LUPAŞ-DURRMEYER OPERATORS BASED ON POLYA DISTRIBUTION

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Abstract. The generalization of the Bernstein polynomials based on Polya distribution is considered in the present article. Here, we introduce a mixed summation-integral type operators having Polya and Bernstein basis functions in summation and integration respectively. We establish some direct results which include an asymptotic formula, local and global approximation results for these operators in terms of modulus of continuity.

1. Introduction and preliminaries

In the year 1968, Stancu [13] introduced a sequence of positive linear operators $P_n^{(\alpha)} : C[0,1] \to C[0,1]$, depending on a non-negative parameter α given by

$$P_n^{(\alpha)}(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}^{(\alpha)}(x), \qquad (1.1)$$

where $p_{n,k}^{(\alpha)}(x)$ is the Polya distribution with density function given by

$$p_{n,k}^{(\alpha)}(x) = \binom{n}{k} \frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)}, x \in [0,1].$$

In case $\alpha = 0$ these operators reduce to the classical Bernstein polynomials. For $\alpha = 1/n$ a special case of the operators (1.1) was considered by Lupaş and Lupaş

[4], which can be represented in an alternate form as

$$P_n^{(1/n)}(f,x) = \frac{2(n!)}{(2n)!} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) (nx)_k (n-nx)_{n-k}, \qquad (1.2)$$

where the rising factorial is given as $(x)_n = x(x+1)(x+2)...(x+n-1)$. Recently Miclaus [12] established some approximation results for the operators (1.1) and for the case (1.2). In order to approximate Lebesgue integrable functions Durrmeyer [7] proposed the integral modification of usual Bernstein polynomials, which was later studied in different forms by Agrawal and Gupta [2], Abel et al. [1], Gupta–Ispir [10] and Gupta–Maheshwari [11] etc. We now propose the Durrmeyer type integral modification of the operators (1.2) as follows:

$$D_n^{(1/n)}(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}^{(1/n)}(x) \int_0^1 p_{n,k}(t) f(t) \, dt, \qquad (1.3)$$

where

$$p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!} \binom{n}{k} (nx)_k (n-nx)_{n-k}$$

and

$$p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}.$$

Some approximation properties related the present paper can be found in [8], [14] and in the recent books [3], [9]. In this paper, we consider the operator (1.3) and obtain a Voronovskaja type asymptotic formula and local and global direct estimates.

2. Auxiliary Results

We start this section with the following useful lemmas, which will be used in the sequel.

Lemma 2.1. *(Miclaus [12]) For $e_i = t^i, i = 0,1,2$ we have*

$$P_n^{(1/n)}(e_0, x) = 1, P_n^{(1/n)}(e_1, x) = x$$

and

$$P_n^{(1/n)}(e_2, x) = \frac{nx^2 + 2x - x^2}{n+1} = x^2 + \frac{2x(1-x)}{n+1}.$$

Lemma 2.2. *For $e_i = t^i, i = 0,1,2$ we have*

$$D_n^{(1/n)}(e_0, x) = 1, D_n^{(1/n)}(e_1, x) = \frac{nx+1}{n+2}$$

$$D_n^{(1/n)}(e_2, x) = \frac{n^3 x^2 + 5n^2 x - n^2 x^2 + 3nx + 2n + 2}{(n+1)(n+2)(n+3)}.$$

Proof. By simple computation, we have

$$\int_0^1 p_{n,k}(t) t^r dt = \frac{n!(k+r)!}{k!(n+r+1)!}. \qquad (2.1)$$

Obviously $D_n^{(1/n)}(e_0, x) = 1$ as $\int_0^1 p_{n,k}(t)dt = \frac{1}{n+1}$. Next using (2.1) and applying Lemma 2.1, we have

$$
\begin{aligned}
D_n^{(1/n)}(e_1, x) &= \sum_{k=0}^n p_{n,k}^{(1/n)}(x) \frac{k+1}{n+2} \\
&= \frac{1}{n+2} \left[nP_n^{(1/n)}(e_1, x) + P_n^{(1/n)}(e_0, x) \right] = \frac{nx+1}{n+2}.
\end{aligned}
$$

