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PSEUDOQUOTIENTS ON COMMUTATIVE BANACH ALGEBRAS

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Abstract. We consider pseudoquotient extensions of positive linear functionals on a commutative Banach algebra $\mathcal{A}$ and give conditions under which the constructed space of pseudoquotients can be identified with all Radon measures on the structure space $\hat{\mathcal{A}}$.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with involution. The structure space of $\mathcal{A}$, denoted by $\hat{\mathcal{A}}$, is the set of all multiplicative linear functionals on $\mathcal{A}$. We use $\hat{x}$ to denote the Gelfand transform of x, that is $\hat{x}(\xi) = \xi(x)$ for any $x \in \mathcal{A}$ and $\xi \in \hat{\mathcal{A}}$.

A linear functional $f : \mathcal{A} \to \mathbb{C}$ is called positive, if

$$f(x^*x) \geq 0, \text{ for all } x \in \mathcal{A}.$$

The set of all positive linear functionals on an algebra $\mathcal{A}$ is denoted by $\mathcal{P}(\mathcal{A})$. The following theorem (attributed to Maltese in [5]) describes $\mathcal{P}(\mathcal{A})$ in terms of measures on $\hat{\mathcal{A}}$.

Theorem 1.1. *Let $\mathcal{A}$ be a commutative Banach algebra with a bounded approximate identity and an isometric and symmetric involution. Let f be a linear*

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functional on $\mathcal{A}$. Then $f \in \mathcal{P}(\mathcal{A})$ if and only if

$$f(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu_f(\xi),$$

for all $x \in \mathcal{A}$, with respect to a unique positive Radon measure on $\hat{\mathcal{A}}$ of total variation $\|f\|$.

By a Radon measure on $\hat{\mathcal{A}}$ we mean a continuous linear functional on the space $\mathcal{K}(\hat{\mathcal{A}})$ of continuous complex-valued functions on $\hat{\mathcal{A}}$ with compact support equipped with the standard inductive limit topology. The space of all such measures will be denoted by $\mathcal{M}(\hat{\mathcal{A}})$. The set of all positive Radon measures on $\hat{\mathcal{A}}$ will be denoted by $\mathcal{M}_+(\hat{\mathcal{A}})$ and the set of all bounded positive Radon measures on $\hat{\mathcal{A}}$ will be denoted by $\mathcal{M}_+^b(\hat{\mathcal{A}})$. The topology of $\mathcal{M}_+^b(\hat{\mathcal{A}})$ is the topology of uniform convergence on $\hat{\mathcal{A}}$ and the topology of $\mathcal{M}_+(\hat{\mathcal{A}})$ is the topology of uniform convergence on compact subsets of $\hat{\mathcal{A}}$.

Let $\mathcal{F} : \mathcal{P}(\mathcal{A}) \to \mathcal{M}_+^b(\hat{\mathcal{A}})$ be the map defined by Maltese's theorem, that is, $\mathcal{F}(f) = \mu_f$. In terms of the introduced notation, Theorem 1.1 states that $\mathcal{F}$ is an isometry between $\mathcal{P}(\mathcal{A})$ and $\mathcal{M}_+^b(\hat{\mathcal{A}})$. In this paper we give conditions under which $\mathcal{P}(\mathcal{A})$ can be extended to a space of pseudoquotients $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$ such that $\mathcal{F}$ can be extended to a bijection between $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$ and $\mathcal{M}_+(\hat{\mathcal{A}})$.

In the next section we recall the construction of pseudoquotients and its basic properties. The construction of pseudoqutients was introduced in [8] under the name of "generalized quotients." The motivation for the idea, early developments, and later modifications, are discussed in [9]. The construction of pseudoquotients has desirable properties. For instance, it preserves the algebraic structure of X and has good topological properties. There is growing evidence that pseudoquotients can be a useful tool (see, for example, [1], [2], or [3]).

In Section 3 we formulate and prove the main result of the paper. In the final section we discuss some examples. We also show that the result in [2] is a special case of the construction presented here.

2. Pseudoquotients

Let X be a nonempty set and let S be a commutative semigroup acting on X injectively. The relation

$$(x, \varphi) \sim (y, \psi) \quad \text{if} \quad \psi x = \varphi y$$

is an equivalence in $X \times S$. We define $\mathcal{B}(X, S) = (X \times S)/\!\sim$. Elements of $\mathcal{B}(X, S)$ are called pseudoquotients. The equivalence class of (x, φ) will be denoted by $\frac{x}{\varphi}$. Thus $\frac{x}{\varphi} = \frac{y}{\psi}$ means $\psi x = \varphi y$.

