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COMPACT OPERATORS IN THE COMMUTANT OF ESSENTIALLY NORMAL OPERATORS

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Abstract. Let T be a bounded, linear operator on a complex, separable, infinite dimensional Hilbert space H. We assume that T is an essentially isometric (resp. normal) operator, that is, $I_H - T^*T$ (resp. $TT^* - T^*T$) is compact. For the compactness of S from the commutant of T, some necessary and sufficient conditions are found on S. Some related problems are also discussed.

1. Introduction and preliminaries

Let H be a complex, separable, infinite dimensional Hilbert space and let $B(H)$ be the algebra of all bounded, linear operators on H. As usual, we denote the spectrum (resp. left, right) of $T \in B(H)$ by $\sigma(T)$ (resp. $\sigma_l(T), \sigma_r(T)$). The unit circle in the complex plane will be denoted by Γ, whereas D indicates the open unit disk. The disc-algebra and the algebra of all bounded analytic functions on D are denoted by $A(D)$ and $H^\infty := H^\infty(D)$, respectively.

If $T \in B(H)$, we let A_T denote the closure in the uniform operator topology of all polynomials in T. Notice that A_T is a commutative unital Banach algebra. The Gelfand space of A_T can be identified with $\sigma_{A_T}(T)$, the spectrum of T with respect to the algebra A_T. Since $\sigma(T)$ is a (closed) subset of $\sigma_{A_T}(T)$, for every $\lambda \in \sigma(T)$ there exists a multiplicative functional ϕ_λ on A_T such that $\phi_\lambda(T) = \lambda$. By $\widehat{S}$, we will denote the Gelfand transform of $S \in A_T$. Here and in the sequel,

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instead of $\widehat{S}(\phi_\lambda)\,(=\phi_\lambda(S))$, where $\lambda\in\sigma(T)$, we will use the notation $\widehat{S}(\lambda)$. Notice that $\lambda\mapsto\widehat{S}(\lambda)$ is a continuous function on $\sigma(T)$.

Recall that $\sigma(T)\cap\Gamma$ is called the *unitary spectrum* of $T\in B(H)$. It follows from the Shilov's Theorem [7, Theorem 2.3.1] that if T is a contraction, then

$$\sigma_{A_T}(T)\cap\Gamma=\sigma(T)\cap\Gamma.$$

A contraction T on H is said to be *completely nonunitary* (c.n.u.) if it has no proper reducing subspace on which it acts as a unitary operator. If T is a c.n.u. contraction, then $f(T)$ $(f\in H^\infty)$ can be defined by the Nagy–Foias functional calculus [13, Chapter III]. We put $H^\infty(T):=\{f(T):f\in H^\infty\}$. A c.n.u. contraction T is called a C_0-*contraction* if there exists a nonzero function $f\in H^\infty$ such that $f(T)=0$. B. Sz.-Nagy [12] proved that if T is a C_0-contraction, then the commutant $\{T\}':=\{S\in B(H):TS=ST\}$ of T contains a nonzero compact operator, but there exists a C_0-contraction T such that zero is the unique compact operator contained in $H^\infty(T)$. An operator $T\in B(H)$ is said to be *essentially unitary* if both I_H-T^*T and I_H-TT^* are compact. Nordgren [16] proved that if T is an essentially unitary C_0-contraction, then $H^\infty(T)$ contains a nonzero compact operator.

If T is a contraction on H, then it follows from the von Neumann inequality that there exists a contractive algebra homomorphism $h:A(D)\to A_T$ (with dense range) such that $h(1)=I_H$ and $h(z)=T$. We will use the notation $f(T):=h(f)$, $f\in A(D)$. Thus we have $\|f(T)\|\le\|f\|_\infty$ for all $f\in A(D)$.

Recall that $T\in B(H)$ is called *essentially isometric operator* if I_H-T^*T is compact. Kellay and Zarrabi [6] proved that if the essentially isometric contraction T satisfies the condition $D\diagdown\sigma(T)\neq\emptyset$ (it follows that T is a compact perturbation of a unitary operator and therefore it is essentially unitary) and if $f\in A(D)$ vanishes on $\sigma(T)\cap\Gamma$, then $f(T)$ is compact. Notice that under the above conditions the Lebesgue measure of $\sigma(T)\cap\Gamma$ is necessarily zero. In [6], it is also shown that if T is an essentially isometric C_0-contraction, then $f(T)$ $(f\in H^\infty)$ is compact if and only if $\lim_{n\to\infty}\|T^nf(T)\|=0$. The proofs of these results essentially use the Beurling–Rudin theorem about the structure of closed ideals of $A(D)$ and the corona theorem.

By $K(H)$ we will denote the ideal of compact operators on H. The quotient algebra $B(H)\diagup K(H)$ is a C^*-algebra called the *Calkin algebra*. Let $\pi:B(H)\to B(H)\diagup K(H)$ be the canonical map. The *essential spectrum* $\sigma_e(T)$ of $T\in B(H)$ is the spectrum of $\pi(T)$ in the Calkin algebra. As is well known, $\sigma_e(T)$ is a nonempty compact subset of $\sigma(T)$. Similarly, the *left* and *right essential spectrum* of T are defined by $\sigma_{le}(T):=\sigma_l(\pi(T))$ and $\sigma_{re}(T):=\sigma_r(\pi(T))$. Recall also that $T\in B(H)$ is a (*left, right*) *Fredholm operator* if $\pi(T)$ is (left, right) invertible in the Calkin algebra.

