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ABSOLUTE SUMS OF BANACH SPACES AND SOME GEOMETRIC PROPERTIES RELATED TO ROTUNDITY AND SMOOTHNESS

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Abstract. We study the notions of acs, luacs and uacs Banach spaces which were introduced by Kadets et al. and form common generalisations of the usual rotundity and smoothness properties of Banach spaces. In particular, we are interested in (mainly infinite) absolute sums of such spaces. We also introduce some new classes of spaces that lie inbetween those of acs and uacs spaces and study their behaviour under the formation of absolute sums as well.

1. Introduction

First let us fix some notation. Where not otherwise stated, X denotes a real Banach space, X^* its dual, B_X its unit ball and S_X its unit sphere.

Since we will deal with various generalisations of rotundity and smoothness properties for Banach spaces, we start by recalling the most important of these notions.

Definition 1.1. A Banach space X is called

(i) *rotund* (R in short) if for any two elements $x, y \in S_X$ the equality $\|x+y\| = 2$ implies $x = y$,

(ii) *locally uniformly rotund* (LUR in short) if for every $x \in S_X$ the implication

$$\|x_n + x\| \to 2 \;\Rightarrow\; \|x_n - x\| \to 0$$

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holds for every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X,

(iii) *weakly locally uniformly rotund* (WLUR in short) if for every $x \in S_X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X we have

$$\|x_n + x\| \to 2 \ \Rightarrow \ x_n \to x \text{ weakly},$$

(iv) *uniformly rotund* (UR in short) if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X the following implication holds:

$$\|x_n + y_n\| \to 2 \ \Rightarrow \ \|x_n - y_n\| \to 0,$$

(v) *weakly uniformly rotund* (WUR in short) if for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ the following implication holds:

$$\|x_n + y_n\| \to 2 \ \Rightarrow \ x_n - y_n \to 0 \text{ weakly}.$$

UR spaces where introduced by Clarkson in [4], LUR spaces by Lovaglia in [22]. The obvious implications between all these notions are summarised in the chart below and no other implications are valid in general, as is shown by the examples in [26].

$$\begin{tikzcd}
 & \text{WUR} \arrow[dr] & & \\
\text{UR} \arrow[ur] \arrow[dr] & & \text{WLUR} \arrow[r] & \text{R} \\
 & \text{LUR} \arrow[ur] & &
\end{tikzcd}$$

FIG. 1

Note that, by standard normalisation arguments, X is UR iff for all bounded sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X which fulfil the conditions $\|x_n + y_n\| - \|x_n\| - \|y_n\| \to 0$ and $\|x_n\| - \|y_n\| \to 0$ we have that $\|x_n - y_n\| \to 0$ and further that the two conditions $\|x_n + y_n\| - \|x_n\| - \|y_n\| \to 0$ and $\|x_n\| - \|y_n\| \to 0$ can be replaced by the single equivalent condition $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$. Similar remarks apply to the definitions of LUR, WUR and WLUR spaces. Also, for a finite-dimensional space X all the above notions coincide (by compactness of the unit ball).

Recall also that the modulus of convexity of the space X is defined by $\delta_X(\varepsilon) = \inf\{1 - 1/2\|x + y\| : x, y \in B_X \text{ and } \|x - y\| \geq \varepsilon\}$ for every ε in the interval $]0, 2]$. Then X is UR iff $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leq 2$.

Concerning notions of smoothness, the space X is called *smooth* (S in short) if its norm is Gâteaux-differentiable at every non-zero point (equivalently at every point of S_X), which is the case iff for every $x \in S_X$ there is a unique functional $x^* \in S_{X^*}$ with $x^*(x) = 1$ (cf. [14, Lemma 8.4 (ii)]). X is called *Fréchet-smooth* (FS in short) if the norm is Frécht-differentiable at every non-zero point. Finally, X is called *uniformly smooth* (US in short) if $\lim_{\tau\to 0} \rho_X(\tau)/\tau = 0$, where ρ_X denotes the modulus of smoothness of X defined by $\rho_X(\tau) = \sup\{1/2(\|x + \tau y\| + \|x - \tau y\| - 2) : x, y \in S_X\}$ for every $\tau > 0$.

Obviously, FS implies S and from [14, Fact 9.7] it follows that US implies FS. It is also well known that X is US iff X^* is UR and X is UR iff X^* is US (cf. [14, Theorem 9.10]).

There is yet another notion of smoothness, namely the norm of the space X is said to be *uniformly Gâteaux-differentiable* (UG in short) if for each $y \in S_X$ the limit $\lim_{\tau \to 0} (\|x + \tau y\| - 1) / \tau$ exists uniformly in $x \in S_X$. The property UG lies between US and S. It is known (cf. [9, Theorem II.6.7]) that X^* is UG iff X is WUR and X is UG iff X^* is WUR* (which means that X^* fulfils the definition of WUR with weak- replaced by weak*-convergence).

In [18] the following notions were introduced.

Definition 1.2. A Banach space X is called

(i) *alternatively convex or smooth* (acs in short) if for every $x, y \in S_X$ with $\|x + y\| = 2$ and every $x^* \in S_{X^*}$ with $x^*(x) = 1$ we have $x^*(y) = 1$ as well,

(ii) *locally uniformly alternatively convex or smooth* (luacs in short) if for every $x \in S_X$, every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and every functional $x^* \in S_{X^*}$ we have
$$\|x_n + x\| \to 2 \text{ and } x^*(x_n) \to 1 \;\Rightarrow\; x^*(x) = 1,$$

(iii) *uniformly alternatively convex or smooth* (uacs in short) if for all sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in S_X and $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have
$$\|x_n + y_n\| \to 2 \text{ and } x_n^*(x_n) \to 1 \;\Rightarrow\; x_n^*(y_n) \to 1.$$

Clearly, R and S both imply acs, WLUR implies luacs and UR and US both imply uacs. Again by standard normalisation arguments one can easily check that X is uacs iff for all bounded sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in X and $(x_n^*)_{n\in\mathbb{N}}$ in X^* with $x_n^*(x_n) - \|x_n^*\| \|x_n\| \to 0$, $\|x_n + y_n\| - \|x_n\| - \|y_n\| \to 0$ and $\|x_n\| - \|y_n\| \to 0$ (or equivalently $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$) we also have $x_n^*(y_n) - \|x_n^*\| \|y_n\| \to 0$ and a similar characterisation holds for luacs spaces. Note also that, again by compactness, in the case $\dim X < \infty$ the notions of acs, luacs and uacs spaces coincide.

The acs, luacs and uacs spaces were originally introduced in [18] to obtain geometric characterisations of the so called Anti-Daugavet property, which was introduced in the same paper. We will briefly recall this property and the connection to the geometric notions.

Firstly, it is well-known and easy to see that a bounded linear operator T on any Banach space X with $\|T\| \in \sigma(T)$ satisfies the so called Daugavet–equation $\|\mathrm{id} + T\| = 1 + \|T\|$ (here, $\sigma(T)$ denotes the spectrum of T). In fact, the following more general statement holds (this is surely known as well, but a proof is included here since the author was not able to find it explicitly in the literature).

Lemma 1.3. *For any (real) Banach space X and every $T \in L(X)$ the inequality*
$$\|\mathrm{id} + T\| \geq 1 + \|T\| - d(\|T\|, \sigma(T))$$
holds, where $d(\|T\|, \sigma(T))$ denotes the distance of $\|T\|$ to $\sigma(T)$.

Proof. If the claim was not true, there would be $\lambda \in \sigma(T)$ such that $|\|T\| - \lambda| < 1 + \|T\| - \|\mathrm{id} + T\|$, hence $\|\mathrm{id} + T\| < 1 + \lambda$. Consequently, the operator $S := (1+\lambda)^{-1}(\mathrm{id} + T)$ has norm less than 1, so $\mathrm{id} - S$ is invertible. But then the operator $(1+\lambda)(\mathrm{id} - S) = \lambda \mathrm{id} - T$ would be invertible as well, contradicting $\lambda \in \sigma(T)$. □

Now X is said to have the Anti-Daugavet property with respect to some class $M \subseteq L(X)$ of operators, if the implication

$$\|\mathrm{id} + T\| = 1 + \|T\| \ \Rightarrow\ \|T\| \in \sigma(T)$$

holds for every $T \in M$.

The results from [18] then read as follows: X has the Anti-Daugavet property for rank-1-operators iff X has the Anti-Daugavet property for compact operators iff X is luacs (see [18, Theorem 4.3]); if X is even uacs, then it has the Anti-Daugavet property with respect to all operators (see [18, Theorem 4.5], it is not known whether the converse of this stament is true). For more information about the Daugavet–equation, the reader is referred to [18] and [27].

Let us now discuss the acs spaces and their relatives a little further. First note the following reformulation of the definition of acs spaces, which was observed in [18]: A Banach space X is acs iff whenever $x, y \in S_X$ such that $\|x + y\| = 2$ then the norm of $\mathrm{span}\{x, y\}$ is Gâteaux-differentiable at x and y.

Recall that a Banach space X is said to be uniformly non-square if there is some $\delta > 0$ such that for all $x, y \in B_X$ we have $\|x + y\| \leq 2(1-\delta)$ or $\|x - y\| \leq 2(1-\delta)$. It is easily seen that uacs spaces are uniformly non-square and hence by a well-known theorem of James (cf. [2, p.261]) they are superreflexive, as was observed in [18, Lemma 4.4].

Actually, to prove the superreflexivity of uacs spaces it is not necessary to employ the rather deep theorem of James, as we will see in the next section.