Finally

$$
\begin{aligned}
D_n^{(1/n)}(e_2, x) &= \sum_{k=0}^n p_{n,k}^{(1/n)}(x) \frac{(k+1)(k+2)}{(n+2)(n+3)} \\
&= \frac{1}{(n+2)(n+3)} \left[n^2 P_n^{(1/n)}(e_2, x) + 3nP_n^{(1/n)}(e_1, x) + 2P_n^{(1/n)}(e_0, x) \right] \\
&= \frac{1}{(n+2)(n+3)} \left[\frac{n^3 x^2 + 2n^2 x - n^2 x^2}{n+1} + 3nx + 2 \right] \\
&= \frac{1}{(n+2)(n+3)} \left[\frac{n^3 x^2 + 2n^2 x - n^2 x^2 + 3n^2 x + 3nx + 2n + 2}{n+1} \right] \\
&= \frac{n^3 x^2 + 5n^2 x - n^2 x^2 + 3nx + 2n + 2}{(n+1)(n+2)(n+3)}.
\end{aligned}
$$

$\square$

Remark 2.3. By simple applications of Lemma 2.2, we have

$$
D_n^{(1/n)}(t - x, x) = \frac{nx+1}{n+2} - x = \frac{1-2x}{n+2}
$$

and

$$
\begin{aligned}
& D_n^{(1/n)}((t - x)^2, x) \\
&= D_n^{(1/n)}(t^2, x) - 2x D_n^{(1/n)}(t, x) + x^2 \\
&= \frac{n^3 x^2 + 5n^2 x - n^2 x^2 + 3nx + 2n + 2}{(n+1)(n+2)(n+3)} - 2x \frac{nx+1}{n+2} + x^2 \\
&= \frac{(x - x^2)(3n^2 - 5n - 6) + 2(n+1)}{(n+1)(n+2)(n+3)}.
\end{aligned}
$$

Lemma 2.4. *For $f \in C[0, 1]$, we have $\left\| D_n^{(1/n)}(f, x) \right\| \le \|f\|.$*

Proof. From the definition of operator and Lemma 2.2, we get

$$
\begin{aligned}
\left| D_n^{(1/n)}(f, x) \right| &\le (n+1) \sum_{k=0}^n p_{n,k}^{(1/n)}(x) \int_0^1 p_{n,k}(t) |f(t)| dt \\
&\le \|f\| D_n^{(1/n)}(1, x) = \|f\|.
\end{aligned}
$$

$\square$

Lemma 2.5. *For $n \in \mathbb{N}$, we have*

$$D_n^{(1/n)}\left((t-x)^2,x\right) \leq \frac{3}{n+1}\delta_n^2(x),$$

where $\delta_n^2(x) = \varphi^2(x) + \frac{1}{n+2}$, where $\varphi^2(x) = x(1-x)$.

Proof. By Lemma 2.2, we have

$$D_n^{(1/n)}\left((t-x)^2,x\right)$$
$$= \frac{(x-x^2)(3n^2-5n-6)+2(n+1)}{(n+1)(n+2)(n+3)}$$
$$\leq \frac{3}{n+1}\left[\varphi^2(x)+\frac{1}{n+2}\right]$$

which is desired. □

3. DIRECT ESTIMATES

In this section, we first present the following asymptotic formula for the operators $D_n^{(1/n)}(f,x)$.

