_0(\cdot) + \alpha$ there exists a family $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathcal{Y}^n}$ of BSS for which $L^{p(\cdot)}(\mathbb{R})$ satisfies uniformly lower and upper $l-$estimate (see [20, Proposition 3.2]). From (1.3) we have $l_{\mathcal{Q}} \cong l^{\mathcal{Q},p(\cdot)}$ and consequently we have

$$\|f\|_{p(\cdot)} \asymp \|\sum_{Q\in\mathcal{Q}} \|f\chi_Q\|_{p(\cdot)} e_Q\|_{l^{\mathcal{Q},p(\cdot)}}. \tag{1.6}$$

Note that for all $1 > \beta > \frac{1}{p_-}$

$$\|f^{\frac{1}{\beta}}\|^{\beta}_{\beta p(\cdot)} = \|f\|_{p(\cdot)} \tag{1.7}$$

and

$$\|\{t_Q\}\|_{l^{\mathcal{Q},p(\cdot)}} = \left\|\left\{|t_Q|^{\frac{1}{\beta}}\right\}\right\|^{\beta}_{l^{\mathcal{Q},p(\cdot)}}. \tag{1.8}$$

From (1.6), (1.7) and (1.8) we have

$$\|g\|_{\beta p(\cdot)} \asymp \|\sum_{Q\in\mathcal{Q}} \|g\chi_Q\|_{\beta p(\cdot)} e_Q\|_{l^{\mathcal{Q},\beta p(\cdot)}}.$$

for $g \in L^{\beta p(\cdot)}(\mathbb{R})$ and the space $L^{\beta p(\cdot)}(\mathbb{R})$ satisfies uniformly lower and upper l^{β}-estimates, where $l^{\beta}_{Q} = l^{Q,\beta p(\cdot)}$.

Note that $\frac{1}{(\beta p(\cdot))_Q} + \frac{1}{((\beta p(\cdot))')_Q} = 1$ and $\left(l^{\mathcal{Q},\beta p(\cdot)}\right)' = l^{\mathcal{Q},(\beta p(\cdot))'}$. Thus the space $\left(L^{\beta p(\cdot)}(\mathbb{R})\right)'$ satisfies uniformly lower and upper $(l^{\beta})'$-estimates, where $(l^{\beta})'_{Q} = l^{Q,(\beta p(\cdot))'}$ and (1.3) and (1.4) are valid for any $\beta p(\cdot)$, $(\beta p(\cdot))'$, where $1 > \beta > \frac{1}{p_-}$. Consequently for exponent $\beta_0 p(\cdot)$ (1.3) and (1.4) are valid but $\beta_0 p(\cdot) \notin \mathcal{B}(\mathbb{R})$.

Remark 1.3. Let $p(\cdot)$ be global log-Hölder continuous function. Then there exists family $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathcal{Y}^n}$ of BSSs for which $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies uniformly lower and upper $l-$estimates (see [20, Proposition 3.4]). As we already mentioned it was show in [6] that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and by Theorem 1.2 (1.3) holds and therefore we have $l_{\mathcal{Q}} \cong l^{\mathcal{Q},p(\cdot)}$ and consequently

$$\|f\|_{p(\cdot)} \asymp \|\sum_{Q\in\mathcal{Q}} \|f\chi_Q\|_{p(\cdot)} e_Q\|_{l^{\mathcal{Q},p(\cdot)}}. \tag{1.9}$$

Remark 1.4. Let $\mathcal{Q} = \{Q_i\}$ be a partition of $\mathbb{R}^n$ into equal sizes cubes, ordered so that $i > j$ if $\operatorname{dist}(0, Q_i) > \operatorname{dist}(0, Q_j)$. Let $p(\cdot)$ be global log-Hölder continuous. Then

$$\|f\|_{p(\cdot)} \approx \left(\sum_i \|f\chi_{Q_i}\|^{p_\infty}_{p(\cdot)}\right)^{1/p_\infty}. \tag{1.10}$$

This was shown in [13, Theorem 2.4]. This statement also follows from Remark 1.3. Indeed, if we have a partition $\mathcal{Q} = \{Q_i\}$ with equal sizes cubes and it is ordered as above by using [24, Theorem 4.3] we can show that $l^{p_\infty} \cong l^{\mathcal{Q},p(\cdot)}$ and consequently from (1.9) we get(1.10).

By $\mathcal{AC}$ we denote the set of exponents $p : \mathbb{R} \to [1, +\infty)$ of the form $p(x) = p + \int_{-\infty}^{x} l(u)du$, where $\int_{-\infty}^{+\infty} |l(u)|du < +\infty$.

Note that example of exponent constructed by Lerner and mentioned above belongs to class $\mathcal{AC}$. In general we have the following

Proposition 1.5. [20, Proposition 3.2] *Let $p(\cdot) \in \mathcal{AC}$. Then exists family $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathcal{Y}^n}$ of BSSs for which $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies uniformly lower and upper $l-$estimate.*

In many applications it is enough to study only boundedness of local Hardy–Littlewood maximal operator rather the Hardy–Littlewood maximal operator. For example in the Littlewood–Paley theory we need local Hardy–Littlewood maximal operator. In the weighted Lebesgue spaces behavior of local Hardy–Littlewood maximal operator was studied by Rychkov in [28].