In [24] it is shown by Sirotkin that for every $1 < p < \infty$ and every measure space (Ω, Σ, μ) the Lebesgue–Bochner space $L^p(\Omega, \Sigma, \mu; X)$ is uacs (resp. luacs, resp. acs) whenever X is an uacs (resp. luacs, resp. acs) Banach space. To get this result, Sirotkin first proves the following characterisation of uacs spaces.

Proposition 1.4 (Sirotkin, cf. [24])**.** *A Banach space X is uacs iff for any two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X and every sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have*

$$\|x_n + y_n\| \to 2 \text{ and } x_n^*(x_n) = 1\ \forall n \in \mathbb{N} \ \Rightarrow\ x_n^*(y_n) \to 1.$$

Instead of repeating the proof from [24] here, we shall give a slightly different proof below (see Proposition 2.1), which—unlike Sirotkin's proof—does not use any reflexivity arguments (but see also the proof of Lemma 2.10).

Now with this characterisation we can define a kind of 'uacs-modulus' of a given Banach space.

Definition 1.5. For a Banach space X we define

$$D_X(\varepsilon) = \{(x, y) \in S_X \times S_X : \exists x^* \in S_{X^*}\ x^*(x) = 1 \text{ and } x^*(y) \leq 1 - \varepsilon\}$$

$$\text{and } \delta^X_{\mathrm{uacs}}(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : (x, y) \in D_X(\varepsilon)\right\}\ \forall \varepsilon \in]0, 2].$$

Then by Proposition 1.4 X is uacs iff $\delta^X_{\mathrm{uacs}}(\varepsilon) > 0$ for every $\varepsilon \in]0,2]$ and we clearly have $\delta_X(\varepsilon) \leq \delta^X_{\mathrm{uacs}}(\varepsilon)$ for each $\varepsilon \in]0,2]$. For the connection to the modulus of smoothness see Lemma 2.6. The characterisation of uacs spaces given above coincides with the notion of U-spaces introduced by Lau in [20] and our modulus δ^X_{uacs} is the same as the modulus of u-convexity from [15]. Also, the notion of u-spaces which was introduced in [11] coincides with the notion of acs spaces. The interested reader may also have a look at [13], where two notions of local U-convexity are introduced and studied quantitatively. The U-spaces (= uacs spaces) are of particular interest, because they possess normal structure (cf. [16, Theorem 3.2] or [24, Theorem 3.1]) and hence (since they are also reflexive) they enjoy the fixed point property (the reader is referred to [17, Section 2] for definitions and background).

It seems natural to introduce two more notions related to uacs spaces, namely the following.

Definition 1.6. A Banach space X is called

(i) *strongly locally uniformly alternatively convex or smooth* (sluacs in short) if for every $x \in S_X$ and all sequences $(x_n)_{n\in\mathbb{N}}$ in S_X and $(x^*_n)_{n\in\mathbb{N}}$ in S_{X^*} we have
$$\|x_n + x\| \to 2 \text{ and } x^*_n(x_n) \to 1 \ \Rightarrow\ x^*_n(x) \to 1,$$

(ii) *weakly uniformly alternatively convex or smooth* (wuacs in short) if for any two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in S_X and every functional $x^* \in S_{X^*}$ we have
$$\|x_n + y_n\| \to 2 \text{ and } x^*(x_n) \to 1 \ \Rightarrow\ x^*(y_n) \to 1.$$

With these definitions we get the following implication chart.

$$\begin{tikzcd}
 & \text{wuacs} \arrow[dr] & & \\
\text{uacs} \arrow[ur] \arrow[dr] & & \text{luacs} \arrow[r] & \text{acs} \\
 & \text{sluacs} \arrow[ur] & &
\end{tikzcd}$$

FIG. 2

Including the rotundity properties finally leaves us with the diagram below.

$$\begin{tikzcd}
 & \text{WUR} \arrow[d] \arrow[dr] & & \\
\text{UR} \arrow[ur] \arrow[d] \arrow[dr] & \text{wuacs} \arrow[dr] & \text{WLUR} \arrow[r] \arrow[d] & \text{R} \arrow[d] \\
\text{uacs} \arrow[ur] \arrow[dr] & \text{LUR} \arrow[ur] \arrow[d] & \text{luacs} \arrow[r] & \text{acs} \\
 & \text{sluacs} \arrow[ur] & &
\end{tikzcd}$$

FIG. 3

Let us further remark that every space whose norm is UG is also sluacs, thus we have the following diagram illustrating the connection to smoothness properties.

$$\begin{tikzcd} \mathrm{US} \arrow[r] \arrow[d] & \mathrm{UG} \arrow[r] \arrow[d] & \mathrm{S} \arrow[d] \\ \mathrm{uacs} \arrow[r] & \mathrm{sluacs} \arrow[r] & \mathrm{acs} \end{tikzcd}$$

FIG. 4

In the next section we collect some general results on uacs spaces and their relatives.

2. Some general facts

We start with the promised alternative proof of Proposition 1.4 which does not rely on reflexivity. Instead, we shall employ the Bishop–Phelps–Bollobás theorem (cf. [3, Chap. 8, Theorem 11]), an argument that will also work for the case of sluacs spaces. This idea was suggested to the author by Dirk Werner.

Proposition 2.1. *A Banach space X is uacs iff for any two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in S_X and every sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have*

$$\|x_n + y_n\| \to 2 \text{ and } x_n^*(x_n) = 1 \ \forall n \in \mathbb{N} \ \Rightarrow \ x_n^*(y_n) \to 1. \tag{2.1}$$

X is sluacs iff for every $x \in S_X$ and all sequences $(x_n)_{n\in\mathbb{N}}$, $(x_n^*)_{n\in\mathbb{N}}$ in S_X resp. S_{X^*} we have

$$\|x_n + x\| \to 2 \text{ and } x_n^*(x_n) = 1 \ \forall n \in \mathbb{N} \ \Rightarrow \ x_n^*(x) \to 1. \tag{2.2}$$

Proof. We only prove the statement for uacs spaces, the proof for the sluacs case is completely analogous. Furthermore, only the 'if' part of the stated equivalence requires proof. So suppose (2.1) holds for any two sequences in S_X and all sequences in S_{X^*}.

Now if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are sequences in S_X and $(x_n^*)_{n\in\mathbb{N}}$ is a sequence in S_{X^*} such that $\|x_n + y_n\| \to 2$ and $x_n^*(x_n) \to 1$ we can choose a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $x_{n_k}^*(x_{n_k}) > 1 - 2^{-2k-2}$ holds for all $k \in \mathbb{N}$. By the already cited Bishop–Phelps–Bollobás theorem we can find sequences $(\tilde{x}_k)_{k\in\mathbb{N}}$ in S_X and $(\tilde{x}_k^*)_{k\in\mathbb{N}}$ in S_{X^*} such that $\tilde{x}_k^*(\tilde{x}_k) = 1$, $\|\tilde{x}_k - x_{n_k}\| \leq 2^{-k}$ and $\left\|\tilde{x}_k^* - x_{n_k}^*\right\| \leq 2^{-k}$ for all $k \in \mathbb{N}$.

It follows that $\|\tilde{x}_k - x_{n_k}\| \to 0$ and $\left\|\tilde{x}_k^* - x_{n_k}^*\right\| \to 0$ and since $\|x_n + y_n\| \to 2$ we get that $\|\tilde{x}_k + y_{n_k}\| \to 2$.

But then we also have $\tilde{x}_k^*(y_{n_k}) \to 1$, by our assumption, which in turn implies $x_{n_k}^*(y_{n_k}) \to 1$.

In the same way we can show that every subsequence of $(x_n^*(y_n))_{n\in\mathbb{N}}$ has another subsequence that tends to one and hence $x_n^*(y_n) \to 1$ which completes the proof. □

Next we would like to give characterisations of acs/sluacs/uacs spaces that do not explicitly involve the dual space. As mentioned before, a Banach space X is acs iff x and y are smooth points of the unit ball of the two-dimensional subspace $\operatorname{span}\{x, y\}$ whenever $x, y \in S_X$ are such that $\|x + y\| = 2$.

It is possible to reformulate and refine this statement in the following way.

Proposition 2.2. *For any Banach space X the following assertions are equivalent:*

(i) *X is acs.*

(ii) *For all $x, y \in S_X$ with $\|x+y\| = 2$ we have*
$$\lim_{t\to 0^+} \frac{\|x+ty\| + \|x-ty\| - 2}{t} = 0.$$

(iii) *For all $x, y \in S_X$ with $\|x+y\| = 2$ we have*
$$\lim_{t\to 0^+} \frac{\|x-ty\| - 1}{t} = -1.$$

(iv) *For all $x, y \in S_X$ with $\|x+y\| = 2$ there is some $1 \le p < \infty$ such that*
$$\lim_{t\to 0^+} \frac{\|x+ty\|^p + \|x-ty\|^p - 2}{t^p} = 0.$$

(v) *For all $x, y \in S_X$ with $\|x+y\| = 2$ there is some $1 \le p < \infty$ such that*
$$\lim_{t\to 0^+} \frac{(1+t)^p + \|x-ty\|^p - 2}{t^p} = 0.$$

The analogous characterisation for sluacs spaces reads as follows.