The main results of this note can be summarized as follows. If T is a c.n.u. contraction and $S\in A_T$ is compact, then $\widehat{S}$ vanishes on $\sigma(T)\cap\Gamma$. If T is an essentially isometric operator and if the Gelfand transform of $S\in A_T$ vanishes on $\sigma_{le}(T)$ (or on $\sigma_{re}(T)\cap\Gamma$), then S is compact. In addition if T is a c.n.u.

contraction, then $S \in \{T\}'$ is compact if and only if

$$\lim_{n\to\infty} \|T^n S\| = 0.$$

Furthermore, the compactness of $S \in \{T\}'$ characterized via the ergodic conditions. If

$$\varliminf_{n\to\infty} \frac{1}{n} \left\| \sum_{k=1}^{n} \overline{\xi}^k T^k S \right\| = 0$$

holds for every $\xi \in \sigma_{le}(T)$ (or $\xi \in \sigma_{re}(T) \cap \Gamma$), then S is compact.

Similar results for essentially normal operators are also obtained. Let T be an essentially normal operator. If the Gelfand transform of $S \in A_T$ vanishes on $\sigma_e(T)$, then S is compact. In addition if T is a Fredholm operator and if

$$\varliminf_{n\to\infty} \frac{1}{n} \left\| \sum_{k=1}^{n} \lambda^{-k} T^k S \right\| = 0$$

holds for every $\lambda \in \sigma_e(T)$, then $S \in \{T\}'$ is compact.

2. Essentially isometric operators

Let T be an essentially isometric operator, that is, $I_H - T^*T$ is compact. In this section, for the compactness of the operator S from the commutant of T, we give some necessary and sufficient conditions on S.

We start with the following result.

Proposition 2.1. *Let T be a c.n.u. contraction on H and let $S \in A_T$. If S is compact, then its Gelfand transform vanishes on $\sigma(T) \cap \Gamma$.*

Proof. We know [9, Lemma 3.3] that if T is a c.n.u. contraction, then $T^n \to 0$ in the weak operator topology. If $S \in A_T$ is compact, then for arbitrary $x \in H$ we can write

$$\lim_{n\to\infty} \|T^n Sx\| = \lim_{n\to\infty} \|ST^n x\| = 0.$$

Since the set $\{Sx : \|x\| \leq 1\}$ is relatively compact, for a given $\varepsilon > 0$ it has a finite ε-mesh, say $\{Sx_1, \cdots, Sx_k\}$, where $\|x_i\| \leq 1$ $(i = 1, \cdots, k)$. Consequently, we have

$$\|T^n S\| \leq \max_i \{\|T^n Sx_i\|\} + \varepsilon \ \ (n \in \mathbb{N}).$$

It follows that $\lim_{n\to\infty} \|T^n S\| = 0$. On the other hand, for every $\xi \in \sigma(T) \cap \Gamma$ there exists a multiplicative functional ϕ_ξ on A_T such that $\phi_\xi(T) = \xi$. Since ϕ_ξ has norm one, we have

$$\left|\widehat{S}(\xi)\right| = |\phi_\xi(T^n S)| \leq \|T^n S\| \to 0 \ \ (n \to \infty).$$

∎

Next, we have the following

Theorem 2.2. *Let T be an essentially isometric operator. If the Gelfand transform of $S \in A_T$ vanishes on $\sigma_{le}(T)$ (or on $\sigma_{re}(T) \cap \Gamma$), then S is compact.*

For the proof we need some preliminary results.

Let A be a C^*-algebra with the unit element e and let S_A be the set of all pure states on A. We know [14, Corollary V.23.3] that if $a \in A$, then $\sigma_l(a)$ consists of all $\lambda \in \mathbb{C}$ for which there exists $f \in S_A$ such that $\lambda = f(a)$ and $f(a^*a) = f(a^*) f(a)$. Assume that $a^*a = e$. If $\lambda \in \sigma_l(a)$, then we have

$$|\lambda|^2 = \overline{f(a)} f(a) = f(a^*) f(a) = f(a^*a) = f(e) = 1.$$

This shows that $\sigma_l(a) \subset \Gamma$. Similarly, we can see that if a is a normal element of A, then $\sigma_l(a) = \sigma_r(a) = \sigma(a)$. In particular, if a is a unitary element of A, then $\sigma_l(a) = \sigma_r(a) = \sigma(a) \subset \Gamma$.

Let T be an essentially isometric operator on H. Since $\pi(T)^* \pi(T) = \pi(I_H)$, it follows from what is showed above that $\sigma_{le}(T) = \sigma_l(\pi(T)) \subset \Gamma$. Notice also that if T is essentially unitary, then $\sigma_{le}(T) = \sigma_{re}(T) = \sigma_e(T) \subset \Gamma$.

The following result is probably known. Not being able to find a ready reference, we include a proof of it.

Proposition 2.3. (a) *If V is a nonunitary isometry on H, then*

$$\sigma_l(V) = \Gamma; \ \sigma_r(V) = \sigma(V) = \overline{D}.$$

(b) *If V is an arbitrary isometry on H, then*

$$\sigma_l(V) = \sigma_r(V) \cap \Gamma = \sigma(V) \cap \Gamma.$$

Proof. (a) As we have seen above, $\sigma_l(V) \subset \Gamma$. On the other hand, we know that if V is nonunitary isometry, then $\sigma(V) = \overline{D}$. It follows that $\Gamma = \partial\sigma(V) \subset \sigma_l(V)$.

Let $\lambda \in D$. Since $\lambda \in \sigma(V)$, from the relation

$$\|(V - \lambda I_H) x\| \geq (1 - |\lambda|) \|x\| \quad (x \in H)$$

we deduce that the range of $V - \lambda I_H$ is closed and $(V - \lambda I_H) H \neq H$. Consequently, $V^*x = \overline{\lambda} x$ for some $x \in H \setminus \{0\}$. On the other hand, we know that for any $T \in B(H)$,

$$\sigma_r(T) = \left\{\lambda \in \mathbb{C} : \inf \left\| \left(T^* - \overline{\lambda}\right) x \right\| = 0, \ \ \|x\| = 1\right\}$$

[3, p.200]. It follows that $\lambda \in \sigma_r(V)$ and therefore, $D \subset \sigma_r(V)$. Since $\sigma_r(V)$ is closed, we have $\sigma_r(V) = \overline{D}$.