Elements of X can be identified with elements of $\mathcal{B}(X, S)$ via the embedding $\iota : X \to \mathcal{B}(X, S)$ defined by $\iota(x) = \frac{\varphi x}{\varphi}$, where φ is an arbitrary element of S. The action of S can be extended to $\mathcal{B}(X, S)$ via $\varphi \frac{x}{\psi} = \frac{\varphi x}{\psi}$. If $\varphi \frac{x}{\psi} = \iota(y)$, for some $y \in X$, we simply write $\varphi \frac{x}{\psi} \in X$ and $\varphi \frac{x}{\psi} = y$. For instance, we have $\varphi \frac{x}{\varphi} = x$.

In the case X is a topological space or a convergence space and S is a commutative semigroup of continuous injections acting on X, then we can define a convergence in $\mathcal{B}(X,S)$ as follows: If, for a sequence $F_n \in \mathcal{B}(X,S)$, there exist a $\varphi \in S$ and an $F \in \mathcal{B}(X,S)$ such that $\varphi F_n, \varphi F \in X$, for all $n \in \mathbb{N}$, and $\varphi F_n \to \varphi F$ in X, then we write $F_n \xrightarrow{I} F$ in $\mathcal{B}(X,S)$. In other words, $F_n \xrightarrow{I} F$ in $\mathcal{B}(X,S)$ if $F_n = \frac{x_n}{\varphi}$, $F = \frac{x}{\varphi}$, and $x_n \to x$ in X, for some $x_n, x \in X$ and $\varphi \in S$.

This convergence is sometimes referred to as *type I* convergence. It is quite natural, but it need not be topological. For this reason we prefer to use the convergence defined as follows: $F_n \to F$ in $\mathcal{B}(X,S)$ if every subsequence (F_{p_n}) of (F_n) has a subsequence (F_{q_n}) such that $F_{q_n} \xrightarrow{I} F$.

It is easy to show that the embedding $\iota : X \to \mathcal{B}(X,S)$, as well as the extension of any $\varphi \in S$ to a map $\varphi : \mathcal{B}(X,S) \to \mathcal{B}(X,S)$ defined above, are continuous.

3. The main result

In this section we will assume $\mathcal{A}$ to be a nonunital commutative Banach algebra with bounded approximate identities and an isometric and symmetric involution. In addition, we assume that $\mathcal{A}$ satisfies the following condition:

Σ There exists a sequence $a_1, a_2, \ldots \in \mathcal{A}$ such that $\hat{a}_1, \hat{a}_2, \ldots \in \mathcal{K}(\hat{\mathcal{A}})$ and for every $\xi \in \hat{\mathcal{A}}$ there is an n such that $\hat{a}_n(\xi) \neq 0$.

For $a \in \mathcal{A}$, by Λ_a we denote the operation on linear functionals on $\mathcal{A}$ defined by $(\Lambda_a f)(x) = f(ax)$. Let

$$\mathcal{S} = \left\{ \Lambda_a : \hat{a} > 0 \text{ on } \hat{\mathcal{A}} \right\}.$$

Lemma 3.1. *If $\mathcal{A}$ satisfies* **Σ**, *then $\mathcal{S}$ is a nonempty commutative semigroup of injective maps acting on $\mathcal{P}(\mathcal{A})$.*

Proof. Without loss of generality, we may assume that $\hat{a}_n \geq 0$ and that for every $\xi \in \hat{\mathcal{A}}$ there exists an n such that $\hat{a}_n(\xi) > 0$ (otherwise, we take $a_n a_n^*$ instead of a_n). If we choose $\lambda_n > 0$ such that $\sum_{n=1}^{\infty} \|\lambda_n a_n\| < \infty$ and define $a = \sum_{n=1}^{\infty} \lambda_n a_n$, then $\Lambda_a \in \mathcal{S}$.

Clearly, $\mathcal{S}$ is a commutative semigroup. Let $f \in \mathcal{P}(\mathcal{A})$ and $\Lambda_a \in \mathcal{S}$. By Maltese's theorem [5], $f(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu(\xi)$ for some $\mu \in \mathcal{M}_+^b(\hat{\mathcal{A}})$. Thus

$$(\Lambda_a f)(x) = f(ax) = \int_{\hat{\mathcal{A}}} \widehat{ax}(\xi) d\mu(\xi) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi)\hat{a}(\xi) d\mu(\xi).$$

Since $\hat{a}(\xi) > 0$ for all $\xi \in \hat{\mathcal{A}}$, $\hat{a}$ is a positive bounded function on $\hat{\mathcal{A}}$. Thus $\tilde{\mu} = \hat{a}\mu \in \mathcal{M}_+^b(\hat{\mathcal{A}})$ and $\Lambda_a f(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\tilde{\mu}(\xi)$. Consequently $\Lambda_a f \in \mathcal{P}(\mathcal{A})$.