Proposition 2.3. *For any Banach space X the following assertions are equivalent:*

(i) *X is sluacs.*

(ii) *For every $\varepsilon > 0$ and every $y \in S_X$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and each $x \in S_X$ with $\|x+y\| \ge 2(1-t)$ we have*
$$\|x+ty\| + \|x-ty\| \le 2 + \varepsilon t.$$

(iii) *For every $\varepsilon > 0$ and every $y \in S_X$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and each $x \in S_X$ with $\|x+y\| \ge 2 - t\delta$ we have*
$$\|x-ty\| \le 1 + t(\varepsilon - 1).$$

(iv) *For every $y \in S_X$ there is some $1 \le p < \infty$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in [0, \delta]$ and each $x \in S_X$ with $\|x+y\| \ge 2(1-t)$ we have*
$$\|x+ty\|^p + \|x-ty\|^p \le 2 + \varepsilon t^p.$$

(v) *For every $y \in S_X$ there is some $1 \le p < \infty$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \in [0, \delta]$ and each $x \in S_X$ with $\|x+y\| \ge 2 - t\delta$ we have*
$$(1+t)^p + \|x-ty\|^p \le 2 + \varepsilon t^p.$$

Finally, we have the following characterisation for uacs spaces.

Proposition 2.4. *For any Banach space X the following assertions are equivalent:*

(i) *X is uacs.*

(ii) *For every* $\varepsilon > 0$ *there exists some* $\delta > 0$ *such that for every* $t \in [0, \delta]$ *and all* $x, y \in S_X$ *with* $\|x + y\| \geq 2(1 - t)$ *we have*

$$\|x + ty\| + \|x - ty\| \leq 2 + \varepsilon t.$$

(iii) *For every* $\varepsilon > 0$ *there exists some* $\delta > 0$ *such that for every* $t \in [0, \delta]$ *and all* $x, y \in S_X$ *with* $\|x + y\| \geq 2 - \delta t$ *we have*

$$\|x - ty\| \leq 1 + t(\varepsilon - 1).$$

(iv) *There exists some* $1 \leq p < \infty$ *such that for every* $\varepsilon > 0$ *there is some* $\delta > 0$ *such that for all* $t \in [0, \delta]$ *and all* $x, y \in S_X$ *with* $\|x + y\| \geq 2(1 - t)$ *we have*

$$\|x + ty\|^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

(v) *There exists some* $1 \leq p < \infty$ *such that for every* $\varepsilon > 0$ *there is some* $\delta > 0$ *such that for all* $t \in [0, \delta]$ *and all* $x, y \in S_X$ *with* $\|x + y\| \geq 2 - t\delta$ *we have*

$$(1 + t)^p + \|x - ty\|^p \leq 2 + \varepsilon t^p.$$

Proof. We will only explicitly prove the characterisation for uacs spaces. First we show (i) $\Rightarrow$ (ii). So suppose X is uacs and fix $\varepsilon > 0$. Then there exists some $\tilde{\delta} > 0$ such that for all $x, y \in S_X$ and $x^* \in S_{X^*}$ we have

$$\|x + y\| \geq 2(1 - \tilde{\delta}) \text{ and } x^*(x) \geq 1 - \tilde{\delta} \;\Rightarrow\; x^*(y) \geq 1 - \varepsilon.$$

Now if we put $\delta = \tilde{\delta}/2$ and take $t \in [0, \delta]$ and $x, y \in S_X$ such that $\|x+y\| \geq 2(1-t)$ then we can find a functional $x^* \in S_{X^*}$ such that $x^*(x - ty) = \|x - ty\|$ and conclude that

$$x^*(x) = \|x - ty\| + tx^*(y) \geq 1 - t - t = 1 - 2t \geq 1 - \tilde{\delta}.$$

By the choice of $\tilde{\delta}$ this implies $x^*(y) \geq 1 - \varepsilon$ and hence

$$\|x + ty\| + \|x - ty\| = \|x + ty\| + x^*(x - ty) \leq 1 + t + 1 - tx^*(y) \leq 2 + t\varepsilon.$$

Now let us prove (ii) $\Rightarrow$ (iii). For a given $\varepsilon > 0$ choose $\delta > 0$ to the value $\varepsilon/2$ according to (ii). We may assume $\delta \leq \min\{1, \varepsilon/2\}$.

Then if $t \in [0, \delta]$ and $x, y \in S_X$ such that $\|x + y\| \geq 2 - \delta t$ we in particular have $\|x + y\| \geq 2(1 - t)$ and hence

$$\|x + ty\| + \|x - ty\| \leq 2 + t\frac{\varepsilon}{2}.$$

But on the other hand

$$\|x + ty\| \geq \|x + y\| - (1 - t)\|y\| \geq 2 - \delta t - 1 + t = 1 - \delta t + t \geq 1 - \frac{\varepsilon}{2}t + t.$$

It follows that $\|x - ty\| \leq 1 + t(\varepsilon - 1)$.

Next we prove that (iii) $\Rightarrow$ (i). Fix sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S_X such that $\|x_n + y_n\| \to 2$ and a sequence $(x_n^*)_{n\in\mathbb{N}}$ of norm-one functionals with $x_n^*(x_n) \to 1$. Also, for every $n \in \mathbb{N}$ we fix $y_n^* \in S_{X^*}$ such that $y_n^*(y_n) = 1$.

For given $\varepsilon > 0$ we choose $\delta > 0$ according to (iii). For sufficiently large n we have $\|x_n + y_n\| \geq 2 - \delta^2$ and $x_n^*(x_n) \geq 1 - \varepsilon\delta$ and hence

$$\begin{aligned}(y_n^* - x_n^*)(\delta y_n) &= x_n^*(x_n - \delta y_n) - x_n^*(x_n) + \delta \leq \|x_n - \delta y_n\| + \delta - x_n^*(x_n)\\ &\leq \|x_n - \delta y_n\| + \delta - 1 + \varepsilon\delta \leq 1 + \delta(\varepsilon - 1) + \delta - 1 + \varepsilon\delta = 2\delta\varepsilon,\end{aligned}$$

where the last inequality holds because of $\|x_n + y_n\| \geq 2 - \delta^2$ and the choice of δ.

It follows that $x_n^*(y_n) \geq y_n^*(y_n) - 2\varepsilon = 1 - 2\varepsilon$ for sufficiently large n.

The implications (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v) are clear. To prove (iv) $\Rightarrow$ (ii) recall the inequalities

$$(a+b)^p \leq 2^{p-1}(a^p + b^p) \quad \forall a, b \geq 0, \forall p \in [1, \infty[$$
$$(a+b)^\alpha \leq a^\alpha + b^\alpha \quad \forall a, b \geq 0, \forall \alpha \in]0, 1].$$

They imply that for all $x, y \in S_X$, every $t > 0$ and each $1 \leq p < \infty$ one has

$$\frac{\|x+ty\| + \|x-ty\| - 2}{t} \leq \frac{\left(2^{p-1}\left(\|x+ty\|^p + \|x-ty\|^p\right)\right)^{1/p} - 2}{t}$$
$$\leq \left(\frac{2^{p-1}\left(\|x+ty\|^p + \|x-ty\|^p\right) - 2^p}{t^p}\right)^{1/p}$$
$$= 2^{1-1/p}\left(\frac{\|x+ty\|^p + \|x-ty\|^p - 2}{t^p}\right)^{1/p},$$

which shows (iv) $\Rightarrow$ (ii). If we replace $\|x+ty\|$ by $1+t$ in the above calculation, we also obtain a proof for (v) $\Rightarrow$ (iii). $\square$

If we define the modulus ρ_{uacs}^X by

$$\rho_{\mathrm{uacs}}^X(\tau) = \sup\left\{1/2(\|x+\tau y\| + \|x-\tau y\|) - 1 : (x,y) \in S_X(\tau)\right\},$$

where $\tau > 0$ and $S_X(\tau) = \{(x,y) \in S_X \times S_X : \|x+y\| \geq 2(1-\tau)\}$ then because of the equivalence of (i) and (ii) in Proposition 2.4 X is uacs iff $\lim_{\tau\to 0} \rho_{\mathrm{uacs}}^X(\tau)/\tau = 0$ and obviously $\rho_{\mathrm{uacs}}^X(\tau) \leq \rho_X(\tau)$.

Let us also define

$$\tilde{\delta}_{\mathrm{uacs}}^X(\varepsilon) = \inf\left\{\max\left\{1 - \frac{1}{2}\|x+y\|, 1 - x^*(x)\right\} : x, y \in S_X, x^* \in A_\varepsilon(y)\right\},$$

where $0 < \varepsilon \leq 2$ and $A_\varepsilon(y) = \{x^* \in S_{X^*} : x^*(y) \leq 1 - \varepsilon\}$.

From the very definition of the uacs spaces it follows that X is uacs iff $\tilde{\delta}_{\mathrm{uacs}}^X(\varepsilon) > 0$ for every $0 < \varepsilon \leq 2$.

Examining the proof of the implication (i) $\Rightarrow$ (ii) in Proposition 2.4 we see that the following holds.

Lemma 2.5. *If X is a Banach space and $0 < \varepsilon \leq 2$ such that $\tilde{\delta}_{\mathrm{uacs}}^X(\varepsilon) > 0$ then for every $\tau > 0$ with $2\tau < \tilde{\delta}_{\mathrm{uacs}}^X(\varepsilon)$ we have $2\rho_{\mathrm{uacs}}^X(\tau) \leq \tau\varepsilon$.*

The reverse connection between ρ_{uacs}^X and δ_{uacs}^X is given by the following lemma.

Lemma 2.6. *Let X be any Banach space and $\tau > 0$ as well as $0 < \varepsilon \leq 2$. Then the inequality*

$$\delta_{\mathrm{uacs}}^X(\varepsilon) \geq \frac{\varepsilon\tau - 2\rho_{\mathrm{uacs}}^X(\tau)}{2(\tau+1)}.$$

holds.