In this paper we investigate the class $\mathcal{B}^{loc}(\mathbb{R}^n)$ of exponents $p(\cdot)$ for which the local Hardy–Littlewood maximal operator is bounded in variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$. Using the obtained results we give Littlewood–Paley square-function characterization of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ with the above class of exponent.

The paper is organized as follows. In Section 2 we give main results. In Section 3 we give application in the Littlewood–Paley theory and in last section we give outlines of the proof of the Theorem 2.2 which is local version of the Diening's theorem from [9].

2. Main results

For any family of pairwise disjoint cubes $\mathcal{Q}$ and $f \in L^1_{loc}$ we define an averaging operator

$$T_{\mathcal{Q}}f = \sum_{Q\in\mathcal{Q}} \chi_Q M_Q f$$

where $M_Q f = |Q|^{-1} \int_Q f(x)dx$.

We say that exponent $p(\cdot)$ is of the class $\mathcal{A}$ (class $\mathcal{A}^{loc}$) if and only if there exists $C > 0$ such that for all $\mathcal{Q} \in \mathcal{Y}^n$ ($\mathcal{Q} \in \mathcal{Y}^n_{loc}$) and all $f \in L^{p(\cdot)}(\mathbb{R}^n)$

$$\|T_{\mathcal{Q}}\|_{p(\cdot)} \le C\|f\|_{p(\cdot)},$$

i.e. the averaging operators $T_{\mathcal{Q}}$ are uniformly continuous on $L^{p(\cdot)}(\mathbb{R}^n)$.

A necessary and sufficient condition on $p(\cdot)$ for which the operator M is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ is given by Diening in [9]. It states that $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ if the averaging operators $T_{\mathcal{Q}}$ are uniformly continuous on $L^{p(\cdot)}(\mathbb{R}^n)$ with respect to all families $\mathcal{Q}$ of disjoint cubes. This concept provides the following characterization of when the maximal operator is bounded.

Theorem 2.1. *([9, Theorem 8.1]). Let $1 < p_- \le p_+ < \infty$. The following are equivalent:*

1) $p(\cdot)$ is of class $\mathcal{A}$;

2) M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$;

3) $(M(|f|^q))^{1/q}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for some $q > 1$, ("left-openness");

4)M is bounded on $L^{p(\cdot)/q}(\mathbb{R}^n)$ for some $q>1$, ("left-openness");
5) M is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$.

Main important results is the corresponding theorem to the Dienings theorem's Theorem 2.1 for local maximal function M^{loc}. The proof is presented in the Section 4. In the proof we are following to the idea of the proof of the Theorem 2.1 with some technical modifications.

Theorem 2.2. *Let $1<p_-\le p_+<\infty$. The following are equivalent:*
1) $p(\cdot)$ is of class $\mathcal{A}^{loc}$;
2) M^{loc} is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$;
3) $(M^{loc}(|f|^q))^{1/q}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for some $q>1$, ("left-openness");
4)M^{loc} is bounded on $L^{p(\cdot)/q}(\mathbb{R}^n)$ for some $q>1$, ("left-openness");
5) M^{loc} is bounded on $L^{p'(\cdot)}(\mathbb{R}^n)$.

We say that dx satisfies the condition $A_{p(\cdot)}$ (condition $A^{loc}_{p(\cdot)}$) if there exists $C>0$ such that for any cube Q (for any cube Q with $|Q|\le 1$)

$$\frac{1}{|Q|}\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)}\le C.$$

Using Theorem 2.1-2.2 we obtain some subclass of $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}^{loc}(\mathbb{R}^n)$.

Theorem 2.3. *Let $1<p_-\le p_+<\infty$ and there exists family $l=\{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathcal{Y}^n}$ (family $l=\{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathcal{Y}^n_{loc}}$) of BSSs for which $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies uniformly lower and upper $l-$estimate ($l_{loc}-$ estimate). Then operator M (operator M^{loc}) is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $dx\in A_{p(\cdot)}$ ($dx\in A^{loc}_{p(\cdot)}$).*

Proof. The proof for the operator M^{loc} is the same as for the operator M. Let $dx\in A^{loc}_{p(\cdot)}$. Using Hölders inequality we get

$$\frac{1}{|Q|}\int_Q|f(x)|dx\le C\frac{\|f\chi_Q\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}}.$$

For $\mathcal{Q}\in\mathcal{Y}^n_{loc}$ and $f\in L^{p(\cdot)}(\mathbb{R}^n)$ we have

$$\left\|\sum_{Q\in\mathcal{Q}}\chi_Q\frac{1}{|Q|}\int_Q f(x)dx\right\|_{p(\cdot)}\le\|\sum_{Q\in\mathcal{Q}}\|f\chi_Q\|_{p(\cdot)}e_Q\|_{l_{\mathcal{Q}}}\le C\|f\|_{p(\cdot)}.$$

The necessary part of theorem is obvious. □

Theorem 2.4. $\mathcal{B}(\mathbb{R}^n)\ne\mathcal{B}^{loc}(\mathbb{R}^n)$

Proof. Let us consider the exponent $p(\cdot)=\beta_0(p_0(\cdot)+\alpha)$ where $p_0(\cdot)$ is defined by (1.5). Fix $\alpha>1$ and β_0 $(1/\alpha<\beta_0<1)$ such that the exponent $p(\cdot)$ does not belong to the class $\mathcal{B}(\mathbb{R})$. Since $p(\cdot)\in\mathcal{AC}$ we can conclude that for $L^{p(\cdot)}(\mathbb{R})$ there exists family $l=\{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathcal{Y}^n}$ of BSSs for which $L^{p(\cdot)}(\mathbb{R})$ satisfies uniformly lower and upper $l-$estimates. For $p(\cdot)$ the condition (1.1) is fulfilled, so it is easy to show that for this exponent $p(\cdot)$ condition $A^{loc}_{p(\cdot)}$ is satisfied. Therefore by Theorem 2.2 $p(\cdot)\in\mathcal{B}^{loc}(\mathbb{R})$.