Theorem 3.1. *Let $f \in C[0,1]$ and If f'' exists at a point $x \in [0,1]$, then*

$$\lim_{n\to\infty} n\left[D_n^{(1/n)}(f,x)-f(x)\right] = (1-2x)f'(x)+\frac{3x(1-x)}{2}f''(x).$$

Proof. By Taylor's expansion of f, we have

$$f(t) = f(x)+(t-x)f'(x)+(t-x)^2\frac{f''(x)}{2}+\varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Applying $D_n^{(1/n)}$ on above Taylor's expansion and using Remark 2.3, we have

$$D_n^{(1/n)}(f,x)-f(x) = f'(x)D_n^{(1/n)}((t-x),x)+\frac{1}{2}f''(x)D_n^{(1/n)}((t-x)^2,x)$$
$$+D_n^{(1/n)}(\varepsilon(t,x)(t-x)^2,x),$$

$$\lim_{n\to\infty} n\left[D_n^{(1/n)}(f;x)-f(x)\right]$$
$$= \lim_{n\to\infty} nf'(x)D_n^{(1/n)}((t-x),x)+\lim_{n\to\infty} n\frac{1}{2}f''(x)D_n^{(1/n)}((t-x)^2,x)$$
$$+\lim_{n\to\infty} nD_n^{(1/n)}(\varepsilon(t,x)(t-x)^2,x)$$

$$\lim_{n\to\infty} n\left[D_n^{(1/n)}(f;x)-f(x)\right]$$
$$= (1-2x)f'(x)+\frac{3x(1-x)}{2}f''(x)+\lim_{n\to\infty} nD_n^{(1/n)}\left(\varepsilon(t,x)(t-x)^2,x\right)$$
$$=: (1-2x)f'(x)+\frac{3x(1-x)}{2}f''(x)+F.$$

In order to complete the proof, it is sufficient to show that $F = 0$. By Cauchy–Schwarz inequality, we have

$$F = \lim_{n \to \infty} n D_n^{(1/n)} \left(\varepsilon^2 (t, x), x \right)^{1/2} D_n^{(1/n)} \left((t - x)^4, x \right)^{1/2}. \qquad (3.1)$$

Furthermore, since $\varepsilon^2 (x, x) = 0$ and $\varepsilon^2 (., x) \in C[0, 1]$, it follows that

$$\lim_{n \to \infty} n D_n^{(1/n)} \left(\varepsilon^2 (t, x), x \right) = 0, \qquad (3.2)$$

uniformly with respect to $x \in [0, 1]$. So from (3.1) and (3.2) we get

$$\lim_{n \to \infty} n D_n^{(1/n)} \left(\varepsilon^2 (t, x), x \right)^{1/2} D_n^{(1/n)} \left((t - x)^4, x \right)^{1/2} = 0.$$

Thus, we have

$$\lim_{n \to \infty} n \left[D_n^{(1/n)} (f; x) - f(x) \right] = (1 - 2x) f'(x) + \frac{3x(1-x)}{2} f''(x),$$

which completes the proof. $\qquad \square$

We begin by recalling the following $K-$functional:

$$K_2 (f, \delta) = \inf \left\{ \|f - g\| + \delta \left\| g'' \right\| : g \in W^2 \right\} \quad (\delta > 0),$$

where $W^2 = \left\{ g \in C[0, 1] : g', g'' \in C[0, 1] \right\}$ and $\|.\|$ is the uniform norm on $C[0, 1]$. By [5], there exists a positive constant $C > 0$ such that

$$K_2 (f, \delta) \le C \omega_2 \left(f, \sqrt{\delta} \right), \qquad (3.3)$$

where the second order modulus of smoothness for $f \in C[0, 1]$ is defined as

$$\omega_2 \left(f, \sqrt{\delta} \right) = \sup_{0 < h \le \sqrt{\delta}} \sup_{x, x+2h \in [0,1]} |f(x + 2h) - 2f(x + h) + f(x)|.$$

We define the usual modulus of continuity for $f \in C[0, 1]$ as

$$\omega(f, \delta) = \sup_{0 < h \le \delta} \sup_{x, x+h \in [0,1]} |f(x + h) - f(x)|.$$

Now we present direct local approximation theorem for the operator $D_n^{(1/n)}(f, x)$.