Proof. We may assume $\varepsilon\tau - 2\rho^X_{\mathrm{uacs}}(\tau) > 0$, because otherwise the inequality is trivially satisfied. Let us put $R = (\varepsilon\tau - 2\rho^X_{\mathrm{uacs}}(\tau))(2(\tau+1))^{-1}$ and take $x, y \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and $\|x+y\| > 2(1-R)$.

Then we can find $z^* \in S_{X^*}$ with $z^*(x+y) > 2(1-R)$ and hence $z^*(x) > 1-2R$ and $z^*(y) > 1-2R$.

It follows that

$$\begin{aligned}
&(z^* - x^*)(\tau y) = z^*(x+\tau y) + x^*(x-\tau y) - x^*(x) - z^*(x)\\
&\leq \|x+\tau y\| + \|x-\tau y\| - 1 - z^*(x) \leq 2\rho^X_{\mathrm{uacs}}(\tau) + 1 - z^*(x)\\
&\leq 2(\rho^X_{\mathrm{uacs}}(\tau) + R).
\end{aligned}$$

Hence

$$x^*(y) \geq z^*(y) - \frac{2}{\tau}\big(\rho^X_{\mathrm{uacs}}(\tau) + R\big) > 1 - 2R - \frac{2}{\tau}\big(\rho^X_{\mathrm{uacs}}(\tau) + R\big) = 1 - \varepsilon$$

and we are done. □

Now we turn to the proof of the superreflexivity of uacs spaces without using James's result on uniformly non-square Banach spaces. A key ingredient to James's proof is the following lemma of his, which may be found in [2, p.51].

Lemma 2.7. *A Banach space X is* not *reflexive iff for every $0 < \theta < 1$ there is a sequence $(x_k)_{k\in\mathbb{N}}$ in B_X and a sequence $(x^*_n)_{n\in\mathbb{N}}$ in B_{X^*} such that for every $n \in \mathbb{N}$ we have*

$$x^*_n(x_k) = \begin{cases} \theta & \text{if } n \leq k \\ 0 & \text{if } n > k. \end{cases}$$

Even armed with this lemma it is still difficult to prove the superreflexivity of uniformly non-square Banach spaces, but it easily yields the result for uacs spaces. We can even prove a stronger result: it is a well known fact that a Banach space X is reflexive if it satisfies $\liminf_{t\to 0^+} \rho_X(t)/t < 1/2$ (cf. [25, Theorem 2]).[1] We will see that the same holds if we replace ρ_X by ρ^X_{uacs}, even a bit more is true.

Proposition 2.8. *If there is some $0 < t$ such that $\rho^X_{\mathrm{uacs}}(t) < t/2$, then X is superreflexive (actually, it is uniformly non-square).*

Proof. Put $\theta = 2\rho^X_{\mathrm{uacs}}(t)/t < 1$ and choose $\varepsilon > 0$ such that $\theta + \varepsilon < 1$. Also, put $\eta = \min\{t\varepsilon/5, \varepsilon/5\}$.

If $x, y \in S_X$ such that $\|x+y\| \geq 2(1-\eta)$ and $x^* \in S_{X^*}$ with $x^*(x) \geq 1-\eta$ fix $y^* \in S_{X^*}$ such that $y^*(x+y) \geq 2(1-\eta)$. Then $y^*(x) \geq 1-2\eta$ and $y^*(y) \geq 1-2\eta$ and hence

$$\begin{aligned}
&(y^* - x^*)(ty) = y^*(x+ty) + x^*(x-ty) - x^*(x) - y^*(x)\\
&\leq \|x+ty\| + \|x-ty\| - 2 + 3\eta \leq 2\rho^X_{\mathrm{uacs}}(t) + 3\eta = t\theta + 3\eta \leq (\theta + \frac{3}{5}\varepsilon)t.
\end{aligned}$$

Consequently, $x^*(y) \geq y^*(y) - \theta - \frac{3}{5}\varepsilon \geq 1 - 2\eta - \theta - \frac{3}{5}\varepsilon \geq 1 - \frac{2}{5}\varepsilon - \theta - \frac{3}{5}\varepsilon = 1 - (\theta + \varepsilon)$.

Next we fix $0 < \tau < 1/2$ such that $\tau(1 + (1-2\tau)^{-1}) \leq \eta$ and put $\beta = 1 - (1-\tau)(1-2\tau)(1-\theta-\varepsilon)$. Then $0 < \beta < 1$.

[1] Note that the definition of ρ_X given there differs from our definition by a factor $1/2$.

Claim. If $x, y \in B_X$ such that $\|x+y\| \geq 2(1-\tau)$ and $x^* \in B_{X^*}$ such that $x^*(x) \geq 1-\tau$ then $x^*(y) \geq 1-\beta$.

To see this, take x, y and x^* as above and observe $\|x\|, \|y\| \geq 1-2\tau$. Hence

$$\begin{aligned}\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\| &\geq \frac{\|x+y\|}{\|x\|}-\left|\frac{1}{\|x\|}-\frac{1}{\|y\|}\right|\|y\| \\ &\geq \|x+y\|-\left|\frac{1}{\|x\|}-\frac{1}{\|y\|}\right| \geq 2(1-\tau)-\frac{2\tau}{1-2\tau} \geq 2(1-\eta)\end{aligned}$$

and moreover, since $\|x^*\|, \|x\| \leq 1$,

$$\frac{x^*}{\|x^*\|}\left(\frac{x}{\|x\|}\right) \geq 1-\tau \geq 1-\eta.$$

Thus by our previous considerations we must have

$$x^*(y) \geq \|x^*\|\|y\|(1-\theta-\varepsilon) \geq (1-\tau)(1-2\tau)(1-\theta-\varepsilon) = 1-\beta.$$

From the above claim together with the fact that $\beta < 1$ it could be easily deduced that X is uniformly non-square and hence superreflexive, but if we just want to prove the superreflexivity an application of Lemma 2.7 is enough. For if X was not reflexive then by said Lemma we could find sequences $(x_k)_{k\in\mathbb{N}}$ in B_X and $(x_n^*)_{n\in\mathbb{N}}$ in B_{X^*} such that $x_n^*(x_k) = 0$ for $n > k$ and $x_n^*(x_k) = 1-\tau$ for $n \leq k$.

We only need the first two members of the sequences to derive a contradiction, namely we have $\|x_1+x_2\| \geq x_1^*(x_1)+x_1^*(x_2) = 2(1-\tau)$ and $x_2^*(x_2) = 1-\tau$ but $x_2^*(x_1) = 0 < 1-\beta$ contradicting our just established claim.

Thus X must be reflexive and to prove the superreflexivity it only remains to show that for every Banach space Y which is finitely representable in X there exists $0 < t'$ such that $\rho_{\text{uacs}}^Y(t') < t'/2$ which we will do in the next Lemma. ☐

Lemma 2.9. *If there is some $0 < t$ such that $\rho_{\text{uacs}}^X(t) < t/2$ and Y is finitely representable in X then there is $0 < t'$ such that $\rho_{\text{uacs}}^Y(t') < t'/2$.*

Proof. Let $\theta, \varepsilon, \eta, \tau$ and β be as in the previous proof. Put $\nu = \tau/4$.

Claim. If $x, y \in B_X$ such that $\|x+y\| \geq 2(1-\nu)$ then $\|x+\nu y\|+\|x-\nu y\| \leq 2+\nu\beta$.

To establish this, take $x, y \in B_X$ as above and also fix $x^* \in S_{X^*}$ such that $x^*(x-\nu y) = \|x-\nu y\|$. Observe as before that $\|x\|, \|y\| \geq 1-\tau/2$. Hence we have

$$x^*(x) = \|x-\nu y\|+x^*(\nu y) \geq \|x\|-\nu\|y\|+\nu x^*(y) \geq \|x\|-2\nu \geq 1-\tau.$$

The claim we established in the previous proof now gives us $x^*(y) \geq 1-\beta$. It follows that

$$\|x+\nu y\|+\|x-\nu y\| = \|x+\nu y\|+x^*(x-\nu y) \leq 2+\nu(1-x^*(y)) \leq 2+\nu\beta.$$

Next fix $\beta < \alpha < 1$ and $0 < \tilde{\eta} < \nu$ such that $(\beta\nu+3\tilde{\eta})(\nu-\tilde{\eta})^{-1} < \alpha$. Put $t' = \nu-\tilde{\eta}$. Finally, choose $\tilde{\varepsilon} > 0$ such that $(1-t')(1+\tilde{\varepsilon})^{-1} > 1-\nu$ and $(1+\tilde{\varepsilon})(2+\nu\beta) \leq 2+\nu\beta+\tilde{\eta}$.

Now take $y_1, y_2 \in S_Y$ with $\|y_1+y_2\| \geq 2(1-t')$ and put $F = \operatorname{span}\{y_1, y_2\}$. Since Y is finitely representable in X there is a subspace $E \subseteq X$ and an isomorphism $T : F \to E$ such that $\|T\| = 1$ and $\|T^{-1}\| \leq 1+\tilde{\varepsilon}$. Let $x_i = Ty_i$ for $i = 1, 2$.

