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GENERALIZATION OF SOME RESULTS ON $p\alpha$-DUALS

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Abstract. We will find the $p\alpha-$dual for X_T, where X is one of the spaces c, c_0, ℓ_∞ and T is a triangle matrix. This will be achieved in two ways: firstly, under some conditions for the inverse matrix S of T and secondly, for arbitrary triangles T.

1. Notation, motivation and known results

Before we explain the motivation for the paper, we give the notations which will be used in the paper.

As usual, let ω, ℓ_∞, c and c_0 denote the sets of all complex, bounded, convergent and null sequences. We also write $\ell_p = \{x \in \omega \mid \sum_{k=0}^\infty |x_k|^p < \infty\}$.

Let X and Y be subsets of ω and $z \in \omega$. Then we use the notation $z^{-1} * Y = \{x \in \omega \mid xz = (x_k z_k)_{k=0}^\infty \in Y\}$ and write $M(X,Y) = \cap_{x\in X} x^{-1} * Y$ for multiplier space of X and Y.

The definition of the $p\alpha$-dual for $1 \le p < \infty$ of a sequence space X was given in [2] as

$$X^{p\alpha} = M(X, \ell_p) = \Big\{a = (a_k) \mid \sum_k |a_k x_k|^p < \infty, \text{ for each } x \text{ in X}\Big\}.$$

It can be shown (see [2]), that $c_0^{p\alpha} = c^{p\alpha} = \ell_\infty^{p\alpha} = \ell_p$ for $1 \le p < \infty$.

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As mentioned, the idea for this paper arises from the results obtained in [3]. In [3, 2], the authors deal with difference sequence spaces and find their $p\alpha$-duals. Also, in [1], the authors consider some classical sequence spaces and their generalized Köthe–Toeplitz duals. All these results inspired us to generalize the existing results and determine the $p\alpha$-duals for the matrix domains of triangles T in the classical sequence spaces c_0, c and ℓ_∞. This will be achieved under some conditions on the matrix T, but we will also establish some results without restrictions on T. This generalizes the results in [3, 2, 1].

Let us recall that if we denote by $A = (a_{nk})_{n,k=0}^{\infty}$ an infinite matrix with complex entries and by A_n its n-th row, we write $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ and $Ax = (A_n x)_{n=0}^{\infty}$ (provided all the series converge); the set $X_A = \{a \in \omega \mid A(x) \in X\}$ is called the matrix domain of A in X. Furthermore, a matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is said to be a triangle if $t_{nk} = 0$ for all $k > n$ and $t_{nn} \neq 0$ $(n = 0, 1 \ldots)$. Throughout, we will write T for a triangle and S for its inverse.

Hence, our task is to find $M(X_T, \ell_p)$ for $1 \leq p < \infty$, that is, the $p\alpha$-dual for X_T where T is an arbitrary triangle and $X \in \{c, c_0, \ell_\infty\}$. This generalizes the results in [3, 2, 1].

2. Main results

We start this section with a theorem whose results are based on the assumption, that the terms of each of the rows of the inverse S of the triangle T have the same sign. This is the case for the matrix of the m–th difference. Furthermore, we will establish a more general result without that restriction on T.

Theorem 2.1. *Let $1 \leq p < \infty$, T be triangle such that its inverse S has the property that the entries in each row of S have constant sign, and S^t denote the transpose of S. Then we have*

$$((c_0)_T)^{p\alpha} = (c_T)^{p\alpha} = ((\ell_\infty)_T)^{p\alpha} = B = (\ell_p)_{S^t}\,,$$

that is,

$$B = \left\{ a \in \omega : \sum_{k=0}^{\infty} \left| a_k \sum_{j=0}^{k} s_{kj} \right|^p < \infty \right\}. \tag{2.1}$$

Proof. Let $\mathrm{e}^{i\alpha_k}$ $(k = 0, 1, \ldots)$ be the constant sign of all non-zero term in the k^{th} row of S, that is, $s_{kj} = \mathrm{e}^{i\alpha_k}|s_{kj}|$ $(0 \leq j \leq k; k = 0, 1, \ldots)$. We know by [5, Theorem 4.3.12, 4.3.14] that $c_0 \subset c \subset \ell_\infty$ implies

$$(c_0)_T \subset c_T \subset (\ell_\infty)_T,$$

and also by [1, Lemma 1(ii)] that

$$((\ell_\infty)_T)^{p\alpha} \subset (c_T)^{p\alpha} \subset ((c_0)_T)^{p\alpha}\,. \tag{2.2}$$