(b) follows from (a) and the fact that if V is unitary, then $\sigma_l(V) = \sigma_r(V) = \sigma(V)$. □

As we have seen above if V is a nonunitary isometry, then $\sigma(V) = \overline{D}$. It follows from the von Neumann inequality and the spectral theorem that for an arbitrary isometry V on H,

$$\|f(V)\| = \sup_{\xi \in \sigma(V) \cap \Gamma} |f(\xi)|, \ \forall f \in A(D). \tag{2.1}$$

Let H_0 be the linear space of all weakly null sequences $\{x_n\}$ in H. Let us define a semi-inner product on H_0 by

$$\langle \{x_n\}, \{y_n\} \rangle = \text{l.i.m.}_n \langle x_n, y_n \rangle,$$

where l.i.m. is a Banach limit. Let

$$E=\left\{\{x_n\}\in H_0 : \text{l.i.m.}_n \left\|x_n\right\|^2=0\right\}.$$

Then, $H_0\diagup E$ becomes a pre-Hilbert space with respect to the inner product defined by

$$\langle\{x_n\}+E,\{y_n\}+E\rangle=\text{l.i.m.}_n\langle x_n,y_n\rangle .$$

Let $\mathcal{H}$ be the completion of $H_0\diagup E$ with respect to the induced norm given by

$$\|\{x_n\}+E\|=\left(\text{l.i.m.}_n\|x_n\|^2\right)^{\frac{1}{2}}.$$

Then, $\mathcal{H}$ is a Hilbert space.

For a given $T\in B(H)$, define the operator $\mathcal{T}$ on $H_0\diagup E$ by

$$\mathcal{T}:\{x_n\}+E\mapsto\{Tx_n\}+E.$$

Then we have

$$\begin{aligned}\|\mathcal{T}(\{x_n\}+E)\| &= \left(\text{l.i.m.}_n\|Tx_n\|^2\right)^{\frac{1}{2}}\\ &\leq \|T\|\left(\text{l.i.m.}_n\|x_n\|^2\right)^{\frac{1}{2}}\\ &= \|T\|\,\|\{x_n\}+E\|.\end{aligned}$$

Since $H_0\diagup E$ is dense in $\mathcal{H}$, the operator $\mathcal{T}$ can be extended to the whole $\mathcal{H}$ which we also denote by $\mathcal{T}$. Clearly, $\|\mathcal{T}\|\leq\|T\|$. The pair $(\mathcal{H},\mathcal{T})$ (sometimes the operator $\mathcal{T}$) will be called the *limit operator* associated with T (see also [11]).

Proposition 2.4. *Let $T\in B(H)$ and let $(\mathcal{H},\mathcal{T})$ be the limit operator associated with T. The following assertions hold:*

(a) The mapping $T\mapsto\mathcal{T}$ is a contractive algebra $*-$homomorphism.

(b) T is compact if and only if $\mathcal{T}=0$.

(c) $\sigma_l(\mathcal{T})\subset\sigma_{le}(T)$, $\sigma_r(\mathcal{T})\subset\sigma_{re}(T)$, and $\sigma(\mathcal{T})\subset\sigma_e(T)$.

(d) T is an essentially isometric (resp. essentially unitary, essentially normal) operator if and only if $\mathcal{T}$ is an isometry (resp. unitary, normal).

(e) If T is an essentially isometric operator and if $\sigma_{le}(T)\neq\Gamma$ $\left(\text{or } \sigma_{re}(T)\neq\overline{D}\right)$, then T is essentially unitary.

Proof. The proof of (a) being very easy is omitted.

(b) It is obvious that if T is compact, then $\mathcal{T}=0$. If $\mathcal{T}=0$, then for every weakly null sequence $\{x_n\}_{n\in\mathbb{N}}$ in H, we have $\text{l.i.m.}_n\|Tx_n\|^2=0$. Consequently, there is a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that

$$\overline{\lim_{n\to\infty}}\|Tx_n\|^2=\lim_{k\to\infty}\|Tx_{n_k}\|^2=\text{l.i.m.}_k\|Tx_{n_k}\|^2=0.$$

It follows that $\lim_{n\to\infty}\|Tx_n\|=0$ and therefore T is compact.

(c) If $\lambda\notin\sigma_{le}(T)$, then λI_H-T is a left Fredholm operator. So, there exists $S\in B(H)$ such that $S(\lambda I_H-T)-I_H\in K(H)$. It follows from (a) and (b) that $\mathcal{S}(\lambda I_{\mathcal{H}}-\mathcal{T})=I_{\mathcal{H}}$, where $\mathcal{S}$ is the limit operator associated with S. This shows that $\lambda\notin\sigma_l(\mathcal{T})$. The proofs of the second and third parts of (c) are similar.

(d) is an immediate consequence of (a) and (b).

(e) Assume that T is an essentially isometric operator and $\sigma_{le}(T)\neq\Gamma$ (or $\sigma_{re}(T)\neq\overline{D}$). By (c) we have $\sigma_l(\mathcal{T})\neq\Gamma$ (or $\sigma_r(\mathcal{T})\neq\overline{D}$). Since $\mathcal{T}$ is an isometry, it follows from Proposition 2.3 that $\mathcal{T}$ is unitary. Consequently, both $I - T^*T$ and $I - TT^*$ are compact. □

We are now able to prove Theorem 2.2.