It easily follows that $\|x_1 + x_2\| \geq 2(1-t')(1+\tilde{\varepsilon})^{-1} > 2(1-\nu)$, whence $\|x_1 + \nu x_2\| + \|x_1 - \nu x_2\| \leq 2 + \nu\beta$ which implies $\|y_1 + \nu y_2\| + \|y_1 - \nu y_2\| \leq (1+\tilde{\varepsilon})(2+\nu\beta)$. Thus we have

$$\begin{aligned}\|y_1 + t'y_2\| + \|y_1 - t'y_2\| &\leq \|y_1 + \nu y_2\| + \|y_1 - \nu y_2\| + 2\,|\nu - t'| \\ &\leq (1+\tilde{\varepsilon})(2+\nu\beta) + 2\tilde{\eta} \leq 2 + \nu\beta + 3\tilde{\eta} \leq 2 + \alpha(\nu - \tilde{\eta}) = 2 + \alpha t'.\end{aligned}$$

So we have proved $2\rho^Y_{\mathrm{uacs}}(t')/t' \leq \alpha < 1$. □

We remark that the uniform non-squareness of a space X satisfying $2\rho^X_{\mathrm{uacs}}(t) < t$ for some $0 < t$ could also be deduced from our Lemma 2.6 and [15, Theorem 2], where it is observed that $\delta^X_{\mathrm{uacs}}(1) > 0$ is sufficient to ensure that X is uniformly non-square.

Now let us have a look at the quantitative connection between the moduli δ^X_{uacs} and $\tilde{\delta}^X_{\mathrm{uacs}}$.

Lemma 2.10. *If X is uacs then*

$$\tilde{\delta}^X_{\mathrm{uacs}}(\varepsilon) \geq \delta^X_{\mathrm{ucas}}\left(\delta^X_{\mathrm{uacs}}(\varepsilon)\right)$$

for every $0 < \varepsilon \leq 2$.

Proof. Here we can adopt Sirotkin's idea from the proof of Proposition 1.4 in [24]. Put $\delta = \delta^X_{\mathrm{uacs}}\left(\delta^X_{\mathrm{uacs}}(\varepsilon)\right)$ and take $x, y \in S_X$ and $x^* \in S_{X^*}$ such that $\|x + y\| > 2(1-\delta)$ and $x^*(x) > 1 - \delta$.

Since X is reflexive, there is some $z \in S_X$ with $x^*(z) = 1$. It follows that $\|x + z\| \geq x^*(x+z) > 2(1-\delta)$.

Now fix $y^* \in S_{X^*}$ such that $y^*(x) = 1$. Then by the definition of δ we must have $y^*(z) > 1 - \delta^X_{\mathrm{uacs}}(\varepsilon)$ and $y^*(y) > 1 - \delta^X_{\mathrm{uacs}}(\varepsilon)$ and hence $\|y + z\| > 2(1 - \delta^X_{\mathrm{uacs}}(\varepsilon))$.

Because of $x^*(z) = 1$ this implies $x^*(y) > 1 - \varepsilon$ and the proof is finished. □

It is claimed in [10, Lemma 3.10] that the modulus of U-convexity, which coincides with our modulus δ^X_{uacs}, is continuous on $]0, 2[$, but it seems that the proof given there only works in the case $\varepsilon < 1$ (this is not a major drawback since one is usually interested in small values of ε). We wish to point out that for values between 0 and 1 even more is true, namely δ^X_{uacs} is Lipschitz continuous on $[a, 1[$ for every $0 < a < 1$.

Lemma 2.11. *For every Banach space X and all $0 < \varepsilon, \varepsilon' < 1$ we have*

$$\left|\delta^X_{\mathrm{uacs}}(\varepsilon) - \delta^X_{\mathrm{uacs}}(\varepsilon')\right| \leq \frac{|\varepsilon - \varepsilon'|}{\min\{\varepsilon, \varepsilon'\}}.$$

In particular, δ^X_{uacs} is Lipschitz continuous on $[a, 1[$ for all $0 < a < 1$.

Proof. Let $0 < \varepsilon < 1$ and $0 < \beta < 1 - \varepsilon$. Put $\tau = \beta/(\varepsilon + \beta)$ and take $x, y \in S_X$ and $x^* \in S_{X^*}$ such that $x^*(x) = 1$ and $x^*(y) \leq 1 - \varepsilon$. Let $z = (y - \tau x)/\|y - \tau x\|$. Note that, since $\|y - \tau x\| \geq 1 - \tau$ and $\varepsilon + \tau < 1$, we have

$$x^*(z) \leq \frac{1 - \varepsilon - \tau}{1 - \tau} = 1 - \varepsilon\left(1 + \frac{\tau}{1-\tau}\right) = 1 - (\varepsilon + \beta)$$

and hence

$$1-\left\|\frac{x+z}{2}\right\| \geq \delta_{\mathrm{uacs}}^{X}(\varepsilon+\beta).$$

Furthermore, we have

$$\|y-z\| \leq \frac{\|(\|y-\tau x\|-1)y+\tau x\|}{1-\tau} \leq \frac{2\tau}{1-\tau} = \frac{2\beta}{\varepsilon}.$$

It follows that

$$1-\left\|\frac{x+y}{2}\right\| \geq \delta_{\mathrm{uacs}}^{X}(\varepsilon+\beta)-\frac{\beta}{\varepsilon}.$$

Thus we have

$$\delta_{\mathrm{uacs}}^{X}(\varepsilon+\beta) \geq \delta_{\mathrm{uacs}}^{X}(\varepsilon) \geq \delta_{\mathrm{uacs}}^{X}(\varepsilon+\beta)-\frac{\beta}{\varepsilon}$$

for all $0<\varepsilon<1$ and every $0<\beta<1-\varepsilon$, which finishes the proof. □

Next we will deal with some duality results. In [20, Theorem 2.4] a proof of the fact that a Banach space X is a U-space iff its dual X^* is a U-space is proposed and in [13, Theorem 2.6] the stronger statement that the moduli of u-convexity of X and X^* coincide is claimed. Both proofs make use of the following claim from [20, Remark after Definition 2.2]:

Claim. X is a U-space iff for every $\varepsilon>0$ there is some $\delta>0$ such that whenever $x,y \in S_X$ and $x^*,y^* \in S_{X^*}$ with $x^*(x)=1=y^*(y)$ and $\|x+y\|>2(1-\delta)$ then $\|x^*+y^*\|>2(1-\varepsilon)$.

A U-space certainly has the above property. However, the converse need not be true, not even in a two-dimensional space.

To see this, first note that if X is finite-dimensional then by an easy compactness argument the condition of the claim is equivalent to the following one: whenever $x,y \in S_X$ and $x^*,y^* \in S_{X^*}$ with $x^*(x)=1=y^*(y)$ and $\|x+y\|=2$ we also have $\|x^*+y^*\|=2$.

Therefore, if X is finite-dimensional it fulfils the condition of the claim if for each $x,y \in S_X$ with $\|x+y\|=2$ at least one of the two points x and y is a smooth point of the unit ball. But as we have mentioned before, a two-dimensional space is acs (equivalently a U-space) iff whenever $x,y \in S_X$ with $\|x+y\|=2$ then *both* points x and y are smooth points of the unit ball.

Taking all this into account, we see that the space $\mathbb{R}^2$ endowed with the norm whose unit ball is sketched below will be an example of a space which fulfils the condition of the claim but is not a U-space.

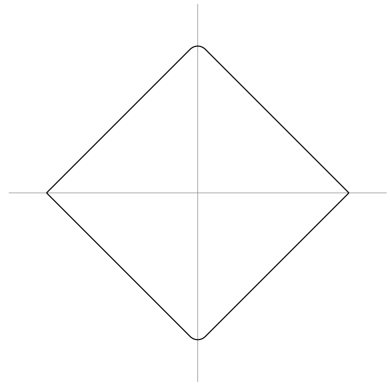

FIG. 5

However, it is possible to modify the proof from [20, Theorem 2.4] to show that the desired self-duality result is true nonetheless.

Proposition 2.12. *Let X be a Banach space whose dual X^* is uacs. Then we have*

$$\delta_{\text{uacs}}^{X}(\varepsilon) \geq \delta_{\text{uacs}}^{X^*}\left(\delta_{\text{uacs}}^{X^*}(\varepsilon)\right) \;\; \forall \varepsilon \in]0,2].$$

In particular, X is also uacs.

Proof. Take any $\varepsilon \in]0,2]$ and put $\delta = \delta_{\text{uacs}}^{X^*}(\varepsilon)$ and $\tilde{\delta} = \delta_{\text{uacs}}^{X^*}(\delta)$.

Now if $x, y \in S_X$ and $x^* \in S_{X^*}$ with $x^*(x) = 1$ and $\|x + y\| > 2(1 - \tilde{\delta})$ choose $y^*, z^* \in S_{X^*}$ such that $y^*(y) = 1$ and $z^*(x + y) = \|x + y\|$.

Then we must have $z^*(x) > 1 - 2\tilde{\delta}$ and $z^*(y) > 1 - 2\tilde{\delta}$. It follows that $(z^* + x^*)(x) > 2 - 2\tilde{\delta}$ and $(z^* + y^*)(y) > 2 - 2\tilde{\delta}$ and hence

$$\left\|\frac{z^* + x^*}{2}\right\| > 1 - \tilde{\delta} \text{ and } \left\|\frac{z^* + y^*}{2}\right\| > 1 - \tilde{\delta}. \tag{2.3}$$

Next we pick any $z^{**} \in S_{X^{**}}$ with $z^{**}(z^*) = 1$. Then from (2.3) and the definition of $\tilde{\delta}$ we get that $z^{**}(x^*) > 1 - \delta$ and $z^{**}(y^*) > 1 - \delta$.

It follows that $\|x^* + y^*\| > 2(1 - \delta)$ and because of $y^*(y) = 1$ and the definition of δ this implies $x^*(y) > 1 - \varepsilon$ and thus we have shown $\delta_{\text{uacs}}^{X}(\varepsilon) \geq \tilde{\delta} = \delta_{\text{uacs}}^{X^*}\left(\delta_{\text{uacs}}^{X^*}(\varepsilon)\right)$. □

Taking into account that uacs spaces are reflexive we finally get that being uacs is a self-dual property.