In the class $\mathcal{B}^{loc}(\mathbb{R})$ there exist exponents that have arbitrary slow decreasing order in infinity. To show this fact we rely on the simple observation. Indeed, let for $L^{p(\cdot)}(\mathbb{R})$ there exists family $l = \{l_{\mathcal{Q}}\}_{\mathcal{Q}\in\mathcal{Y}^n}$ of BSS for which $L^{p(\cdot)}(\mathbb{R}^n)$ satisfies uniformly lower and upper $l-$ estimates and $\omega : \mathbb{R} \to \mathbb{R}$; $\omega(-\infty) = -\infty$, $\omega(+\infty) = +\infty$ is strictly increasing absolutely continuous mapping. Then there exists family l_ω of BSS for which $L^{p(\omega(\cdot))}(\mathbb{R})$ satisfies uniformly lower and upper $l_\omega-$ estimates (see [20]).

Consider the exponent from Theorem 2.4. Let $t_k = e^{k^3}$, $m_k = e^{k^3}e^{1/k^2}$, $k \geq 1$. Let us construct new points t'_k, m'_k, $k \geq 1$ so that $m'_k - t'_k = m_k - t_k$ and $t'_{k+1} > m'_k$. Let us now construct the pairwise linear continuous function ω in the following way: $\omega(x) = x$ if $x \leq 0$, $\omega(t_k) = t'_k$, $\omega(m_k) = m'_k$; $k \geq 1$. We can choose the points t'_k, m'_k so that $(m'_{k+1} - t'_k)/(m_{k+1} - t_k)$ was arbitrary large. Note that exponents $p(w^{-1}(\cdot))$ and $p(\cdot)$ has the same local behavior but the decreasing order in infinity of $p(w^{-1}(\cdot))$ is very slow.

Let now consider the case $n \geq 2$. Let $D = \cup_{k=1}^{\infty}[2k-1, 2k] \times [0,1]^{n-1}$. Consider non-trivial exponent $p(\cdot)$ that satisfies global log-Hölder condition and is constant on the set $\mathbb{R}^n \backslash D$.

Let $\{m_k\}$ be the strictly increasing sequence of integers. Consider the bijection $\omega : \mathbb{R}^n \to \mathbb{R}^n$ that for each $k \in \mathbb{N}$ has the form $\omega(x) = x-(m_k, 0, \cdots, 0)$ on the set $[2k-1, 2k] \times [0,1]^{n-1}$. We can choose the sequence $\{m_k\}$ so that $p(\omega(\cdot)) \notin \mathcal{B}(\mathbb{R}^n)$ but $p(\omega(\cdot)) \in \mathcal{B}^{loc}(\mathbb{R}^n)$. $\square$

Note that only the condition $dx \in A^{loc}_{p(\cdot)}$ (even $dx \in A_{p(\cdot)}$) does not guarantee in general $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$. For the corresponding example see [19].

3. Some applications

In this section, we give Littlewood–Paley square-function characterization of $L^{p(\cdot)}(\mathbb{R}^n)$ when $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$. Let us recall the definition of local Muckenhoupt weights. The weight class A^{loc}_p $(1 < p < \infty)$ consists of all nonnegative locally integrable functions w on $\mathbb{R}^n$ for which

$$A^{loc}_p(w) = \sup_{|Q|\leq 1} \frac{1}{|Q|^p} \int_Q w(x)dx \left(w(x)^{-p'/p}dx\right)^{p/p'} < \infty.$$

Extending the suprema from $|Q| \leq 1$ to all Q gives the definition of the usual classes A_p. It follows directly from definition that $A_p \subset A^{loc}_p$. The Littlwood–Paley theory for weight Lebesgue space L^p_w with local Muckenhoupt weights was investigated by Rychkov in [28]. For more details for A^{loc}_p weights we refer paper [28].

Below we formulate analog of Rubio de Francia theorem for variable exponent case. Hereafter, $\mathcal{F}$ will denote a family of ordered pairs of non-negative, measurable functions (f, g). If we say that for some p, $1 < p < \infty$, and $w \in A^{loc}_p$

$$\int_{\mathbb{R}^n} f(x)^p w(x)dx \leq C \int_{\mathbb{R}^n} g(x)^p w(x)dx, \quad (f,g) \in \mathcal{F}, \tag{3.1}$$

we mean that this inequality holds for any $(f, g) \in \mathcal{F}$ such that the left-hand side is finite, and that the constant C depends only on p and the constant $A^{loc}_p(w)$.