If $\Lambda_a f = 0$, then

$$0 = f(ax) = \int_{\hat{\mathcal{A}}} \widehat{ax}(\xi) d\mu(\xi) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi)\hat{a}(\xi) d\mu(\xi),$$

for all x in $\mathcal{A}$. Therefore $\hat{a}\mu = 0$ and $\mu = 0$, because $\hat{a} > 0$. Thus

$$f(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu(\xi) = 0.$$

Hence Λ_a is injective.

□

The map $\mathcal{F} : \mathcal{P}(\mathcal{A}) \to \mathcal{M}_+^b(\hat{\mathcal{A}})$ defined by Maltese's theorem, can be extended to a map $\mathcal{F} : \mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S}) \to \mathcal{M}_+(\hat{\mathcal{A}})$ in the natural way:

$$\mathcal{F}\left(\frac{f}{\Lambda_a}\right) = \frac{\mathcal{F}(f)}{\hat{a}} = \frac{1}{\hat{a}}\mu_f. \tag{3.1}$$

It is clear that $\mathcal{F}$ is well-defined and that it is injective.

Theorem 3.2. *Let $\mathcal{A}$ be a nonunital commutative Banach algebra with a bounded approximate identity and an isometric and symmetric involution. If $\mathcal{A}$ satisfies $\mathbf{\Sigma}$, then the extended $\mathcal{F}$ defined by (3.1) is an bijection from $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$ to $\mathcal{M}_+(\hat{\mathcal{A}})$.*

Proof. It suffices to show that $\mathcal{F}$ is surjective. Let $\mu \in \mathcal{M}_+(\hat{\mathcal{A}})$. There are $a_n \in \mathcal{A}$ such that $\hat{a}_n \geq 0$, $\operatorname{supp} \hat{a}_n$ is compact, and such that for every $\xi \in \hat{\mathcal{A}}$ there exists an n such that $\hat{a}_n(\xi) > 0$. Then $\hat{a}_n\mu$ is a finite positive Radon measure on $\hat{\mathcal{A}}$ for every $n \in \mathbb{N}$. There exist positive numbers $\lambda_1, \lambda_2, \dots$ such that $\sum_{n=1}^{\infty} \lambda_n \hat{a}_n \mu$ defines a finite positive Radon measure on $\hat{\mathcal{A}}$. By Maltese's theorem there exist $f \in \mathcal{P}(\mathcal{A})$ such that

$$\mu_f = \sum_{n=1}^{\infty} \lambda_n \hat{a}_n \mu.$$

Without loss of generality, we can assume that the numbers $\lambda_1, \lambda_2, \dots$ are chosen such that

$$\sum_{n=1}^{\infty} \|\lambda_n a_n\| < \infty.$$

Let $a = \sum_{n=1}^{\infty} \lambda_n a_n$. Then $\Lambda_a \in \mathcal{S}$ and $\sum_{n=1}^{\infty} \lambda_n \hat{a}_n \mu = \hat{a}\mu$. Thus

$$\mathcal{F}\left(\frac{f}{\Lambda_a}\right) = \frac{\mu_f}{\hat{a}} = \frac{\hat{a}\mu}{\hat{a}} = \mu.$$

□

Theorem 3.3. *The map $\mathcal{F} : \mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S}) \to \mathcal{M}_+(\hat{\mathcal{A}})$ is a sequential homeomorphism.*

Proof. If $F_n \xrightarrow{I} F$ in $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$, then $F_n = \frac{f_n}{\Lambda_a}$, $F = \frac{f}{\Lambda_a}$, and $f_n \to f$ in $\mathcal{P}(\mathcal{A})$ for some $f_n, f \in \mathcal{P}(\mathcal{A})$, where $f_n \to f$ means $f_n(x) \to f(x)$ for all $x \in \mathcal{A}$. Consequently,

$$\int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu_{f_n}(\xi) \to \int_{\hat{\mathcal{A}}} \hat{x}(\xi) d\mu_f(\xi)$$