First we show

$$B \subset ((\ell_\infty)_T)^{p\alpha}\,. \tag{2.3}$$

Let $a \in B$ and $x \in (\ell_\infty)_T$, hence $y = Tx \in \ell_\infty$ and so $x = Sy$. We obtain

$$\sum_{k=0}^{\infty} |a_k x_k|^p = \sum_{k=0}^{\infty} |a_k \cdot S_k y|^p = \sum_{k=0}^{\infty} \left| a_k \sum_{j=0}^{k} s_{kj} y_j \right|^p \leq \sum_{k=0}^{\infty} \left(|a_k| \cdot \left| \mathrm{e}^{i\alpha_k} \sum_{j=0}^{k} |s_{kj}| y_j \right| \right)^p$$
$$\leq \sum_{k=0}^{\infty} \left(|a_k| \sup_j |y_j| \sum_{j=0}^{k} |s_{kj}| \right)^p < \infty,$$

that is, $a \in ((\ell_\infty)_T)^{p\alpha}$. Thus we have shown (2.3).
Now we show

$$((c_0)_T)^{p\alpha} \subseteq B. \tag{2.4}$$

We assume $a \notin B$. Then there is a sequence $(k(r))_{r=0}^{\infty}$ of integers with $0 = k(0) < k(1) < k(2) < \cdots$ such that

$$\sum_{k=k(r)}^{k(r+1)-1} |a_k|^p \cdot \left(\sum_{j=0}^{k} |s_{kj}| \right)^p > (r+1)^p \ (r = 0, 1, \dots).$$

We define sequence $x = (x_k)_{k=0}^{\infty}$ and $y = (y_n)_{n=0}^{\infty}$ by

$$x_k = \sum_{\ell=0}^{r-1} \frac{1}{\ell+1} \sum_{j=k(\ell)}^{k(\ell+1)-1} s_{kj} + \frac{1}{r+1} \cdot \sum_{j=k(r)}^{k} s_{kj} \text{ for } (k(r) \leq k \leq k(r+1)-1; r = 0, 1, \cdots)$$

and

$$y_n = \frac{1}{r+1} \text{ for } k(r) \leq k \leq k(r+1) - 1; r = 0, 1, \dots.$$

Then we have $y = Tx$. To see this let $k \in \mathbb{N}_0$ be given. Then there exists a unique $r \in \mathbb{N}_0$ such that $k(r) \leq k \leq k(r+1) - 1$ and then

$$S_k y = \sum_{j=0}^{k} s_{kj} y_j = \sum_{\ell=0}^{r-1} \sum_{j=k(\ell)}^{k(\ell+1)-1} s_{kj} y_j + \sum_{j=k(r)}^{k} s_{kj} y_j$$
$$= \sum_{\ell=0}^{r-1} \frac{1}{\ell+1} \sum_{j=k(\ell)}^{k(\ell+1)-1} s_{kj} + \frac{1}{r+1} \sum_{j=k(r)}^{k} s_{kj} = x_k,$$

that is, $x = Sy$, and so $y = Tx$.
Since obviously $y \in c_0$, it follows that $x \in (c_0)_T$. Furthermore,

$$\sum_{k=k(r)}^{k(r+1)-1} |a_k x_k|^p = \sum_{k=k(r)}^{k(r+1)-1} \left(|a_k| \cdot \left| \sum_{j=0}^{k} s_{kj} \cdot \frac{1}{r+1} \right| \right)^p$$
$$= \left(\frac{1}{r+1} \right)^p \sum_{k=k(r)}^{k(r+1)-1} |a_k| \cdot \left(|\mathrm{e}^{i\alpha_k}| \cdot \sum_{j=0}^{k} |s_{jk}| \right)^p$$
$$= \left(\frac{1}{r+1} \right)^p \sum_{k=k(r)}^{k(r+1)-1} |a_k| \left(\sum_{j=0}^{k} |s_{jk}| \right)^p > 1 \text{ for } r = 0, 1, \dots$$

implies

$$\sum_{k=0}^{\infty}|a_k x_k|^p = \sum_{r=0}^{\infty}\sum_{k=k(r)}^{k(r+1)-1}|a_k x_k|^p > \sum_{r=0}^{\infty}1 = \infty.$$

Thus we have shown (2.4).
Now the statement of the theorem follows from (2.2), (2.3) and (2.4). □

Now, using the theory of matrix transformations between classical sequence spaces [4, 5], we give a general result without conditions on T and S. We use standard arguments for the triangle T.