Theorem 3.1. *Given a family* $\mathcal{F}$, *assume that* (3.1) *holds for some* $1 < p_0 < \infty$, *for every weight* $\omega \in A_{p_0}^{loc}$ *and for all* $(f,g) \in \mathcal{F}$. *Let* $p(\cdot)$ *be such that there exists* $1 < p_1 < p_-$, *with* $(p(\cdot)/p_1)' \in \mathcal{B}^{loc}(\mathbb{R}^n)$. *Then*

$$\|f\|_{p(t)} \le C\|g\|_{p(t)}$$

for all $(f,g) \in \mathcal{F}$ *such that* $f \in L^{p(t)}(\mathbb{R}^n)$. *Furthermore, for every* $0 < q < \infty$ *and sequence* $\{(f_j, g_j)\}_j \subset \mathcal{F}$,

$$\left\| \left(\sum_j (f_j)^q \right)^{1/q} \right\|_{p(t)} \le C \left\| \left(\sum_j (g_j)^q \right)^{1/q} \right\|_{p(t)}.$$

In the case when $w \in A_{p_0}$ and $(p(\cdot)/p_1)' \in \mathcal{B}(\mathbb{R}^n)$ Theorem 3.1 was proved in [5, Theorem 1.3], (see also proof Theorem 3.25 in [7]). Note that the collection of all cubes Q with $|Q| \le 1$ form the Muckenhoupt basis , that is for each p, $1 < p < \infty$, and for every $w \in A_p^{loc}$, the maximal operator M_{loc} is bounded on $L_w^p(\mathbb{R}^n)$ ([28, Lemma 2.11]. Theorem 3.1 follows from Theorem 2.2 and extrapolation theorem for general Banach function spaces ([7, Theorem 3.5]).

We give a number of applications of Theorem 3.1. It is well known (see [28]) that for $1 < p < \infty$ and for $w \in A_p^{loc}$,

$$\int_{\mathbb{R}^n} M^{loc} f(x)^p w(x)dx \le C \int_{\mathbb{R}^n} f(x)^p w(x)dx.$$

From Theorem 3.1 with the pairs $(M^{loc}f, |f|)$, we get vector-valued inequalities for M^{loc} on $L^{p(\cdot)}(\mathbb{R}^n)$, provided there exists $1 < p_1 < p_-$ with $(p(\cdot)/p_1)' \in \mathcal{B}^{loc}(\mathbb{R}^n)$; by Theorem 2.2, this is equivalent to $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$. We obtain following local version of the Fefferman–Stein vector-valued maximal theorem:

Corollary 3.2. *Let* $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$. *Then for all* $1 < q < \infty$,

$$\left\| \left(\sum_j (M^{loc} f_j)^q \right)^{1/q} \right\|_{p(t)} \le C \left\| \left(\sum_j (g_j)^q \right)^{1/q} \right\|_{p(t)}.$$

Let $1 < p < \infty$ and $w \in A_p^{loc}$. Let $\varphi_0 \in C_0^\infty$ have nonzero integral, and $\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2})$, $x \in \mathbb{R}^n$. Consider the square operator $S = S_{\varphi_0,\varphi}$ given by

$$S(f) = \left(\sum_{j=0}^{+\infty} |\varphi_j * f|^2 \right)^{1/2} \quad (f \in L_w^p(\mathbb{R}^n)), \tag{3.2}$$

where $\varphi_j(x) = 2^{jn}\varphi(2^j x)$, $j \in \mathbb{N}$. Then

$$\|S(f)\|_{L_w^p} \approx \|f\|_{L_w^p}, \text{ all } f \in L_w^p(\mathbb{R}^n).$$

(For details, see [28]). Therefore by Theorem 3.1 we have following Littlewood–Paley square-function characterization of $L^{p(\cdot)}(\mathbb{R}^n)$.

Corollary 3.3. *Let* $p(\cdot) \in \mathcal{B}^{loc}(\mathbb{R}^n)$. *Let* $\varphi_0 \in C_0^\infty$ *have nonzero integral, and* $\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2})$. *Consider the square operator* $S = S_{\varphi_0,\varphi}$ *given by equation* (3.2). *Then*

$$\|S(f)\|_{p(\cdot)} \approx \|f\|_{p(\cdot)}, \ \ \textit{all} \ \ f \in L^{p(\cdot)}(\mathbb{R}^n).$$

4. The proof of the Theorem 2.2

Let $\varphi(x,t) = t^{p(x)}$ $t \geq 0$, $x \in \mathbb{R}^n$, $1 < p_- \leq p_+ < \infty$. We need some notations. For $t \geq 0$, $s \geq 1$, we define

$$\varphi(f)(x) : \mathbb{R}^n \to [0, +\infty) = \mathbb{R}^{\geq 0}, \ \ (\varphi(f))(x) = \varphi(x, \, |f(x)|),$$

$$M_{s,Q\varphi} : \mathbb{R}^n \to \mathbb{R}^{\geq 0}, \ \ M_{s,Q\varphi}(t) = \left(\frac{1}{|Q|} \int_Q (\varphi(x,t))^s dx \right)^{1/s}$$

$$M_{Q\varphi} : \mathbb{R} \to \mathbb{R}^{\geq 0}, \ \ M_{Q\varphi}(t) = \left(M_{1,Q\varphi}\right)(t).$$

Analogously we will use notation for the complementary function of φ given by $\varphi^*(x,t) = (p(x)-1)p(x)^{-p'(x)}t^{p'(x)}$.