for all $x \in \mathcal{A}$. Since the involution in $\mathcal{A}$ is symmetric and $\Gamma(\mathcal{A}) = \{\hat{x} : x \in \mathcal{A}\}$ strongly separates points in $\hat{\mathcal{A}}$ (see, for example, Theorem 2.2.7 in [6]), we obtain

$$\int_{\hat{\mathcal{A}}} \varphi(\xi) d\mu_{f_n}(\xi) \to \int_{\hat{\mathcal{A}}} \varphi(\xi) d\mu_f(\xi)$$

for all $\varphi \in \mathcal{K}(\hat{\mathcal{A}})$. Therefore,

$$\int_{\hat{\mathcal{A}}} \varphi(\xi) \frac{d\mu_{f_n}(\xi)}{\hat{a}(\xi)} \to \int_{\hat{\mathcal{A}}} \varphi(\xi) \frac{d\mu_f(\xi)}{\hat{a}(\xi)}$$

for all $\varphi \in \mathcal{K}(\hat{\mathcal{A}})$, which means that $\mathcal{F}(F_n) \to \mathcal{F}(F)$ in $\mathcal{M}_+(\hat{\mathcal{A}})$.

Now assume $\mu_n, \mu \in \mathcal{M}_+(\hat{\mathcal{A}})$ and

$$\int_{\hat{\mathcal{A}}} \varphi(\xi) d\mu_n(\xi) \to \int_{\hat{\mathcal{A}}} \varphi(\xi) d\mu(\xi)$$

for all $\varphi \in \mathcal{K}(\hat{\mathcal{A}})$. There exist $\lambda_k > 0$, $k \in \mathbb{N}$, such that $\sum_{k=1}^{\infty} \lambda_k \hat{a}_k \mu_n$ is a finite measure for all $n \in \mathbb{N}$ and $\Lambda_a \in \mathcal{S}$, where $a = \sum_{k=1}^{\infty} \lambda_k a_k$. Let

$$f_n = \mathcal{F}^{-1}\left(\sum_{k=1}^{\infty} \lambda_k \hat{a}_k \mu_n\right) = \mathcal{F}^{-1}(\hat{a}\mu_n)$$

and

$$f = \mathcal{F}^{-1}\left(\sum_{k=1}^{\infty} \lambda_k \hat{a}_k \mu\right) = \mathcal{F}^{-1}(\hat{a}\mu).$$

Then $\dfrac{f_n}{\Lambda_a} = \mathcal{F}^{-1}(\mu_n)$ and $\dfrac{f}{\Lambda_a} = \mathcal{F}^{-1}(\mu)$. Moreover,

$$f_n(x) = \int_{\hat{\mathcal{A}}} \hat{x}(\xi)\hat{a}(\xi) d\mu_n(\xi) \to \int_{\hat{\mathcal{A}}} \hat{x}(\xi)\hat{a}(\xi) d\mu(\xi) = f(x)$$

for every $x \in \mathcal{A}$. Therefore $\dfrac{f_k}{\Lambda_a} \xrightarrow{I} \dfrac{f}{\Lambda_a}$ in $\mathcal{B}(\mathcal{P}(\mathcal{A}), \mathcal{S})$. □

4. Examples

In this section we give some examples of spaces where the assumptions of Theorem 3.2 are satisfied.

4.1. **Normal algebras.** Let $\mathcal{A}$ be a commutative Banach algebra. We say that $\mathcal{A}$ is normal [7], if for every compact $K \subset \hat{\mathcal{A}}$ and closed $E \subset \hat{\mathcal{A}}$ such that $K \cap E = \emptyset$, there exists $x \in \mathcal{A}$ such that

$$\hat{x}(\xi) = 1 \text{ for } \xi \in K \quad \text{and} \quad \hat{x}(\xi) = 0 \text{ for } \xi \in E.$$

If $\mathcal{A}$ is a normal commutative Banach algebra and $\hat{\mathcal{A}}$ is σ-compact, then $\mathcal{A}$ satisfies condition $\mathbf{\Sigma}$. Indeed, if $\hat{\mathcal{A}}$ is σ-compact, there are compact sets $K_n \subset \hat{\mathcal{A}}$ such that $\hat{\mathcal{A}} = \cup_{n=0}^{\infty} K_n$ and $K_n \subset K_{n+1}^{\circ}$ for all $n \in \mathbb{N}$, where K_{n+1}° is the interior of K_{n+1}. Since $\mathcal{A}$ is regular, for every $n \in \mathbb{N}$ there exists $b_n \in \mathcal{A}$ such that

$$\hat{b}_n(\xi) = \begin{cases} 1 & \text{if } \xi \in K_n \\ 0 & \text{if } \xi \notin K_{n+1}^{\circ} \end{cases}.$$

Let $a_n = b_n b_n^*$. Then $\hat{a}_n = |\hat{b}_n|^2 \geq 0$ and $K_n \subset \operatorname{supp} \hat{a}_n \subset K_{n+2}$. Clearly, for every $\xi \in \hat{\mathcal{A}}$, there exists n such that $\hat{a}_n > 0$.