Corollary 2.13. *A Banach space X is uacs iff X^* is uacs.*

The author does not know whether the equality $\delta_{\text{uacs}}^{X} = \delta_{\text{uacs}}^{X^*}$ that was claimed in [13, Theorem 2.6] is actually true.

Alternatively, we could also derive the self-duality from the following lemma (cf. the proof of [14, Lemma 9.9]). The modulus $\tilde{\rho}_{\text{uacs}}^{X}$ is defined exactly as ρ_{uacs}^{X} except that one replaces S_X by B_X. The argument that X is uacs iff $\lim_{\tau \to 0} \tilde{\rho}_{\text{uacs}}^{X}(\tau)/\tau = 0$ is analogous to the one for ρ_{uacs}^{X}.

Lemma 2.14. *If X is any Banach space then for every $\tau > 0$ and every $0 < \varepsilon \leq 2$ the following inequalities hold:*

(i) $\delta_{\text{uacs}}^{X}(\varepsilon) + \rho_{\text{uacs}}^{X^*}(\tau) \geq \tau\frac{\varepsilon}{2}$

(ii) $\delta_{\mathrm{uacs}}^{X^*}(\varepsilon)+\tilde{\rho}_{\mathrm{uacs}}^{X}(\tau)\geq\tau\frac{\varepsilon}{2}$

Proof. We only prove the slightly more difficult inequality (ii). To this end, fix $x^*,y^*\in S_{X^*}$ and $x^{**}\in S_{X^{**}}$ such that $x^{**}(x^*)=1$ and $x^{**}(y^*)\leq 1-\varepsilon$.

If $\|x^*+y^*\|\leq 2(1-\tau)$ then we certainly have $2-\|x^*+y^*\|\geq\tau\varepsilon-2\tilde{\rho}_{\mathrm{uacs}}^{X}(\tau)$.

If $\|x^*+y^*\|>2(1-\tau)$ then take an arbitrary $0<\alpha<\|x^*+y^*\|-2(1-\tau)$. By Goldstine's theorem there is some $x\in B_X$ such that

$$|x^{**}(x^*)-x^*(x)|\leq\frac{\alpha}{2}\text{ and }|x^{**}(y^*)-y^*(x)|\leq\frac{\alpha}{2}.$$

Now choose $y\in S_X$ such that $(x^*+y^*)(y)>\|x^*+y^*\|-\alpha/2$. It follows that $(x^*+y^*)(y)>2(1-\tau)+\alpha/2$ and hence $x^*(y),y^*(y)>1-2\tau+\alpha/2$.

Thus we have

$$\|x+y\|\geq x^*(x+y)\geq x^{**}(x^*)-\frac{\alpha}{2}+1-2\tau+\frac{\alpha}{2}=2(1-\tau)$$

and hence

$$\begin{aligned}&2\tilde{\rho}_{\mathrm{uacs}}^{X}(\tau)\geq\|y+\tau x\|+\|y-\tau x\|-2\geq x^*(y+\tau x)+y^*(y-\tau x)-2\\&=(x^*+y^*)(y)+\tau(x^*(x)-y^*(x))-2\\&\geq\|x^*+y^*\|-\frac{\alpha}{2}+\tau\left(x^{**}(x^*)-x^{**}(y^*)-\alpha\right)-2\\&\geq\|x^*+y^*\|-\frac{\alpha}{2}+\tau(\varepsilon-\alpha)-2.\end{aligned}$$

For $\alpha\to 0$ we get $2-\|x^*+y^*\|\geq\tau\varepsilon-2\tilde{\rho}_{\mathrm{uacs}}^{X}(\tau)$ and we are done. ☐

There are also some duality result on acs, luacs, sluacs and wuacs spaces which we will treat in the following. The proof of the first statement is very easy and will therefore be omitted.

Proposition 2.15. *A Banach space X is acs iff for all $x^*,y^*\in S_{X^*}$ and all $x,y\in S_X$ the implication*

$$(x^*+y^*)(x)=2\text{ and }x^*(y)=1\ \Rightarrow\ y^*(y)=1$$

holds. In particular, if X^* is acs then so is X and the converse is true if X is reflexive.

We will say that a dual space X^* is luacs* resp. wuacs* if it fulfils the definition of an luacs resp. wuacs space with for all weak*-continuous functionals on X^*. With this terminology the following is valid.

Proposition 2.16. *For any Banach space X we have the following implications.*

(i) X^* luacs* $\iff$ X luacs
(ii) X^* wuacs* $\iff$ X sluacs
(iii) X^* sluacs $\iff$ X wuacs

Proof. We only prove (iii). Let us first assume that X^* is sluacs and take sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in S_X and a functional $x^*\in S_{X^*}$ such that $\|x_n+y_n\|\to 2$ and $x^*(x_n)\to 1$.

Choose a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} with $x_n^*(x_n+y_n)=\|x_n+y_n\|$ for every n. It follows that $x_n^*(x_n)\to 1$ and $x_n^*(y_n)\to 1$.

From $x^*(x_n) \to 1$ and $x_n^*(x_n) \to 1$ we get $\|x_n^* + x^*\| \to 2$. Together with $x_n^*(y_n) \to 1$ and the fact that X^* is sluacs this implies $x^*(y_n) \to 1$ and we are done.

Now assume X is wuacs and fix a sequence $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and $x^* \in S_{X^*}$ such that $\|x_n^* + x^*\| \to 2$ as well as a sequence $(x_n^{**})_{n\in\mathbb{N}}$ in $S_{X^{**}}$ with $x_n^{**}(x_n^*) \to 1$.

Because of $\|x_n^* + x^*\| \to 2$ we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in S_X such that $x_n^*(x_n) \to 1$ and $x^*(x_n) \to 1$.

By Goldstine's theorem we can also find a sequence $(y_n)_{n\in\mathbb{N}}$ in B_X which satisfies

$$|x_n^*(y_n) - x_n^{**}(x_n^*)| \leq \frac{1}{n} \text{ and } |x^*(y_n) - x_n^{**}(x^*)| \leq \frac{1}{n} \quad \forall n \in \mathbb{N}.$$

So we have $x_n^*(x_n + y_n) \to 2$ and hence $\|x_n + y_n\| \to 2$. Since X is wuacs and $x^*(x_n) \to 1$ we must also have $x^*(y_n) \to 1$ and consequently $x_n^{**}(x^*) \to 1$. □

If X is reflexive then by (i) and (ii) of the preceding proposition X^* is luacs (resp. wuacs) iff X is luacs (resp. sluacs). Next we would like to give necessary and sufficient conditions for a dual space to be acs resp. luacs resp. wuacs that do not explicitly involve the bidual space. We start with the acs case. The characterisation is inspired by [28, Proposition 3].

Proposition 2.17. *Let X be any Banach space. The dual space X^* is acs iff for all sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in B_X and all functionals $x^*, y^* \in S_{X^*}$ the implication*

$$x^*(x_n + y_n) \to 2 \text{ and } y^*(x_n) \to 1 \;\Rightarrow\; y^*(y_n) \to 1$$

holds.

Proof. To prove the necessity, assume that X^* is acs and take two sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ and functionals x^*, y^* as above. It follows that $\|x^* + y^*\| = 2$. By the weak*-compactness of $B_{X^{**}}$ we can find for an arbitrary subsequence $(y_{n_k})_{k\in\mathbb{N}}$ a subnet $(y_{n_{\phi(i)}})_{i\in I}$ that weak*-converges to some $y^{**} \in B_{X^{**}}$. It follows that $y^{**}(x^*) = 1$ and since X^* is acs we must also have $y^{**}(y^*) = 1$. Thus $y^*(y_{n_{\phi(i)}}) \to 1$ and the proof of the necessity is finished.

Now assume that X^* fulfils the above condition and take $x^*, y^* \in S_{X^*}$ and $x^{**} \in S_{X^{**}}$ such that $\|x^* + y^*\| = 2$ and $x^{**}(x^*) = 1$. Then we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X such that $x^*(x_n) \to 1$ and $y^*(x_n) \to 1$.

By Goldstine's theorem there is a sequence $(y_n)_{n\in\mathbb{N}}$ in B_X such that $x^*(y_n) \to x^{**}(x^*) = 1$ and $y^*(y_n) \to x^{**}(y^*)$.

Thus we have $x^*(x_n + y_n) \to 2$ and $y^*(x_n) \to 1$ and hence by our assumption we get $y^*(y_n) \to 1$, so $x^{**}(y^*) = 1$. □

The characterisations for the dual space to be luacs resp. wuacs are a bit more complicated. They read as follows.

Proposition 2.18. *Let X be a Banach space.*

(i) *X^* is luacs iff for every $x^* \in S_{X^*}$ and all sequences $(x_n^*)_{n\in\mathbb{N}}$ and $(x_k)_{k\in\mathbb{N}}$ in S_{X^*} and B_X respectively, the implication*

$$\|x^* + x_n^*\| \to 2 \text{ and } x_n^*(x_k) \xrightarrow[k\geq n]{k,n\to\infty} 1 \;\Rightarrow\; x^*(x_k) \to 1$$

holds.

(ii) *X^* is wuacs iff for all sequences $(x_n^*)_{n\in\mathbb{N}}, (y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and $(x_k)_{k\in\mathbb{N}}$ in B_X the implication*

$$\|x_n^* + y_n^*\| \to 2 \text{ and } x_n^*(x_k) \xrightarrow[k\geq n]{k,n\to\infty} 1 \;\Rightarrow\; \lim_{n\to\infty}\sup_{k\geq n} y_n^*(x_k) = 1.$$

holds.