Note that for all cubes Q functions $(M_{s,Q\varphi})(t)$, $(M_{s,Q\varphi^*})(t)$ are N-functions and satisfy uniformly $\triangle_2$-condition with respect to Q (see [9, Lemma 3.4]). In addition we mention following properties of functions defined above ([9, Lemma 3.7]): let $s \geq 1$ and $Q \in \mathcal{X}^n$, then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ there holds

$$(M_{s,Q\varphi^*})^* \left(\frac{1}{2} M_{s,Q} f \right) \leq M_{s,Q}(\varphi(f)).$$

Especially, for all $u > 0$

$$(M_{s,Q\varphi^*})^* \left(\frac{1}{2} u \right) \leq M_{s,Q\varphi}(u).$$

On the other hand for all $t > 0$ the function $f_t = \chi_Q \varphi^*(t)/t$ satisfies

$$(M_{s,Q\varphi^*})^*(2M_{s,Q} f_t) \geq M_{s,Q}(\varphi(f_t)).$$

For $\mathcal{Q} \in \mathcal{Y}^n$ we define the space $l^{|Q|M_{Q\varphi}}(\mathcal{Q})$

$$l^{|Q|M_{Q\varphi}}(\mathcal{Q}) = \left\{ \bar{t} = \{t_Q\}_{Q\in\mathcal{Q}} : \sum_{Q\in\mathcal{Q}} |Q|(M_{Q\varphi})(t_Q) < \infty \right\},$$

equipped with the norm

$$\left\|\bar{t}\right\|_{l^{|Q|M_{Q\varphi}}(\mathcal{Q})} = \inf \left\{ \lambda > 0 : \sum_{Q\in\mathcal{Q}} |Q|(M_{Q\varphi})(t_Q/\lambda) < 1 \right\}.$$

Analogously we define the spaces $l^{|Q|M_{Q\varphi^*}}(\mathcal{Q})$, $l^{|Q|M_{s,Q\varphi}}(\mathcal{Q})$, $l^{|Q|M_{s,Q\varphi^*}}(\mathcal{Q})$.

Definition 4.1. Let

$$l^{|Q|(M_{Q\varphi^*})^*}(\mathcal{Q}) \hookrightarrow l^{|Q|M_{Q\varphi}}(\mathcal{Q})$$

be uniformly continuous with respect to $\mathcal{Q} \in \mathcal{Y}^n_{loc}$ ($\mathcal{Q} \in \mathcal{Y}^n$) i.e. for all $A_1 > 0$ there exists $A_2 > 0$ such that for all $\mathcal{Q} \in \mathcal{Y}^n_{loc}$ (all $\mathcal{Q} \in \mathcal{Y}^n$) and all sequences $\{t_Q\}_{Q\in\mathcal{Q}}$ there holds

$$\sum_{Q\in\mathcal{Q}} |Q|(M_{Q\varphi^*})^*(t_Q) \leq A_1 \;\; \Rightarrow \;\; \sum_{Q\in\mathcal{Q}} |Q|(M_{Q\varphi})(t_Q) \leq A_2.$$

Then we say that $M_{Q\varphi}$ is locally dominated (dominated) by $(M_{Q\varphi^*})^*$ and write $M_{Q\varphi} \preceq (M_{Q\varphi^*})^*(loc)$ ($M_{Q\varphi} \preceq (M_{Q\varphi^*})^*$).

Analogously we may define uniformly continuous embedding discrete function spaces defined above with respect to $\mathcal{Q} \in \mathcal{Y}^n_{loc}$ ($\mathcal{Q} \in \mathcal{Y}^n$). The basic property of domination ($\preceq$) in a "pointwise" sense is described in original paper [9]. Analogous properties of local domination is essentially based on the following general lemma (note that if $X = \mathcal{X}^n(loc)$ and $Y = \mathcal{Y}^n(loc)$, then X, Y are admissible for Lemma 4.2)

Lemma 4.2. *([9, Lemma 7.1].) Let X be an arbitrary set. Let Y be a subset of the power set of X such that $M_1 \subset M_2 \in Y$ implies $M_1 \in Y$. Let $\psi_1, \psi_2 : X \to \mathbb{R}^{\geq}$. If there exists $A_1 > 0$ and $A_2, A_3 \geq 0$ such that for all $M \in Y$*

$$\omega\psi_1(\omega) \leq A_1 \;\; \Rightarrow \;\; \sum_{\omega\in M} \psi_2(\omega) \leq A_2 \sum_{\omega\in M} \psi_1(\omega) + A_3$$

then there exists $b : X \to \mathbb{R}^{\geq}$ such that for all $\omega \in X$ holds

$$\psi_1(\omega) \leq \frac{A_1}{4} \quad \Rightarrow \quad \psi_2 \leq \max\left\{\frac{4A_3}{A_1}, 2A_2\right\}\psi_1 + b(\omega) \tag{4.1}$$

and

$$\sup_{M\in Y} \sum_{\omega\in M} b(\omega) \leq A_3. \tag{4.2}$$

If on the other hand there exist $b : X \to \mathbb{R}^{\geq}$, $A_1 > 0$, and $A_2, A_3 \geq 0$ such that (4.1) and (4.2) hold, then for all $M \in Y$

$$\psi_1(\omega) \leq \frac{A_1}{4} \quad \Rightarrow \quad \sum_{\omega\in M} \psi_2(\omega) \leq \max\left\{\frac{4A_3}{A_1}, 2A_2\right\}\omega\psi_1(\omega) + A_3.$$

We can now state characterization of classes $\mathcal{A}^{loc}$ and $\mathcal{A}$.