Theorem 3.2. *For the operators $D_n^{(1/n)}$, there exists a constant $C > 0$ such that*

$$\left| D_n^{(1/n)} (f; x) - f(x) \right| \le C \omega_2 \left(f, (n + 1)^{-1} \delta_n(x) \right) + \omega \left(f, (n + 2)^{-1} \right),$$

where $f \in C[0, 1]$, $\delta_n(x) = \left[\varphi^2(x) + \frac{1}{n+1} \right]^{1/2}$ and $x \in [0, 1]$.

Proof. We are introducing the auxiliary operators as follows

$$\mathcal{D}_n^{(1/n)} (f, x) = D_n^{(1/n)} (f, x) + f(x) - f \left(\frac{nx + 1}{n + 2} \right).$$

Obviously by Lemma 2.2, we have

$$\mathcal{D}_n^{(1/n)} (1, x) = D_n^{(1/n)} (1, x) = 1$$

and

$$\mathcal{D}_n^{(1/n)}(t,x) = D_n^{(1/n)}(t,x) + x - \frac{nx+1}{n+2} = x.$$

Let $g \in W^2$ and $t \in [0,1]$. By Taylor's expansion we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)\, du.$$

Applying the operator $\mathcal{D}_n^{(1/n)}$ to both sides of above equation, we get

$$
\begin{aligned}
\mathcal{D}_n^{(1/n)}(g,x)(g,x) &= g(x) + \mathcal{D}_n^{(1/n)}\left(\int_x^t (t-u)g''(u)\, du\right)\\
&= g(x) + D_n^{(1/n)}\left(\int_x^t (t-u)g''(u)\, du, x\right)\\
&\quad - \int_x^{\frac{nx+1}{n+2}} \left(\frac{nx+1}{n+2} - u\right) g''(u)\, du.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left| \mathcal{D}_n^{(1/n)}(g,x) - g(x) \right|\\
&\leq \mathcal{D}_n^{(1/n)}\left(\int_x^t |t-u|\,|g''(u)|\, du, x\right)\\
&\quad + \int_x^{\frac{nx+1}{n+2}} \left|\frac{nx+1}{n+2} - u\right| |g''(u)|\, du\\
&\leq \mathcal{D}_n^{(1/n)}\left((t-x)^2, x\right)\|g''\| + \left(\frac{nx+1}{n+2} - x\right)^2 \|g''\|.
\end{aligned}
\tag{3.4}
$$

On the other hand, from Lemma 2.5, we have

$$
\begin{aligned}
&D_n^{(1/n)}\left((t-x)^2, x\right) + \left(\frac{1-2x}{n+2}\right)^2\\
&\leq \frac{3}{n+1}\delta_n^2(x) + \left(\frac{1-2x}{n+2}\right)^2\\
&\leq \frac{3}{n+1}\delta_n^2(x) + \frac{1}{(n+1)^2} \leq \frac{4}{n+1}\delta_n^2(x).
\end{aligned}
\tag{3.5}
$$

Thus, by (3.4) and (3.5), we have

$$\left| \mathcal{D}_n^{(1/n)}(g,x) - g(x) \right| \leq \frac{4}{n+1}\delta_n^2(x)\|g''\|,
\tag{3.6}$$

where $x \in [0,1]$. Furthermore, by Lemma 2.4, we get

$$
\begin{aligned}
\left| \mathcal{D}_n^{(1/n)}(f,x) \right| &\leq \left| D_n^{(1/n)}(f,x) \right| + |f(x)| + \left| f\left(\frac{nx+1}{n+2}\right) \right|\\
&\leq 3\|f\|,
\end{aligned}
\tag{3.7}
$$

for all $f \in C[0,1]$.