Proof of Theorem 2.2. Assume that the Gelfand transform of $S \in A_T$ vanishes on $\sigma_{le}(T)$ (or on $\sigma_{re}(T) \cap \Gamma$). Since $S \in A_T$, there exists a sequence of polynomials $\{P_n\}$ such that

$$\lim_{n\to\infty} \|P_n(T) - S\| = 0.$$

Let $\mathcal{T}$ and $\mathcal{S}$ be the limit operators associated with T and S, respectively. In view of Proposition 2.4 (a), we have

$$\lim_{n\to\infty} \|P_n(\mathcal{T}) - \mathcal{S}\| = 0.$$

On the other hand, for every $\xi \in \sigma_{le}(T)$ ($\xi \in \sigma_{re}(T) \cap \Gamma$) there exists a multiplicative functional ϕ_ξ on A_T such that $\phi_\xi(T) = \xi$. Consequently, we have

$$\begin{aligned} |P_n(\xi)| &= \left|P_n(\xi) - \widehat{S}(\xi)\right| \\ &= |\phi_\xi(P_n(T) - S)| \\ &\leq \|P_n(T) - S\|. \end{aligned}$$

From this we deduce that $\lim_{n\to\infty} P_n(\xi) = 0$ uniformly on $\sigma_{le}(T)$ (on $\sigma_{re}(T) \cap \Gamma$). Further, it follows from Proposition 2.4 (d), (c), and Proposition 2.3 that $\mathcal{T}$ is an isometry and

$$\begin{aligned} \sigma(\mathcal{T}) \cap \Gamma &= \sigma_l(\mathcal{T}) \subset \sigma_{le}(T) \\ (\sigma(\mathcal{T}) \cap \Gamma &= \sigma_r(\mathcal{T}) \cap \Gamma \subset \sigma_{re}(T) \cap \Gamma). \end{aligned}$$

Consequently, $\lim_{n\to\infty} P_n(\xi) = 0$ uniformly on $\sigma(\mathcal{T}) \cap \Gamma$. Now, taking into account the identity (2.1), we obtain

$$\lim_{n\to\infty} \|P_n(\mathcal{T})\| = 0.$$

Hence, $\mathcal{S} = 0$. By Proposition 2.4 (b), S is compact. □

The following proposition is an improvement of [6, Proposition 2.5] and shows that the condition "T is essentially isometric" is necessary in the Theorem 2.2.

Proposition 2.5. (*a*) *Let T be a contraction on H and let K be a closed subset of Γ of Lebesgue measure zero. Assume that $f(T)$ is compact for every $f \in A(D)$ vanishing on K. Then, T is essentially unitary and $\sigma_e(T) \subset K$.*

(*b*) *Let T be an essentially unitary, but nonunitary contraction such that $\sigma_e(T)$ is of Lebesgue measure zero. Then, there exists $f \in A(D)$ such that $f(T)$ is a nonzero compact operator.*

Proof. (a) Let $\pi : B(H) \to B(H)/K(H)$ be the canonical map. By Rudin–Carleson Theorem [1, Theorem VIII.7.4], there exists $f \in A(D)$ such that $f(\xi) = \overline{\xi}$ for all $\xi \in K$ and $\|f\|_\infty = 1$. Since the function $zf(z) - 1$ vanishes on K, the operator $Tf(T) - I_H$ is compact. Consequently, we have $\pi(T)\pi(f(T)) = \pi(I_H)$. This imply that $\pi(T)$ is invertible and

$$\left\|\pi(T)^{-1}\right\| = \|\pi(f(T))\| \leq \|f(T)\| \leq \|f\|_\infty \leq 1.$$

Since $\|\pi(T)\| \leq 1$, we have $\|\pi(T)\| = \left\|\pi(T)^{-1}\right\| = 1$. This shows that $\pi(T)$ is unitary and therefore T is essentially unitary.

We have $\sigma_e(T) \subset \Gamma$. Let us show that $\sigma_e(T) \subset K$. Let $\xi_0 \in \Gamma \setminus K$. By Rudin–Carleson Theorem, there exists $f \in A(D)$ such that $f(\xi) = (\xi_0 - \xi)^{-1}$ on K. Since the function $(\xi_0 - z) f(z) - 1$ vanishes on K, the operator $(\xi_0 I_H - T) f(T) - I_H$ is compact. This shows that $\xi_0 \notin \sigma_e(T)$.

(b) By Theorem 2.2, it suffices to show that there exists $f \in A(D)$ such that f vanishes on $\sigma_e(T)$, but $f(T) \neq 0$. Assume on the contrary that $f(T) = 0$ for every $f \in A(D)$ vanishing on $\sigma_e(T)$. By Rudin–Carleson Theorem, there exists $f \in A(D)$ such that $f(\xi) = \overline{\xi}$ for all $\xi \in K$ and $\|f\|_\infty = 1$. Since the function $zf(z) - 1$ vanishes on K, we have $Tf(T) = I_H$. Consequently, T is invertible and

$$\left\|T^{-1}\right\| = \|f(T)\| \leq \|f\|_\infty \leq 1.$$

Thus we have $\|T\| = \|T^{-1}\| = 1$. This shows that T is unitary. This is a contradiction. ☐

Recall [15, III.1] that the *spectrum* $\Sigma(\varphi)$ of an inner function φ is defined by

$$\Sigma(\varphi) = \overline{\varphi^{-1}(0)} \cup \operatorname{supp}\mu,$$

where μ is the singular measure associated to the singular part of φ. As is known [13, Proposition III.4.4] if T is a C_0-contraction, then there exists a minimal inner function m_T that annihilates T, i.e., $m_T(T) = 0$ and we have $\sigma(T) = \Sigma(m_T)$. Now, it follows from Proposition 2.4 (e) that if T is an essentially isometric C_0-contraction, then it is essentially unitary. In fact, T is a compact perturbation of a unitary operator [6, 17].