Proof. To prove (ii) we first assume that X^* is wuacs and fix sequences $(x_n^*)_{n\in\mathbb{N}}$ and $(y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} as well as $(x_k)_{k\in\mathbb{N}}$ in B_X as above. Since $B_{X^{**}}$ is weak*-compact there is a subnet $(x_{\phi(i)})_{i\in I}$ that is weak*-convergent to some $x^{**}\in B_{X^{**}}$. We will show that $x^{**}(x_n^*)\to 1$.

Given any $\varepsilon > 0$ by our assumption on $(x_n^*)_{n\in\mathbb{N}}$ and $(x_k)_{k\in\mathbb{N}}$ we can find an $N\in\mathbb{N}$ such that

$$|x_n^*(x_k) - 1| \leq \varepsilon \;\; \forall k \geq n \geq N.$$

For every $n \geq N$ it is possible to find an index $i \in I$ with $\phi(i) \geq n$ and $\left|x_n^*(x_{\phi(i)}) - x^{**}(x_n^*)\right| \leq \varepsilon$. It follows that $|x^{**}(y_n^*) - 1| \leq 2\varepsilon$ and the convergence is proved.

So we have $\|x_n^* + y_n^*\| \to 2$ and $x^{**}(x_n^*) \to 1$. Since X^* is wuacs this implies $x^{**}(y_n^*) \to 1$. Thus for any $\delta > 0$ there is some $n_0 \in \mathbb{N}$ such that $|x^{**}(y_n^*) - 1| \leq \delta$ for all $n \geq n_0$ and for any such n we find $j \in I$ with $\phi(j) \geq n$ and $\left|y_n^*(x_{\phi(i)}) - x^{**}(y_n^*)\right| \leq \delta$. Hence $\left|y_n^*(x_{\phi(i)}) - 1\right| \leq 2\delta$ and we have shown $\sup_{k\geq n} y_n^*(x_k) \geq 1 - 2\delta$ for all $n \geq n_0$.

Now let us prove the converse. We take sequences $(x_n^*)_{n\in\mathbb{N}}, (y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $\|x_n^* + y_n^*\| \to 2$ and a functional $x^{**}\in S_{X^{**}}$ with $x^{**}(x_n^*) \to 1$.

By means of Goldstine's theorem we find a sequence $(x_k)_{k\in\mathbb{N}}$ in B_X that satisfies

$$|x_n^*(x_k) - x^{**}(x_n^*)| \leq \frac{1}{k} \text{ and } |y_n^*(x_k) - x^{**}(y_n^*)| \leq \frac{1}{k} \;\; \forall n \leq k.$$

It is then easy to see that $(x_n^*(x_k))_{k\geq n}$ tends to 1 and hence our assumption gives us $\lim_{n\to\infty}\sup_{k\geq n} y_n^*(x_k) = 1$.

Thus for any $\varepsilon > 0$ there exists $N\in\mathbb{N}$ with $\sup_{k\geq n} y_n^*(x_k) > 1-\varepsilon$ and $1/n \leq \varepsilon$ for each $n \geq N$.

If we fix $n \geq N$ we find $k \geq n$ with $y_n^*(x_k) \geq 1-\varepsilon$ and because of the inequality $|x^{**}(y_n^*) - y_n^*(x_k)| \leq 1/k \leq \varepsilon$ it follows that $x^{**}(y_n^*) \geq 1 - 2\varepsilon$ and the proof is finished. Part (i) is proved similarly. □

One can also give some more characterisations of acs, luacs and sluacs spaces by apparently stronger properties.

Proposition 2.19. *For a Banach space X, the following assertions are equivalent:*

(i) *X is acs.*

(ii) *For all sequences $(x_n^*)_{n\in\mathbb{N}}, (y_n^*)_{n\in\mathbb{N}}$ in B_{X^*} and all $x, y \in S_X$ the implication*

$$(x_n^* + y_n^*)(x) \to 2 \text{ and } y_n^*(y) \to 1 \;\Rightarrow\; x_n^*(y) \to 1$$

holds.

(iii) *For every sequence* $(x_n^*)_{n\in\mathbb{N}}$ *in* S_{X^*} *and all* $x, y \in S_X$ *the implication*

$$\|x+y\| = 2 \text{ and } x_n^*(x) \to 1 \;\Rightarrow\; x_n^*(y) \to 1$$

holds.

Proof. (i) ⇒ (ii) follows from Proposition 2.15 together with the fact that B_{X^*} is weak*-compact, the implication (iii) ⇒ (i) is trivial and (ii) ⇒ (iii) is also quite easy to see. □

By means of Goldstine's theorem one can also prove the following characterisation of luacs spaces (we omit the details).

Proposition 2.20. *A Banach space* X *is luacs if and only if for every sequence* $(x_n^{**})_{n\in\mathbb{N}}$ *in* $S_{X^{**}}$*, every* $x \in S_X$ *and each* $x^* \in S_{X^*}$ *the implication*

$$\|x_n^{**} + x\| \to 2 \text{ and } x_n^{**}(x^*) \to 1 \;\Rightarrow\; x^*(x) = 1.$$

holds.

Let us denote by $X^{(k)}$ the k-th dual of X. Then X resp. X^* naturally embeds into $X^{(2k)}$ resp. $X^{(2k+1)}$ for each k. For sluacs spaces we have the following stronger result.

Proposition 2.21. *A Banach space* X *is sluacs iff for every* $k \in \mathbb{N}$*, for every sequence* $(z_n)_{n\in\mathbb{N}}$ *in* $B_{X^{(2k)}}$*, every* $x \in S_X$ *and each sequence* $(z_n^*)_{n\in\mathbb{N}}$ *in* $B_{X^{(2k+1)}}$ *the implication*

$$\|z_n + x\| \to 2 \text{ and } z_n^*(z_n) \to 1 \;\Rightarrow\; z_n^*(x) \to 1$$

holds.

Proof. The sufficiency is obvious. To prove the necessity, we first take sequences $(x_n^{**})_{n\in\mathbb{N}}$ in $B_{X^{**}}$ and $(x_n^{***})_{n\in\mathbb{N}}$ in $B_{X^{***}}$ as well as an element $x \in S_X$ such that $\|x_n^{**} + x\| \to 2$ and $x_n^{***}(x_n^{**}) \to 1$. Then we can find a sequence $(y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $x_n^{**}(y_n^*) \to 1$ and $y_n^*(x) \to 1$.

By Goldstine's theorem (applied to X^*) there is a sequence $(x_n^*)_{n\in\mathbb{N}}$ in B_{X^*} such that $x_n^{***}(x_n^{**}) - x_n^{**}(x_n^*) \to 0$ and $x_n^{***}(x) - x_n^*(x) \to 0$. Hence $x_n^{**}(x_n^*) \to 1$.

Again by Goldstine's theorem (now applied to X) there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X such that $x_n^{**}(x_n^*) - x_n^*(x_n) \to 0$ and $x_n^{**}(y_n^*) - y_n^*(x_n) \to 0$. It follows that $x_n^*(x_n) \to 1$ and $y_n^*(x_n) \to 1$.

Taking into account that $y_n^*(x) \to 1$ we get $\|x_n + x\| \to 2$. Since X is sluacs it follows $x_n^*(x) \to 1$ and hence $x_n^{***}(x) \to 1$.

Thus we have proved our claim for $k = 1$. Continuing by induction with the above argument we can show it for all $k \in \mathbb{N}$. □

If we use the preceding proposition and the technique of proof from Proposition 2.4 we see that the following holds.

Proposition 2.22. *For a Banach space* X *the following assertions are equivalent:*

(i) *X is sluacs.*

(ii) *For every* $k \in \mathbb{N}$*, every* $\varepsilon > 0$ *and every* $y \in S_X$ *there is some* $\delta > 0$ *such that for all* $t \in [0, \delta]$ *and each* $z \in S_{X^{(2k)}}$ *with* $\|z + y\| \ge 2(1-t)$ *we have*

$$\|z + ty\| + \|z - ty\| \le 2 + \varepsilon t.$$

(iii) *For every $k \in \mathbb{N}$, every $\varepsilon > 0$ and every $y \in S_X$ there is some $\delta > 0$ such that for all $t \in [0, \delta]$ and each $z \in S_{X^{(2k)}}$ with $\|z + y\| \geq 2 - t\delta$ we have*

$$\|z - ty\| \leq 1 + t(\varepsilon - 1).$$

Finally, let us consider quotient spaces. If U is a closed subpace of X then $(X/U)^*$ is isometrically isomorphic to $U^\perp$ (the annihilator of U in X^*). Using this together with Corollary 2.13 and the obvious fact that closed subspaces of uacs spaces are again uacs, one immediately gets that quotients of uacs spaces are uacs as well. An analogous argument using part (iii) of Proposition 2.16 works for wuacs spaces, so in the summary we have the following proposition.

Proposition 2.23. *Let U be a closed subspace of the Banach space X. If X is uacs (resp. wuacs) then X/U is also uacs (resp. wuacs).*

As for quotients of acs, luacs and sluacs spaces we have the following result which is an analogue of [19, Proposition 3.2].

Proposition 2.24. *If U is a reflexive subspace of the Banach space X then the properties acs, luacs and sluacs pass from X to X/U.*

Proof. Let $\omega : X \to X/U$ be the canonical quotient map. As was observed in the proof of [19, Proposition 3.2] the reflexivity of U implies $\omega(B_X) = B_{X/U}$.