Note that a regular commutative Banach algebra is normal, [7].

4.2. **Algebras with σ-compact-open structure spaces.** For our next example we use Shilov's idempotent theorem [10].

Theorem 4.1 (Shilov)**.** *Let $\mathcal{A}$ be a commutative Banach algebra. If K is a compact and open subset of $\hat{\mathcal{A}}$, then there is a unique idempotent $a \in \mathcal{A}$ such that $\hat{a}$ is the characteristic function of K.*

Let $\mathcal{A}$ be a commutative Banach algebra such that $\hat{\mathcal{A}}$ is σ-compact-open, that is, $\hat{\mathcal{A}} = \cup_{n=0}^{\infty} K_n$ where K_n are disjoint compact and open sets in the Gelfand topology in $\hat{\mathcal{A}}$. Since, by Shilov's idempotent theorem, for every $n \in \mathbb{N}$ there exist a unique idempotent $a_n \in \mathcal{A}$ such that $\operatorname{supp} \hat{a}_n = K_n$, $\mathcal{A}$ satisfies condition $\mathbf{\Sigma}$.

4.3. **Locally compact groups.** Let G be a locally compact abelian group. A continuous function $f : G \to \mathbb{C}$ is called positive definite if

$$\sum_{k,l=1}^{n} c_k \overline{c_l} f(x_l^{-1} x_k) \geq 0$$

for all $c_1, \ldots, c_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in G$ for any $n \in \mathbb{N}$. We denote the cone of positive definite functions on G by $\mathcal{P}_+(G)$. A character α on G is a continuous homomorphism from G into the unit circle group $\mathbb{T}$. Let $\widehat{G}$ denote the group of characters. By Bochner's theorem [4], $f \in \mathcal{P}_+(G)$ if and only if there exists a unique bounded positive Radon measure μ_f on $\widehat{G}$ such that

$$f(x) = \int_{\widehat{G}} \hat{x} d\mu_f.$$

In [2] it was shown that, if $\widehat{G}$ is σ-compact, then the map $f \mapsto \mu_f$ defined by Bochner's theorem can be extended to a map from a space of pseudoquotients to all positive measures on $\widehat{G}$. That space of pseudoquotients was $\mathcal{B}(\mathcal{P}_+(G), \mathcal{S})$ where

$$\mathcal{S} = \left\{ \varphi \in L^1(G) : \widehat{\varphi}(\xi) > 0 \text{ for all } \xi \in \widehat{G} \right\}.$$

We will show that this extension is a special case of the extension presented in this note.

Since the convolution algebra $L^1(G)$ is regular, it satisfies $\mathbf{\Sigma}$, as indicated in 4.1. For $\alpha \in \widehat{G}$ we define $\varphi_\alpha : L^1(G) \to \mathbb{C}$ by

$$\varphi_\alpha(f) = \int_G f(x) \overline{\alpha(x)} dx,$$

where dx indicates the integral with respect to the Haar measure on G. The map $\alpha \mapsto \varphi_\alpha$ is a bijection from $\widehat{G}$ onto $\widehat{L^1(G)}$ (see, for example, [6]). This allows us to identify $\mathcal{M}_+(\widehat{G})$ and $\mathcal{M}_+(\widehat{L^1(G)})$. If f is a positive definite function on G, we define a positive functional on $L^1(G)$ by

$$F(\varphi) = \int_G f(x) \varphi(x) dx$$

and a map from $\mathcal{B}(\mathcal{P}_+(G), \mathcal{S})$ to $\mathcal{B}(L^1(G), \mathcal{S})$ by $\dfrac{f}{\varphi} \mapsto \dfrac{F}{\Lambda_{\tilde{\varphi}}}$, where $\tilde{\varphi}(x) = \varphi(x^{-1})$. Since $\mathcal{B}(L^1(G), \mathcal{S})$ is isomorphic with $\mathcal{M}_+(\widehat{L^1(G)})$, by Theorem 3.2, there is a bijection from $\mathcal{B}(\mathcal{P}_+(G), \mathcal{S})$ to $\mathcal{M}_+(\hat{G})$.

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