Corollary 2.6. *If T is an essentially isometric C_0-contraction on H, then there exist a nonzero $T-$invariant subspace E and $f \in A(D)$ such that $f(T\mid_E)$ is a nonzero compact operator.*

Proof. Let m_T be the minimal inner function that annihilates T. Then, there exists an inner function θ such that θ divides m_T and $\Sigma(\theta) \cap \Gamma$ is of Lebesgue measure zero [6, 16]. Let

$$\psi := \frac{m_T}{\theta};\ E := \overline{\psi(T) H}.$$

The minimality of m_T implies that $E \neq \{0\}$ and $T\mid_E$ is a C_0-contraction with $m_{T\mid_E} = \theta$. Moreover, the operator

$$I_E - (T\mid_E)^* (T\mid_E) = P_E (I_H - T^*T)\mid_E$$

is compact, where P_E is the orthogonal projection from H onto E. As we already noted above, essentially isometric C_0-contractions are essentially unitary. Thus, $T\mid_E$ is an essentially unitary (but nonunitary) contraction and $\sigma(T\mid_E) \cap \Gamma$ $(= \Sigma(\theta) \cap \Gamma)$ is of Lebesgue measure zero. By Proposition 2.5 (b), there exists $f \in A(D)$ such that $f(T\mid_E)$ is a nonzero compact operator. ☐

We already noted in the introduction that if T is an essentially unitary C_0-contraction, then $H^\infty(T)$ contains a nonzero compact operator [16]. Notice that this result can be derived form the preceding corollary as follows. Let ψ, E, and f be as in the proof of Corollary 2.6. Then, $f(T|_E)$ is a nonzero compact operator. Now, from the identity $f(T)\psi(T) = f(T|_E)\psi(T)$ we deduce that $f(T)\psi(T)$ is a nonzero compact operator, which is contained in $H^\infty(T)$.

If T is a contraction on H, then there exists a canonical decomposition of H into two $T-$invariant subspaces $H = H_0 \oplus H_u$ such that $T_0 := T|_{H_0}$ is a c.n.u. contraction and $T_u := T|_{H_u}$ is unitary [13, I.3.2]. It can be seen that $\sigma(T_u) \subset \sigma(T) \cap \Gamma$. For a nonempty closed subset S of Γ, by H_S^∞ we denote the set of all those f in H^∞ that have a continuous extension $\overline{f}$ to $D \cup S$. If $f \in H^\infty_{\sigma(T)\cap\Gamma}$ with continuous extension $\overline{f}$ to $D \cup (\sigma(T) \cap \Gamma)$, then we can define $f(T) \in B(H)$, by

$$f(T) = f(T_0) \oplus \overline{f}(T_u),$$

where $f(T_0)$ is given by the Nagy–Foias functional calculus and

$$\overline{f}(T_u) = \left(\overline{f}|_{\sigma(T)\cap\Gamma}\right)(T_u)$$

is defined by the usual functional calculus for continuous functions of a unitary operator (see also [5]). Notice that

$$\|f(T)\| \leq \|f\|_\infty, \ \forall f \in H^\infty_{\sigma(T)\cap\Gamma}.$$

Now, let $f \in H^\infty_{\sigma(T)\cap\Gamma}$ with continuous extension $\overline{f}$ to $D \cup (\sigma(T) \cap \Gamma)$. By the Gamelin-Garnett Theorem [4], there exists a sequence $\{f_n\}$ in H^∞ such that

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0$$

and each f_n has an analytic extension g_n to an open set O_n containing $D \cup (\sigma(T) \cap \Gamma)$. Then, $g_n(T)$ can be defined by the Riesz-Dunford functional calculus and coincides with $f_n(T)$, where $f_n(T)$ is defined as above. Notice also that $g_n(T)$ is in A_T. Consequently, we have

$$\|g_n(T) - f(T)\| = \|f_n(T) - f(T)\| \leq \|f_n - f\|_\infty \to 0 \ \ (n \to \infty).$$

It follows that $f(T) \in A_T$.

The next corollary is now an immediate consequence of Theorem 2.2.

Corollary 2.7. *Let T be an essentially isometric contraction and let $f \in H^\infty_{\sigma(T)\cap\Gamma}$ with continuous extension $\overline{f}$ to $D \cup (\sigma(T) \cap \Gamma)$. If $\overline{f}(\xi) = 0$ on $\sigma(T) \cap \Gamma$, then $f(T)$ is compact.*

Corollary 2.8. *Let T be an essentially unitary c.n.u. contraction such that $\sigma(T) \cap \Gamma$ is of Lebesgue measure zero. Then,*

$$\sigma_e(T) = \sigma(T) \cap \Gamma.$$

Proof. Assume on the contrary that there exists $\xi_0 \in \sigma(T) \cap \Gamma$, but $\xi_0 \notin \sigma_e(T)$. Then, there exists a continuous function f_0 on $\sigma(T) \cap \Gamma$ such that $f_0(\xi_0) \neq 0$ and $f_0(\xi) = 0$ for all $\xi \in \sigma_e(T)$. Let $f \in A(D)$ be the Rudin–Carleson extension of f_0. By Theorem 2.2, $f(T)$ is compact. On the other hand, it follows from Proposition 2.1 that f vanishes on $\sigma(T) \cap \Gamma$. This contradicts $f_0(\xi_0) \neq 0$. □

Recall that a contraction T on H is said to be of class C_{00} if $T^n x \to 0$ and $T^{*n}x \to 0$ for every $x \in H$.