Theorem 4.3. *Exponent $p(\cdot)$ is of class $\mathcal{A}^{loc}$ (of class $\mathcal{A}$) if and only if $M_{Q\varphi} \preceq (M_{Q\varphi^*})^*(loc)$ ($M_{Q\varphi} \preceq (M_{Q\varphi^*})$)*

The proof of above theorem in the case $p(\cdot)$ is of class $\mathcal{A}$ is based on properties (4.1)-(4.3) of $M_{Q\varphi}$ and $(M_{Q\varphi^*})$ and may use analogously arguments in local variant.

Inspired by the classical Muckenhoupt class A_∞ in [9] was defined a condition $\mathcal{A}_\infty$. The importance of our considerations is analogous of the definition in local case.

Definition 4.4. We say that exponent $p(\cdot)$ is of class $\mathcal{A}_\infty^{loc}$ (class $\mathcal{A}_\infty$) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds: if $N \subset \mathbb{R}^n$ is measurable and $\mathcal{Q} \in \mathcal{Y}^n_{loc}$ ($\mathcal{Q} \in \mathcal{Y}^n$) such that

$$|Q \cap N| \geq \varepsilon |Q| \quad \text{for all } \; Q \in \mathcal{Q},$$

then for any sequence $\{t_Q\}_{Q\in\mathcal{Q}}$

$$\delta \left\| \sum_{Q\in\mathcal{Q}} t_Q \chi_Q \right\|_{p(\cdot)} \leq \left\| \sum_{Q\in\mathcal{Q}} t_Q \chi_{Q\cap N} \right\|_{p(\cdot)}.$$

It is not hard to prove that if exponent $p(\cdot)$ is of class $\mathcal{A}^{loc}$ then exponent $p(\cdot)$ is in $\mathcal{A}_\infty^{loc}$.

The important property of exponents from class $\mathcal{A}_\infty$ is that $\mathcal{A}_\infty$ implies $M_{s,Q\varphi} \preceq M_{Q\varphi}$ for some $s > 1$. The proof of this result is based on the following lemma.

Lemma 4.5. *([9, Lemma 5.5]). Let exponent $p(\cdot)$ is of class A_∞. Then there exists $\delta > 0$ and $A \geq 1$ such that for all $Q \in \mathcal{Y}^n$, all $\{t_Q\}_{Q\in\mathcal{Q}}, t_Q \geq 0$, and all $f \in L^1_{loc}$ with $M_Q f \neq 0$, $Q \in \mathcal{Q}$, holds*

$$\left\| \sum_{Q\in\mathcal{Q}} t_Q \left| \frac{f}{M_Q f} \right|^\delta \chi_Q \right\|_{p(\cdot)} \leq A \left\| \sum_{Q\in\mathcal{Q}} t_Q \chi_Q \right\|_{p(\cdot)}.$$

Note that a very similar argument can be used to obtain a local version of Lemma 4.5. In the original proof of the Lemma 4.5 is used Q-dyadic ($Q \in \mathcal{X}^n$) maximal function $M^{\triangle,Q}$ (as defined in [9, Definition 5.4]). Note that in fact in proof of Lemma 4.5 it is used local Q-dyadic maximal function, where the supremum is taken over all Q-dyadic cube Q' containing x and $|Q'| \leq |Q|$. As a consequence of local variant of Lemma 4.5 we obtain a kind reverse Hölder estimate for exponents from class $\mathcal{A}^{loc}$.

Theorem 4.6. *Let $p(\cdot) \in \mathcal{A}^{loc}$. Then there exists $s > 1$, such that $M_{s,Q\varphi} \preceq M_{Q\varphi}(loc)$.*

From Theorem 4.3 and Theorem 4.6 for local variant we obtain

Theorem 4.7. *The following conditions are equivalent*

(a) $p(\cdot)$ is of class $\mathcal{A}^{loc}$.

(b) $M_{Q\varphi} \preceq (M_{Q\varphi^*})^*(loc)$

(c) There exists $s > 1$, such that $M_{s,Q\varphi} \preceq M_{Q\varphi} \preceq (M_{Q\varphi^*})^* \preceq (M_{s,Q\varphi^*})^*(loc)$.

The key lemma from which was derived original Theorem 3.1 is Lemma 8.7 from [9]. We formulate analogous statement for local variant.

Lemma 4.8. *Let $p(\cdot) \in \mathcal{A}^{loc}$. Then there exists $s > 1$ such that for all $A_1 > 0$ there exist $A_2 > 0$ such that the following holds:*

For all families $\mathcal{Q}_\lambda \in \mathcal{Y}^n_{loc}$, $\lambda > 0$, with

$$\sum_{Q\in\mathcal{Q}_\lambda} |Q| (M_{s,Q\varphi^*})^*(\lambda) \leq A_1$$

and

$$\int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_\lambda} |Q|(M_{s,Q\varphi^*})^*(\lambda) \le A_1,$$

there holds

$$\int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_\lambda} |Q|(M_{s,Q\varphi})(\lambda) \le A_2.$$

Note that the relation described in Lemma 8.7 from [9] is denoted as $M_{Q\varphi} \ll (M_{s,Q\varphi^*})^*$ (strong domination).

The proof of Lemma 4.8 is based on some pointwise estimate of functions $(M_{Q\varphi^*})^*$ and $(M_{s,Q\varphi^*})^*$. These properties we will describe bellow in Lemma 4.9, 4.10.