Theorem 2.2. *Let $1 \leq p < \infty$, $X \in \{c, c_0, \ell_\infty\}$, T be an arbitrary triangle and S be its inverse. Then we have*

$$a = (a_k) \in X_T^{p\alpha} \text{ if and only if } \sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} a_n s_{nk}\right|^p < \infty. \qquad (2.5)$$

Proof. Since $z \in X_T$ if and only if $Tz \in X$ we have $z = Sx$ for some $x \in X$. If we denote by $B^{(a)} = (b_{nk}^{(a)})_{n,k=0}^{\infty}$ triangle matrix with entries $b_{nk}^{(a)} = a_n s_{nk}$, we get

$$a_n z_n = a_n S_n x = \sum_{k=0}^{n} a_n s_{nk} x_k = B_n^{(a)} x \text{ for all } n.$$

Hence

$$a \in (X_T, Y) \Leftrightarrow B^{(a)} \in (X, Y),$$

that is, in our case

$$a \in X_T^{p\alpha} \Leftrightarrow B^{(a)} \in (X, \ell_p).$$

Specially for $X \in \{c, c_0, \ell_\infty\}$, applying [5, Examples 8.4.3B, 8.4.9A and 8.4.8A] we have

$$B^{(a)} \in (X, \ell_p) \Leftrightarrow \sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \sum_{n=0}^{\infty}\left|\sum_{k=0}^{n} a_n s_{nk}\right|^p < \infty.$$

Now, it is clear that (2.5) holds. □

3. Applications

Here, we will cover existing results from [3]. Actually, we will apply our generalized results to some special cases and obtain results from [3] which have been treated separately.

There are a great number of papers on spaces of $m-$th order difference sequences. If we use notation from the beginning, the inverse matrix S of the matrix of the $m-$th difference is with non-negative entries, so results from [3] can be covered just applying our Theorem 2.1. Of course, the same can be done and by Theorem 2.2. but this is not necessary.

Let us start with results from [3, Theorem 2.6]. The set $E(k^m)$ is defined by:

$$E(k^m) = \{x = (x_k) \mid (k^m x_k) \in E\},$$

where E is one of the classical spaces l_∞, c, c_0. It has been claimed that $(E(k^m))^{p\alpha} = U^{(p)}$, where

$$U^{(p)} = \{a = (a_k) \mid \sum_{k=1}^{\infty} \left|\frac{a_k}{k^m}\right|^p < \infty\}.$$

If we go back to our generalized results, first, it can be seen that the set $E(k^m)$ is actually matrix domain of triangle T, given by:

$$t_{nk} = \begin{cases} k^m & (k = n) \\ 0 & (k \neq n) \end{cases} \quad (n = 0, 1, \dots)$$

in classical sequence space $E \in \{l_\infty, c, c_0\}$. It is obvious that the inverse S of T is also triangle defined by:

$$s_{nk} = \begin{cases} k^{-m} & (k = n) \\ 0 & (k \neq n) \end{cases} \quad (n = 0, 1, \dots).$$

Hence, $\sum_{j=0}^{k} s_{kj} = s_{kk} = k^{-m}$ and this implies the following:

$$(E(k^m))^{p\alpha} = (E_T)^{p\alpha};$$

$$\begin{aligned} (E_T)^{p\alpha} &= B = \Big\{a = (a_k) \mid \sum_k |a_k|^p \cdot \big(\sum_{j=0}^{k} s_{kj}\big)^p < \infty\Big\} \\ &= \Big\{a = (a_k) \mid \sum_k |a_k|^p \cdot s_{kk}^p < \infty\Big\}; \end{aligned}$$

$$B = \Big\{a = (a_k) \mid \sum_k |a_k|^p \cdot |k^{-m}|^p < \infty\Big\} = \Big\{a = (a_k) \mid \sum_{k=1}^{\infty} \left|\frac{a_k}{k^m}\right|^p < \infty\Big\} = U^{(p)}.$$

Further, we will consider the sequence space based on difference sequence spaces, defined in [3]:

$$\Delta_{v,r}^{(m)}(E) = \{x = (x_k) \mid (k^r \Delta_v^{(m)} x)_k \in E\}$$

where $E \in \{l_\infty, c, c_0\}$, $v = (v_k)$ is any fixed sequence of non-zero complex numbers, $m \in \mathbb{N}$, $r \in \mathbb{R}$ and

$$\Big(k^r \Delta_v^{(m)} x_k\Big)_k = k^r \cdot \sum_{i=0}^{m} (-1)^m \begin{pmatrix} m \\ i \end{pmatrix} v_{k-i} x_{k-i}.$$

In the mentioned paper in Theorem 2.5 authors have claimed that

$$\Big(\Delta_{v,r}^{(m)}(c_0)\Big)^{p\alpha} = \Big(\Delta_{v,r}^{(m)}(c)\Big)^{p\alpha} = \Big(\Delta_{v,r}^{(m)}(\ell_\infty)\Big)^{p\alpha} = U_1,$$

where

$$U_1 = \{a = (a_k) \mid \sum_{k=1}^{\infty} k^{p(m-r)} |v_k^{-1} a_k|^p < \infty\}.$$

It is clear that the space $\Delta^{(m)}_{v,r}(E)$ is matrix domain of certain triangle T in the space E and its inverse is matrix S with entries defined in the following way:

$$s_{kj} = \frac{1}{k^r v_k}\binom{m+k-j-1}{k-j}.$$

Here we will give general result for $p\alpha-$ dual of sequence space which can be represented as matrix domain of arbitrary triangle in one of the classical sequence spaces c, c_0, ℓ_∞ and after that we will apply our results to the spaces considered in [3, Theorem 2.5].