Assume that the contraction T is of class C_{00}. Moreover, assume that

$$\dim (I - TT^*) H = \dim (I - T^*T) H = 1$$

(consequently, T is essentially unitary). According to the well-known model theorem of Nagy–Foias [13, 15], T is unitary equivalent to its model operator $M_\varphi = P_\varphi S \mid_{K_\varphi}$ acting on the model space $K_\varphi := H^2 \ominus \varphi H^2$, where φ is an inner function, $Sf = zf$ is the shift operator on the Hardy space H^2, and P_φ is the orthogonal projection from H^2 onto K_φ. It follows that for every $f \in H^\infty$, the operator $f(T)$ is unitary equivalent to

$$f(M_\varphi) := P_\varphi f(S) \mid_{K_\varphi} .$$

As is known [15, p.235], $\{T\}' = \{f(T) : f \in H^\infty\}$. By Hartman-Sarason theorem [15, p.235], $f(T)$ $(f \in H^\infty)$ is compact if and only if $\lim_{n\to\infty} \|T^n f(T)\| = 0$.

We have the following

Theorem 2.9. *If T is an essentially isometric c.n.u. contraction, then $S \in \{T\}'$ is compact if and only if*

$$\lim_{n\to\infty} \|T^n S\| = 0.$$

Proof. Assume that $S \in \{T\}'$ is compact. Since T is a c.n.u. contraction, $T^n \to 0$ in the weak operator topology. Consequently, for every $x \in H$ we can write

$$\lim_{n\to\infty} \|T^n Sx\| = \lim_{n\to\infty} \|ST^n x\| = 0.$$

As in the proof of Proposition 2.1, we have

$$\lim_{n\to\infty} \|T^n S\| = 0.$$

Let $\mathcal{T}$ and $\mathcal{S}$ be the limit operators associated with T and S, respectively. By Proposition 2.4 (a),

$$\|\mathcal{T}^n \mathcal{S}\| \le \|T^n S\|, \ \forall n \in \mathbb{N}.$$

Since $\mathcal{T}$ is an isometry, we have

$$\|\mathcal{S}\| \le \lim_{n\to\infty} \|T^n S\| = 0,$$

so that $\mathcal{S} = 0$. By Proposition 2.4 (b), S is compact. □

Note that the preceding theorem contains the main results of [6].

In the proof of the following proposition we use the dilation arguments of Nagy–Foias (see, [13, p.140] and [17, Theorem 3.3]).

Proposition 2.10. *Let T be a c.n.u. contraction on H. Assume that there exists a nonzero function $f \in H^\infty$ such that $f(T)$ is compact. Then for every $S \in K(H)$, we have*

$$\lim_{n\to\infty} \|T^n S\| = \lim_{n\to\infty} \|ST^n\| = 0.$$

Proof. Assume that $f(T)$ is compact for some nonzero $f \in H^{\infty}$. Since T is a c.n.u. contraction, $T^n \to 0$ in the weak operator topology and therefore,

$$\lim_{n\to\infty} \|T^n f(T) x\| = 0, \forall x \in H.$$

Let $f = f_i f_e$ be the canonical inner-outer factorization of f, where f_i is inner and f_e is outer function. Since $f_e(T)$ has dense range [13, Proposition III.3.1], we have

$$\lim_{n\to\infty} \|T^n f_i(T) x\| = 0, \ \forall x \in H.$$

If U is the minimal unitary dilation of T, then

$$\lim_{n\to\infty} U^{-n} T^n x = Px,$$

where P is the orthogonal projection onto the residual part of the dilation space [13, Proposition II.3.1]. It follows that

$$\lim_{n\to\infty} \|T^n x\| = \|Px\| \quad (x \in H).$$

Let us show that $Px = 0$. We can write

$$U^{-m} P T^m x = \lim_{n\to\infty} U^{-m-n} T^{m+n} x = Px,$$

which implies

$$PT^m x = U^m P x \ \ (m \in \mathbb{N}).$$

Consequently, we have

$$PT^n f_i(T) x = U^n f_i(U) Px \ \ (n \in \mathbb{N}).$$

Since $f_i(U)$ is unitary, we can write

$$\begin{aligned} \|Px\| &= \|U^n f_i(U) Px\| \\ &= \|PT^n f_i(T) x\| \\ &\leq \|T^n f_i(T) x\| \to 0 \ \ (n \to \infty). \end{aligned}$$

Hence we have $\lim_{n\to\infty} \|T^n x\| = 0$ which implies that

$$\lim_{n\to\infty} \|T^n S x\| = 0 \text{ for every } S \in B(H) \text{ and } x \in H.$$

As in the proof of Proposition 2.1 we can see that if $S \in K(H)$, then

$$\lim_{n\to\infty} \|T^n S\| = 0.$$

Taking into account the fact that $f(T)^* = \widetilde{f}(T^*)$, where $\widetilde{f}(z) = \overline{f(\overline{z})}$, we can apply the above result to T^* to obtain

$$\lim_{n\to\infty} \|T^{*n} S^* x\| = 0 \text{ for every } S \in B(H) \text{ and } x \in H.$$

It follows that if $S \in K(H)$, then $\lim_{n\to\infty} \|ST^n\| = 0$. □

The following result is of independent interest (for related results see [10]).