If $p(\cdot) \in \mathcal{A}^{loc}$, then $M_{s,Q\varphi} \preceq (M_{s,Q\varphi^*})^*(loc)$ for some $s > 1$. It is not hard to prove that (analogously as the proof of Lemma 8.3 from [9]) uniformly in $Q \in \mathcal{X}^n_{loc}$

$$|Q|(M_{s,Q\varphi})\left(\frac{1}{\|\chi_Q\|_{p(\cdot)}}\right) \sim 1, \quad |Q|(M^*_{s,Q\varphi})^*\left(\frac{1}{\|\chi_Q\|_{p(\cdot)}}\right) \sim 1. \tag{4.3}$$

It is important to investigate for any $Q \in \mathcal{X}^n_{loc}$ the function

$$\alpha_s(Q,t) = \frac{(M_{s,Q\varphi})(t)}{(M_{s,Q\varphi^*})_*(t)}.$$

Lemma 4.9. *Let $p(\cdot) \in \mathcal{A}^{loc}$. Then*

$$\alpha_s(Q, 1/\|\chi_Q\|_{p(\cdot)}) \sim 1, \quad \alpha_s(Q,1) \sim 1$$

uniformly in $Q \in \mathcal{X}^n_{loc}$ and $t > 0$. Moreover, there exists $C \ge 1$ such that for all $Q \in \mathcal{X}^n_{loc}$

$$\alpha_s(Q,t_2) \le C(\alpha_s(Q,t_1)+1) \quad \textit{for } 0 < t_1 \le t_2 \le 1,$$

$$\alpha_s(Q,t_3) \le C(\alpha_s(Q,t_4)+1) \quad \textit{for } 1 < t_3 \le t_4 \le 1.$$

Furthermore, for all C_1, $C_2 > 0$ there exists $C_3 \ge 1$ such that for all $Q \in \mathcal{X}^n_{loc}$

$$t \in \left[C_1 \min\left\{1, \frac{1}{\|\chi_Q\|_{p(\cdot)}}\right\}, C_1 \max\left\{1, \frac{1}{\|\chi_Q\|_{p(\cdot)}}\right\}\right] \Rightarrow \alpha_s(Q,t) \le C_3. \tag{4.4}$$

The proof of analogous statement for nonlocal case ([9, Lemma 8.4])is based on the estimates (4.3) and some properties (not depend on Q) of convex functions $M_{s,Q\varphi}$, $(M_{s,Q\varphi^*})_*$. We can use these arguments in the local variant.

Lemma 4.10. *Let $p(\cdot) \in \mathcal{A}(loc)$. Then there exists $b : \mathcal{X}^n(loc) \to \mathbb{R}^{\ge}$ and $K > 0$ such that*

$$\sup_{\mathcal{Q}\in\mathcal{Y}^n(loc)} \sum_{Q\in\mathcal{Q}} |Q|b(Q) + \sup_{Q\in\mathcal{X}^n(loc)} |Q|b(Q) < \infty$$

and for all $Q \in \mathcal{X}^n(loc)$ and all $t \ge 0$ holds

$$|Q|(M_{s,Q\varphi^*})^*(t) \le 1 \quad \Rightarrow \quad (M_{s,Q\varphi})(t) \le K(M_{s,Q\varphi^*})^*(t) + b(Q).$$

Moreover, for all $Q \in \mathcal{X}^n(loc)$ and all $t \ge 1$ there holds

$$|Q|(M_{s,Q\varphi^*})^*(t) \le 1 \quad \Rightarrow \quad (M_{s,Q\varphi})(t) \le K(M_{s,Q\varphi^*})^*(t).$$

The proof may be obtained from the general Lemma 4.2 and by using the estimate (4.4) (see [9], proof Lemma 8.5).

Lemma 4.11. *Assume* $M_{s_2,Q\varphi} \preceq M^*_{s_2,Q\varphi^*}(loc)$ *for some* $s_2 > 1$ *and* $1 \le s_1 \le s_2$. *Then*

$$\left(\alpha_{s_2}(Q, t^{\frac{s_1}{s_2}})\right)^{\frac{s_2}{s_1}} \sim \alpha_{s_1}(Q,t)$$

uniformly in $Q \in \mathcal{X}^n(loc)$ *and* $t > 0$.

The proof of Lemma 4.11 is basically based on the Lemma 4.10 and may be proved as an analogous lemma from [9, Lemma 8.6].

Let f be a locally integrable function. For $q \ge 1$ we consider the local maximal operator given by

$$M_q^{loc} f(x) = \sup_{Q \ni x, |Q| \le 1} \left(\frac{1}{|Q|}\int_Q |f(y)|^q dy\right)^{1/q}.$$

We define a local dyadic maximal operator $M_{q,d}^{loc}$ witch restricted supremmum in definition of M_q^{loc} to dyadic cubes (cubes of the form $Q = 2^{-z}((0,1)^n + k),\ k = (k_1, \cdots, k_n) \in \mathbb{Z}^n,\ z \in \mathbb{N}_0$).

For fixed $t \in \mathbb{R}^n$ we define also a maximal operator $M_{q,d}^{loc,t}$ with restricted supremum in definition of $M_{q,d}^{loc}$ on the cubes $Q - t$, where Q dyadic cubes.

Note that there is a constant $C > 0$ such that (see [29])

$$M_q^{loc} f(x) \le C \int_{[-4,4]^n} M_{q,d}^{loc,t} f(x) dt. \tag{4.5}$$

The main step to proof Theorem 2.2 (as in proof of original Theorem 2.1) is the following Theorem.