For $f \in C[0,1]$ and $g \in W^2$, using (3.6) and (3.7) we obtain

$$
\begin{aligned}
\left| D_n^{(1/n)}(f,x) - f(x) \right| &= \left| \mathcal{D}_n^{(1/n)}(f,x) - f(x) + f\left(\frac{nx+1}{n+2}\right) - f(x) \right| \\
&\leq \left| \mathcal{D}_n^{(1/n)}(f-g,x) \right| + \left| \mathcal{D}_n^{(1/n)}(g,x) - g(x) \right| \\
&\quad + |g(x) - f(x)| + \left| f\left(\frac{nx+1}{n+2}\right) - f(x) \right| \\
&\leq 4\|f-g\| + \frac{4}{n+1}\delta_n^2(x)\|g''\| + \omega\left(f, \left|\frac{1-2x}{(n+2)}\right|\right).
\end{aligned}
$$

Taking infimum over all $g \in W^2$, we obtain

$$
\left| D_n^{(1/n)}(f,x) - f(x) \right| \leq 4K_2\left(f, \frac{1}{n+1}\delta_n^2(x)\right) + \omega\left(f, \left|\frac{1-2x}{n+2}\right|\right)
$$

and by inequality (3.3), we get

$$
\left| D_n^{(1/n)}(f;x) - f(x) \right| \leq C\omega_2\left(f, (n+1)^{-1}\delta_n(x)\right) + \omega\left(f, (n+2)^{-1}\right)
$$

so proof is completed. $\qquad\square$

Let $f \in C[0,1]$ and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. The second order Ditzian–Totik modulus of smoothness and corresponding $K-$functional are given by, respectively,

$$
\begin{aligned}
\omega_2^\varphi\left(f, \sqrt{\delta}\right) &= \sup_{0<h\leq\sqrt{\delta}} \sup_{x\pm h\varphi(x)\in[0,1]} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|, \\
\bar{K}_{2,\varphi}(f,\delta) &= \inf\left\{\|f-g\| + \delta\left\|\varphi^2 g''\right\| + \delta^2\left\|g''\right\| : g \in W^2(\varphi)\right\} \quad (\delta>0),
\end{aligned}
$$

where $W^2(\varphi) = \left\{g \in C[0,1] : g' \in AC_{loc}[0,1], \; \varphi^2 g'' \in C[0,1]\right\}$ and $g' \in AC_{loc}[0,1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b] \subset [0,1]$. We know from Theorem 1.3.1 of [6] that there exists a positive constant $C > 0$, such that

$$
\bar{K}_{2,\varphi}(f,\delta) \leq C\omega_2^\varphi\left(f, \sqrt{\delta}\right). \tag{3.8}
$$

Also, the Ditzian–Totik modulus of first order is given by

$$
\overrightarrow{\omega}_\psi(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\pm h\varphi(x)\in[0,1]} |f(x+h\psi(x)) - f(x)|,
$$

where ψ is an admissible step-weight function on $[0,1]$.

Our last direct estimate is following global theorem in terms of weighted modulus of continuity.

Theorem 3.3. *Let $f \in C[0,1]$. Then for $x \in [0,1]$, we have*

$$
\left\| D_n^{(1/n)}f - f \right\| \leq C\omega_2^\varphi\left(f, (n+1)^{-1/2}\right) + \overrightarrow{\omega}_\psi\left(f, (n+2)^{-1}\right),
$$

where $C > 0$ is an absolute constant, $\psi(x) = 1 - 2x$ and $\varphi(x) = \sqrt{x(1-x)}$.

Proof. Again we consider the auxiliary operators

$$\mathcal{D}_n^{(1/n)}(f,x) = D_n^{(1/n)}(f,x) + f(x) - f\left(\frac{nx+1}{n+2}\right).$$

Using the definition of the operator $D_n^{(1/n)}$ and Lemma 2.2, we obtained the inequality (3.5) in the proof of Theorem 3.2 as

$$\left|\mathcal{D}_n^{(1/n)}(g;x) - g(x)\right|$$

$$\leq D_n^{(1/n)}\left(\int_x^t |t-u|\,|g''(u)|\,du; x\right)$$

$$+ \int_x^{\frac{nx+1}{n+2}} \left|\frac{nx+1}{n+2} - u\right| |g''(u)|\,du. \tag{3.9}$$