Proposition 2.11. *Let T be an essentially unitary c.n.u. contraction on H such that $\sigma_e(T)$ is of Lebesgue measure zero. For every $S \in A_T$, we have*

$$dist\,(S, A_T \cap K(H)) = \sup_{\xi \in \sigma_e(T)} \left|\widehat{S}(\xi)\right|.$$

Proof. Let $S \in A_T$, $K \in A_T \cap K(H)$, and $\xi \in \sigma_e(T)$ be given. There exists a multiplicative functional ϕ_ξ on A_T such that $\phi_\xi(T) = \xi$. Consequently, we have

$$\begin{aligned} \left|\widehat{S}(\xi)\right| &= |\phi_\xi(T^n S)| \leq \|T^n S\| \\ &\leq \|T^n S - T^n K\| + \|T^n K\| \\ &\leq \|S - K\| + \|T^n K\|. \end{aligned}$$

Since $T^n \to 0$ in the weak operator topology, as in the proof of Proposition 2.1 we have

$$\lim_{n\to\infty} \|T^n K\| = 0.$$

Letting $n \to \infty$ in the preceding inequality, we obtain $\left|\widehat{S}(\xi)\right| \leq \|S - K\|$. It follows that

$$\sup_{\xi \in \sigma_e(T)} \left|\widehat{S}(\xi)\right| \leq \text{dist}\,(S, A_T \cap K(H)).$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Then there exists $f \in A(D)$ such that $\|S - f(T)\| \leq \varepsilon$. It follows that

$$\sup_{\xi \in \sigma_e(T)} |f(\xi)| \leq \sup_{\xi \in \sigma_e(T)} \left|\widehat{S}(\xi)\right| + \varepsilon.$$

By Rudin–Carleson Theorem, there exists $g \in A(D)$ such that $g(\xi) = f(\xi)$ on $\sigma_e(T)$ and

$$\|g\|_\infty = \sup_{\xi \in \sigma_e(T)} |f(\xi)|.$$

Since $g - f$ vanishes on $\sigma_e(T)$, by Theorem 2.2, $g(T) - f(T)$ is compact. Hence, we can write

$$\begin{aligned} \text{dist}\,(S, A_T \cap K(H)) &\leq \|S + g(T) - f(T)\| \\ &\leq \|g(T)\| + \varepsilon \\ &\leq \|g\|_\infty + \varepsilon \\ &= \sup_{\xi \in \sigma_e(T)} |f(\xi)| + \varepsilon \\ &\leq \sup_{\xi \in \sigma_e(T)} \left|\widehat{S}(\xi)\right| + 2\varepsilon. \end{aligned}$$

Since ε was arbitrary, we obtain that

$$\text{dist}\,(S, A_T \cap K(H)) \leq \sup_{\xi \in \sigma_e(T)} \left|\widehat{S}(\xi)\right|.$$

□

Next, we characterize the compactness via the ergodic conditions. The following lemma was proved in [8, Lemma 2.4].

Lemma 2.12. *Let V be an isometry on H and let $S \in \{V\}'$. If*

$$\varlimsup_{n\to\infty} \frac{1}{n} \left\| \sum_{k=1}^{n} \overline{\xi}^k V^k S \right\| = 0$$

holds for every $\xi \in \sigma(V) \cap \Gamma$, then $S = 0$.

As an application, we have the following

Theorem 2.13. *Let T be an essentially isometric operator and let $S \in \{T\}'$. If*

$$\varlimsup_{n\to\infty} \frac{1}{n} \left\| \sum_{k=1}^{n} \overline{\xi}^k T^k S \right\| = 0$$

holds for every $\xi \in \sigma_{le}(T)$ (or $\xi \in \sigma_{re}(T) \cap \Gamma$), then S is compact.

Proof. Let $\mathcal{T}$ and $\mathcal{S}$ be the limit operators associated with T and S, respectively. From Proposition 2.4 (d), (c), and Proposition 2.3 we deduce that $\mathcal{T}$ is an isometry and

$$\sigma(\mathcal{T}) \cap \Gamma \subset \sigma_{le}(T) \quad (\text{or } \sigma(\mathcal{T}) \cap \Gamma \subset \sigma_{re}(T) \cap \Gamma).$$

Furthermore, $\mathcal{S} \in \{\mathcal{T}\}'$. Now, it follows from Proposition 2.4 (a) that

$$\varlimsup_{n\to\infty} \frac{1}{n} \left\| \sum_{k=1}^{n} \overline{\xi}^k \mathcal{T}^k \mathcal{S} \right\| = 0$$

holds for every $\xi \in \sigma(\mathcal{T}) \cap \Gamma$. By the preceding lemma, $\mathcal{S} = 0$. Consequently, by Proposition 2.4 (b), S is compact. ☐

3. Essentially normal operators

Let T be an essentially normal operator, that is, $TT^* - T^*T$ is compact. Since $\pi(T)$ is a normal element of the Calkin algebra, we have $\sigma_{le}(T) = \sigma_{re}(T) = \sigma_e(T)$. It will be useful to note that if $\text{ind}(T - \lambda I_H) = 0$ for every $\lambda \in \mathbb{C} \setminus \sigma_e(T)$, then T is a compact perturbation of a normal operator [2].

In this section, for the compactness of S from the commutant of T, some necessary and sufficient conditions are found on S. The compactness of $S \in \{T\}'$ via the ergodic conditions is also characterized.

The first main result of this section is the following

Theorem 3.1. *Let T be an essentially normal operator. If the Gelfand transform of $S \in A_T$ vanishes on $\sigma_e(T)$, then S is compact.*

Proof. Assume that the Gelfand transform of $S \in A_T$ vanishes on $\sigma_e(T)$. Since $S \in A_T$, there exists a sequence of polynomials $\{P_n\}$ such that

$$\lim_{n\to\infty} \|P_n(T) - S\| = 0.$$

Let $\mathcal{T}$ and $\mathcal{S}$ be the limit operators associated with T and S, respectively. In view of Proposition 2.4 (a), we have

$$\lim_{n\to\infty} \|P_n(\mathcal{T}) - \mathcal{S}\| = 0.$$

Further, for every $\lambda \in \sigma_e(T)$, there exists a multiplicative functional ϕ_λ on A_T such that $\phi_\lambda(T) = \lambda$. Consequently, we can write

$$\begin{aligned} |P_n(\lambda)| &= \left|P_n(\lambda) - \widehat{S}(\lambda)\right| \\ &= |\phi_\lambda(P_n(T) - S)| \\ &\leq \|P_n(T) - S\|. \end{aligned}$$