From the definition of the space $\Delta^{(m)}_{v,r}(E)$, we can conclude that $\Delta^{(m)}_{v,r}(E) = v^{-1} * X_T$, where $T = T_1\Delta^{(m)}$, T_1 is diagonal matrix with $t_{kk} = k^r$ and $\Delta^{(m)}$ is matrix of $m-th$ order difference operator.

Hence, we will give general result for $p\alpha-$ dual of the space $v^{-1} * X_{T'}$ for arbitrary triangle T' (with inverse S')and $X \in \{l_\infty, c, c_0\}$ and after that apply that to the special case considered in [3, Theorem 2.5]. In that way we will cover and generalize all existing results.

As we know, $v^{-1} * X = \{x \mid vx \in X\}$ and by the definition of $p\alpha-$ dual of the space, we have that

$$(v^{-1} * X)^{p\alpha} = \big\{a = (a_k) \mid \sum_k |a_k x_k|^p < \infty, \text{ for each } x \in v^{-1} * X \ \big\}.$$

It can be shown easily that $(v^{-1} * X)^{p\alpha} = v * X^{p\alpha}$. Actually, if $a \in (v^{-1} * X)^{p\alpha}$ and $x \in v^{-1} * X$, we have

$$\sum_k |a_k x_k|^p = \sum_k |a_k v_k^{-1}|^p \cdot |v_k x_k|^p < \infty$$

This implies that $av^{-1} \in X^{p\alpha}$, that is $a \in v * X^{p\alpha}$. On the other side, if $a \in v * X^{p\alpha}$ and $x \in v^{-1} * X$, similarly we can conclude that that $a \in (v^{-1} * X)^{p\alpha}$.

Following all noticed, we have that $(v^{-1} * X_{T'})^{p\alpha} = v * (X_{T'})^{p\alpha} = v * B$, where the set B is given by (2.1).

Let us consider the space $\Delta^{(m)}_{v,r}(E)$ and its $p\alpha-$ dual. We have:

$$\Big(\Delta^{(m)}_{v,r}(c_0)\Big)^{p\alpha} = \Big(\Delta^{(m)}_{v,r}(c)\Big)^{p\alpha} = \Big(\Delta^{(m)}_{v,r}(\ell_\infty)\Big)^{p\alpha} = v * B,$$

that is

$$\Big(\Delta^{(m)}_{v,r}(E)\Big)^{p\alpha} = v * \Big\{a = (a_k) \mid \sum_k |a_k|^p \cdot \big(\sum_{j=0}^{k} s'_{kj}\big)^p < \infty\Big\} =$$

$$= \Big\{a = (a_k) \mid \sum_k |a_k v_k^{-1}|^p \cdot \big(\sum_{j=0}^{k} s'_{kj}\big)^p < \infty\Big\}.$$

Since

$$s'_{kj} = \frac{1}{k^r}\binom{m+k-j-1}{k-j},$$

we obtain:

$$\Big(\sum_{j=0}^{k} s'_{kj}\Big)^p = \Big(\sum_{j=0}^{k} \frac{1}{k^r}\binom{m+k-j-1}{k-j}\Big)^p = k^{-pr}\cdot\Big(\sum_{j=0}^{k}\binom{m+k-j-1}{k-j}\Big)^p.$$

Further, as we know that $\sum_{j=0}^{k}\binom{m+j-1}{j} = \binom{m+k}{k}$ [4, (3.11)], we have:

$$\Big(\sum_{j=0}^{k} s'_{kj}\Big)^p = k^{-pr}\cdot\binom{m+k}{k}^p.$$

Also, we apply result [4, (3.12)] that there are positive constants M_1 and M_2 such that $M_1 k^m \le \binom{m+k}{k} \le M_2 k^m$, for all $k = 0, 1, 2\cdots$ Applying that we obtain that

$$\begin{aligned}\left(\Delta_{v,r}^{(m)}(E)\right)^{p\alpha} &= \Big\{a=(a_k) \mid \sum_k |a_k v_k^{-1}|^p \cdot k^{-pr}\cdot\binom{m+k}{k}^p < \infty\Big\} = \\ &= \Big\{a=(a_k) \mid \sum_k |a_k v_k^{-1}|^p \cdot k^{-pr}\cdot k^{pm} < \infty\Big\} = \\ &= \Big\{a=(a_k) \mid \sum_k |a_k v_k^{-1}|^p \cdot k^{p(m-r)} < \infty\Big\} = U_1.\end{aligned}$$

We have covered results from [3, Theorem 2.5].

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