Moreover, δ_n^2 is a concave function on $x \in [0,1]$, for $u = \lambda x + (1-\lambda)t$, $\lambda \in [0,1]$, we get

$$\frac{|t-u|}{\delta_n^2(u)} = \frac{\lambda|t-x|}{\delta_n^2(\lambda x + (1-\lambda)t)} \leq \frac{\lambda|t-x|}{\lambda\delta_n^2(x) + (1-\lambda)\delta_n^2(t)} \leq \frac{|t-x|}{\delta_n^2(x)}.$$

Thus, if we use this inequality in (3.9), we have

$$\left|\mathcal{D}_n^{(1/n)}(g,x) - g(x)\right| \leq D_n^{(1/n)}\left(\int_x^t \frac{|t-u|}{\delta_n^2(u)}du, x\right)\left\|\delta_n^2 g''\right\|$$

$$+ \int_x^{\frac{nx+1}{n+2}} \frac{\left|\frac{nx+1}{n+2} - u\right|}{\delta_n^2(u)}du\left\|\delta_n^2 g''\right\|$$

$$\leq \frac{1}{\delta_n^2(x)}\left\|\delta_n^2 g''\right\|\left[D_n^{(1/n)}\left((t-x)^2,x\right) + \left(\frac{1-2x}{n+2}\right)^2\right]. \tag{3.10}$$

By the inequality (3.5), we have

$$\left|\mathcal{D}_n^{(1/n)}(g,x) - g(x)\right| \leq \frac{3}{n+1}\left\|\delta_n^2 g''\right\|$$

$$\leq \frac{3}{n+1}\left(\left\|\varphi^2 g''\right\| + \frac{1}{n+2}\|g''\|\right).$$

Using (3.7) and (3.10), we have for $f \in C[0,1]$,

$$\left|D_n^{(1/n)}(f,x) - f(x)\right| \leq \left|\mathcal{D}_n^{(1/n)}(f-g,x)\right| + \left|\mathcal{D}_n^{(1/n)}(g,x) - g(x)\right|$$

$$+ |g(x) - f(x)| + \left|f\left(\frac{nx+1}{n+2}\right) - f(x)\right|$$

$$\leq 4\|f-g\| + \frac{4}{n+1}\left\|\varphi^2 g''\right\| + \frac{4}{(n+1)^2}\|g''\|$$

$$+ \left|f\left(\frac{nx+1}{n+2}\right) - f(x)\right|.$$

Taking infimum over all $g \in W^2$, we obtain

$$\left|D_n^{(1/n)}\left(f,x\right) - f\left(x\right)\right| \leq 4\bar{K}_{2,\varphi}\left(f,\frac{1}{n+1}\right) + \left|f\left(\frac{nx+1}{n+2}\right) - f\left(x\right)\right|. \quad (3.11)$$

On the other hand

$$\begin{aligned}
&\left|f\left(\frac{nx+1}{n+2}\right) - f\left(x\right)\right| \\
&= \left|f\left(x + \psi(x).\frac{1-2x}{(n+2)\psi(x)}\right) - f\left(x\right)\right| \\
&\leq \sup_{t,t+\psi(t)\frac{1-2x}{(n+2)\psi(x)}\in[0,1]}\left|f\left(t + \psi\left(t\right)\frac{1-2x}{(n+2)\psi(x)}\right) - f\left(t\right)\right| \\
&\leq \overrightarrow{\omega}_\psi\left(f,\frac{|1-2x|}{(n+2)\psi\left(x\right)}\right) \\
&= \overrightarrow{\omega}_\psi\left(f,\frac{1}{(n+2)}\right).
\end{aligned}$$

Therefore, from (3.8) and (3.11) we obtain

$$\left\|D_n^{(1/n)}f - f\right\| \leq C\omega_2^\varphi\left(f,(n+1)^{-1/2}\right) + \overrightarrow{\omega}_\psi\left(f,(n+2)^{-1}\right),$$

which is the desired result. $\qquad\square$

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