From this, we deduce that $\lim_{n\to\infty} P_n(\lambda) = 0$ uniformly on $\sigma_e(T)$. On the other hand, it follows from Proposition 2.4 (c) and (d) that $\mathcal{T}$ is a normal operator and $\sigma(\mathcal{T}) \subset \sigma_e(T)$. Consequently, $\lim_{n\to\infty} P_n(\lambda) = 0$ uniformly on $\sigma(\mathcal{T})$. It follows that

$$\lim_{n\to\infty} \|P_n(\mathcal{T})\| = 0.$$

Thus we obtain $\mathcal{S} = 0$. By Proposition 2.4 (b), S is compact. □

Corollary 3.2. *If T is an essentially normal operator on H, then the following assertions hold:*

(a) *If $\sigma_e(T) = \{\lambda_1, \cdots, \lambda_n\}$, then $(T - \lambda_1 I_H) \cdot \cdots \cdot (T - \lambda_n I_H)$ is compact.*

(b) *The radical $Rad(A_T)$ of the algebra A_T consists of Volterra operators.*

Proof. (a) The Gelfand transform of $S := (T - \lambda_1 I_H) \cdot \cdots \cdot (T - \lambda_n I_H)$ vanishes on $\{\lambda_1, \cdots, \lambda_n\}$. By Theorem 3.1, S is compact.

(b) If $R \in Rad(A_T)$, then $\widehat{R}$ vanishes on $\sigma(T)$. Since $\sigma_e(T) \subset \sigma(T)$, it follows that $\widehat{R}$ vanishes on $\sigma_e(T)$. By Theorem 3.1, S is compact. □

Next, we will prove the following

Proposition 3.3. *Let T be an essentially normal operator such that*

$$\sigma_e(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}$$

and let $S \in B(H)$. If

$$\varliminf_{n\to\infty} \|T^n S\| = 0,$$

then S is compact.

Proof. Let $\mathcal{T}$ and $\mathcal{S}$ be the limit operators associated with T and S, respectively. It follows from Proposition 2.4 (d) and (c) that $\mathcal{T}$ is normal and

$$\sigma(\mathcal{T}) \subset \sigma_e(T) \subset \{\lambda \in \mathbb{C} : |\lambda| \geq 1\}.$$

On the other hand, we have

$$\|\mathcal{S}\| \leq \left\|\mathcal{T}^{-n}\right\| \|\mathcal{T}^n \mathcal{S}\| = \sup_{\lambda \in \sigma(\mathcal{T})} |\lambda|^{-n} \|\mathcal{T}^n \mathcal{S}\| \leq \|T^n S\|.$$

This clearly implies that $\mathcal{S} = 0$. By Proposition 2.4 (b), S is compact. □

Below, we characterize the compactness via the ergodic conditions.

Theorem 3.4. *Let T be an essentially normal Fredholm operator and let $S \in \{T\}'$. If*

$$\varliminf_{n\to\infty} \frac{1}{n} \left\|\sum_{k=1}^{n} \lambda^{-k} T^k S\right\| = 0$$

holds for every $\lambda \in \sigma_e(T)$, then S is compact.

We shall need the following

Lemma 3.5. *Let N be an invertible normal operator and let $S \in \{N\}'$. If*

$$\varliminf_{n\to\infty} \frac{1}{n} \left\| \sum_{k=1}^{n} \lambda^{-k} N^k S \right\| = 0$$

holds for every $\lambda \in \sigma(N)$, then $S = 0$.

Proof. It suffices to show that $SS^* = 0$. By Fuglede–Putnam theorem, $SN^* = N^*S$ which implies $NS^* = S^*N$. Consequently, we can write

$$N(SS^*) = (NS)S^* = (SN)S^* = S(NS^*) = S(S^*N) = (SS^*)N.$$

So, N commutes with SS^*. Let A be the unital C^*-algebra generated by N and SS^*. Then, A is commutative. Denote by Σ the Gelfand spectrum of A. Since the algebra A is isomprphic to $C(\Sigma)$, it suffices to show that $\phi(SS^*) = 0$ for all $\phi \in \Sigma$. Notice also that A is a full subalgebra of $B(H)$ and therefore, $\sigma(N) = \{\phi(N) : \phi \in \Sigma\}$. If $\phi \in \Sigma$ and if $\lambda := \phi(N)$, then we have

$$\begin{aligned} |\phi(SS^*)| &= \frac{1}{n} \left| \sum_{k=1}^{n} \lambda^{-k} \phi(N)^k \phi(SS^*) \right| \\ &= \frac{1}{n} \left| \langle \phi, \sum_{k=1}^{n} \lambda^{-k} N^k SS^* \rangle \right| \\ &\leq \frac{1}{n} \left\| \sum_{k=1}^{n} \lambda^{-k} N^k S \right\| \|S^*\|. \end{aligned}$$

Taking lower limit as $n \to \infty$, we get $\phi(SS^*) = 0$. □

Proof of Theorem 3.4. Let $\mathcal{T}$ and $\mathcal{S}$ be the limit operators associated with T and S, respectively. In view of Proposition 2.4 (c) and (d), $\mathcal{T}$ is an invertible normal operator and $\sigma(\mathcal{T}) \subset \sigma_e(T)$. Furthermore, $\mathcal{S} \in \{\mathcal{T}\}'$. Now, it follows from Proposition 2.4 (a) that

$$\varliminf_{n\to\infty} \frac{1}{n} \left\| \sum_{k=1}^{n} \lambda^{-k} \mathcal{T}^k \mathcal{S} \right\| = 0$$

holds for every $\lambda \in \sigma(\mathcal{T})$. By the preceding lemma, $\mathcal{S} = 0$. Consequently, by Proposition 2.4 (b), S is compact. □

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