Theorem 4.12. *Let* $p(\cdot) \in \mathcal{A}_{loc}$. *Then there exists* $q > 1$ *such that* M_q^{loc} *is continuous on* $L^{p(\cdot)}(\mathbb{R}^n)$.

Note that using 4.5 estimate it is sufficient to prove Theorem 4.12 for operator $M_{q,d}^{loc}$.

It is suffices to show that there exists $A > 0$ such that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx \le 1 \ \Rightarrow \ \int_{\mathbb{R}^n} |M_{q,d}^{loc} f(x)|^{p(x)} dx \le A.$$

For $\lambda > 0$ define functions

$$f_{0,\lambda} = f\chi_{\{|f| \le \lambda\}},\ \ f_{1,\lambda} = f\chi_{\{|f| > \lambda\}}.$$

Then

$$\{M_{q,d}^{loc} f > \lambda\} \subset \{M_{q,d}^{loc} f_{0,\lambda} > \lambda/2\} \cup \{M_{q,d}^{loc} f_{1,\lambda} > \lambda/2\}.$$

This implies

$$\int_{\mathbb{R}^n} |M_{q,d}^{loc} f(x)|^{p(x)} dx = \int_0^\infty \int_{\mathbb{R}^n} p(x)\lambda^{p(x)-1} \chi_{\{M_{q,d}^{loc} f > \lambda\}} dx d\lambda$$

$$\le C \sum_{j=1}^2 \int_0^\infty \lambda^{-1} \int_{\mathbb{R}^n} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{j,\lambda} > \lambda/2\}} dx d\lambda.$$

For $\lambda > 0$ let $\mathcal{Q}_{0,\lambda}$ be the decomposition of $\{M_{q,d}^{loc} f_{0,\lambda} > \lambda/2\}$ into maximal dyadic cubes. Then for all $Q \in \mathcal{Q}_{0,\lambda}$ there holds (uniformly in Q)

$$M_{q,Q} f_{0,\lambda} \sim \lambda$$

and we have

$$\int_0^\infty \lambda^{-1} \int_{\mathbb{R}^n} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{0,\lambda} > \lambda/2\}} dx d\lambda \leq C \int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_{0,\lambda}} |Q| (M_{Q\varphi})(\lambda) d\lambda.$$

Denote $f_{1,\lambda}^k = \chi_{(0,1)^n + k} f_{1,\lambda}$, $k \in \mathbb{Z}^n$. Note that if $x \in (0,1)^n + k$ then

$$M_{q,d}^{loc} f_{1,\lambda}(x) = M_{q,d}^{loc} f_{1,\lambda}^k(x)$$

and

$$\{M_{q,d}^{loc} f_{1,\lambda} > \lambda/2\} = \cup_{k \in \mathbb{Z}^n} \{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}.$$

We have

$$\int_0^\infty \lambda^{-1} \int_{\mathbb{R}^n} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda} > \lambda/2\}} dx d\lambda$$
$$= \int_0^\infty \lambda^{-1} \sum_{k \in \mathbb{Z}^n} \int_{(0,1)^n + k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda.$$

Define $m_k = 2 \int_{(0,1)^n + k} |f(x)| dx$, we have

$$\int_0^\infty \lambda^{-1} \int_{(0,1)^n + k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda$$
$$= \int_0^{m_k} \lambda^{-1} \int_{(0,1)^n + k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda$$
$$+ \int_{m_k}^\infty \lambda^{-1} \int_{(0,1)^n + k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda.$$

Note that

$$\int_0^{m_k} \lambda^{-1} \int_{(0,1)^n + k} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}} dx d\lambda \leq C \int_{(0,1)^n + k} \left(\int_{(0,1)^n + k} |f(t)| dt \right)^{p(x)} dx.$$

Let $\mathcal{Q}_{1,\lambda}^k$ be the decomposition of $\{M_{q,d}^{loc} f_{1,\lambda}^k > \lambda/2\}$ into maximal dyadic cubes. Then

$$M_{q,Q} f_{1,\lambda}^k = M_{q,Q} f_{1,\lambda} \sim \lambda.$$

holds for all $Q \in \mathcal{Q}_{1,\lambda}^k$. Define $\mathcal{Q}_{1,\lambda} = \cup_{k \in \mathbb{Z}^n} \mathcal{Q}_{1,\lambda}^k$. Then we have

$$\int_0^\infty \lambda^{-1} \int_{\mathbb{R}^n} \lambda^{p(x)} \chi_{\{M_{q,d}^{loc} f_{1,\lambda} > \lambda/2\}} dx d\lambda$$

$$\leq C \sum_{k \in \mathbb{Z}^n} \int_{(0,1)^n + k} \left(\int_{(0,1)^n + k} |f(t)| dt \right)^{p(x)} dx + \int_0^\infty \lambda^{-1} \sum_{Q \in \mathcal{Q}_{1,\lambda}} |Q| (M_{Q\varphi})(\lambda) d\lambda.$$

For the first term we have

$$\sum_{k \in \mathbb{Z}^n} \int_{(0,1)^n + k} \left(\int_{(0,1)^n + k} |f(t)| dt \right)^{p(x)} dx \leq C.$$

The second term $\int_0^\infty \lambda^{-1} \sum_{Q\in\mathcal{Q}_{1,\lambda}} |Q|(M_{Q\varphi})(\lambda)d\lambda$ can be estimated in the same way as in the Theorem 6.2 from [9]. $\square$

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