Now suppose that X is sluacs and take a sequence $(z_n)_{n\in\mathbb{N}}$ in $S_{X/U}$ and an element $z \in S_{X/U}$ such that $\|z_n + z\| \to 2$. Further, take a sequence $(\psi_n)_{n\in\mathbb{N}}$ in $S_{(X/U)^*}$ with $\psi_n(z_n) \to 1$.

Since $\omega(B_X) = B_{X/U}$ we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and a point $x \in S_X$ such that $z_n = \omega(x_n)$ for every n and $z = \omega(x)$.

It easily follows from $\|z_n + z\| \to 2$ that we also have $\|x_n + x\| \to 2$.

We put $x_n^* := \psi_n \circ \omega \in S_{U^\perp}$ for every n and observe that $x_n^*(x_n) = \psi_n(z_n) \to 1$. Since X is sluacs this implies $x_n^*(x) = \psi_n(z) \to 1$.

The proofs for acs and luacs spaces are analogous. □

Using again the relation $(X/U)^* \cong U^\perp$ for every closed subspace U of X we can derive the following from Propositions 2.15 and 2.16.

Proposition 2.25. *If U is a closed subspace of the Banach space X the following implications hold.*

(i) X^* acs $\Rightarrow$ X/U acs
(ii) X^* luacs $\Rightarrow$ X/U luacs
(iii) X^* wuacs $\Rightarrow$ X/U sluacs

It is known (cf. [8, p.145]) that for any Banach space X the dual X^* is R (resp. S) iff every quotient space of X is S (resp. R) iff every two-dimensional quotient space of X is S (resp. R). By an analogous argument we can get the following result.

Proposition 2.26. *For a Banach space X the following assertions are equivalent.*

(i) *X^* is acs.*
(ii) *X/U is acs for every closed subspace U of X.*
(iii) *X/U is acs for every closed subspace U of X with $\dim X/U = 2$.*

Proof. (i) $\Rightarrow$ (ii) holds according to Proposition 2.25 and (ii) $\Rightarrow$ (iii) is trivial, so it only remains to prove (iii) $\Rightarrow$ (i). Obviously it suffices to show that every two-dimensional subspace of X^* is acs, so let us take such a subspace $V = \operatorname{span}\{x^*, y^*\}$. Then $V = U^\perp = (X/U)^*$, where $U = \ker x^* \cap \ker y^*$. The quotient space X/U is two-dimensional and hence by our assumption it is acs. Since X/U is in particular reflexive it follows from Proposition 2.16 that $(X/U)^* = V$ is also acs. □

By [19, Proposition 3.4] there is an equivalent norm $\|\cdot\|$ on ℓ^1 such that $(\ell^1, \|\cdot\|)$ is R and every separable Banach space is isometrically isomorphic to a quotient space of $(\ell^1, \|\cdot\|)$, so in particular ℓ^1 is a quotient of $(\ell^1, \|\cdot\|)$. Thus quotients of acs spaces are in general not acs and it also follows (in view of Proposition 2.26) that the fact that X is acs is not sufficient to ensure that X^* is acs.

There is also an analogue of Proposition 2.26 for uacs spaces which reads as follows. (The corresponding result for UR spaces was proved by Day (cf. [7, Theorem 5.5]).)

Proposition 2.27. *For a Banach space X let $\mathcal{S}(X)$ denote the set of all closed subspaces of X and $\mathcal{S}_2(X)$ the set of all closed subspaces U of X such that* $\dim X/U \leq 2$. *Then the following assertions are equivalent:*

(i) *X is uacs.*
(ii) $\inf\left\{\delta_{\mathrm{uacs}}^{X/U}(\varepsilon) : U \in \mathcal{S}(X)\right\} > 0 \quad \forall \varepsilon \in]0, 2]$.
(iii) $\inf\left\{\delta_{\mathrm{uacs}}^{X/U}(\varepsilon) : U \in \mathcal{S}_2(X)\right\} > 0 \quad \forall \varepsilon \in]0, 2]$.

Proof. (i) $\Rightarrow$ (ii) Let X be uacs. If $U \in \mathcal{S}(X)$ then $(X/U)^* \cong U^\perp$, hence $\delta_{\mathrm{uacs}}^{(X/U)^*}(\varepsilon) \geq \delta_{\mathrm{uacs}}^{X^*}(\varepsilon) \geq \delta_{\mathrm{uacs}}^{X}\left(\delta_{\mathrm{uacs}}^{X}(\varepsilon)\right)$ by Proposition 2.12 and the reflexivity of X.

Using again Proposition 2.12 (now applied to X/U) and the monotonicity of the uacs modulus we obtain

$$\delta_{\mathrm{uacs}}^{X/U}(\varepsilon) \geq \delta_{\mathrm{uacs}}^{X}\left(\delta_{\mathrm{uacs}}^{X}\left(\delta_{\mathrm{uacs}}^{X}\left(\delta_{\mathrm{uacs}}^{X}(\varepsilon)\right)\right)\right) > 0,$$

which finishes our argument.

Since (ii) $\Rightarrow$ (iii) is obvious it only remains to prove (iii) $\Rightarrow$ (i). Denote the infimum in (iii) by $\delta(\varepsilon)$ and take sequence $(x_n^*)_{n\in\mathbb{N}}, (y_n^*)_{n\in\mathbb{N}}$ in S_{X^*} such that $\|x_n^* + y_n^*\| \to 2$ and a sequence $(x_n^{**})_{n\in\mathbb{N}}$ in $S_{X^{**}}$ with $x_n^{**}(x_n^*) \to 1$.

We put $V_n = \operatorname{span}\{x_n^*, y_n^*\}$ and $U_n = \ker x_n^* \cap \ker y_n^*$ for every n. Then $V_n = U_n^\perp = (X/U_n)^*$. Again by Proposition 2.12 (and reflexivity of X/U_n) we get that $\delta_{\mathrm{uacs}}^{V_n}(\varepsilon) \geq \delta_{\mathrm{uacs}}^{X/U_n}\left(\delta_{\mathrm{ucas}}^{X/U_n}(\varepsilon)\right) \geq \delta\left(\delta(\varepsilon)\right)$.

Let φ_n denote the restriction of x_n^{**} to V_n and fix any $\varepsilon_0 > 0$. Because of $\|x_n^* + y_n^*\| \to 2$ we have $1 - 2^{-1}\|x_n^* + y_n^*\| < \delta\left(\delta(\varepsilon_0)\right) \leq \delta_{\mathrm{uacs}}^{V_n}(\varepsilon_0)$ for sufficiently large n.

Since $\varphi_n(x_n^*) = 1$ this implies that we eventually have $\varphi_n(y_n^*) = x_n^{**}(y_n^*) \geq 1 - \varepsilon_0$.

Thus we have shown that X^* is uacs and by Proposition 2.12 X is uacs as well. □

In the next section we will study absolute sums of uacs spaces and their relatives, but first we have to introduce two more definitions that will be needed, namely "symmetrised" versions of the notions of luacs and sluacs spaces.

Definition 2.28. A Banach space X is called

(i) a luacs$^+$ space if for every $x \in S_X$, every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X with $\|x_n + x\| \to 2$ and all $x^* \in S_{X^*}$ we have

$$x^*(x_n) \to 1 \iff x^*(x) = 1,$$

(ii) a sluacs$^+$ space if for every $x \in S_X$, every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X with $\|x_n + x\| \to 2$ and all sequences $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} we have

$$x_n^*(x_n) \to 1 \iff x_n^*(x) \to 1.$$

If we include these two properties in our implication chart we get the following.

$$\begin{tikzcd}
 & \text{WUR} \arrow[d] \arrow[dr] & & \\
\text{UR} \arrow[ur] \arrow[d] \arrow[dr] & \text{wuacs} \arrow[dr] & \text{WLUR} \arrow[r] \arrow[d] & \text{R} \arrow[dd] \\
\text{uacs} \arrow[ur] \arrow[dr] & \text{LUR} \arrow[ur] \arrow[d] & \text{luacs}^+ \arrow[d] & \\
 & \text{sluacs}^+ \arrow[ur] \arrow[d] & \text{luacs} \arrow[r] & \text{acs} \\
 & \text{sluacs} \arrow[ur] & &
\end{tikzcd}$$

FIG. 6

Let us mention that Proposition 2.24 also holds for luacs$^+$ and sluacs$^+$ spaces (with the same argument). Also, Propositions 2.20 resp. 2.21 hold accordingly for luacs$^+$ resp. sluacs$^+$ spaces.

In analogy to Proposition 2.4 one can prove that for any Banach space X the following conditions are equivalent:

(i) For all sequences $(x_n)_{n\in\mathbb{N}}$ in S_X, $(x_n^*)_{n\in\mathbb{N}}$ in S_{X^*} and every $x \in S_X$ with $\|x_n + x\| \to 2$ and $x_n^*(x) \to 1$ one has $x_n^*(x_n) \to 1$.

(ii) For every $x \in S_X$ and every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x + ty\| + \|x - ty\| \leq 2 + \varepsilon t$$

whenever $t \in [0, \delta]$ and $y \in S_X$ with $\|x + y\| \geq 2(1 - t)$.

In particular, every FS space fulfils (i) and hence a space which is FS and sluacs is sluacs$^+$. In the context of FS spaces we also have the following proposition.

Proposition 2.29. *If X is FS and X^* is acs then X is luacs$^+$. In particular, every reflexive FS space is luacs$^+$.*

Proof. By our previous considerations we only have to show that X is luacs.

Take a sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and a point $x \in S_X$ with $\|x_n + x\| \to 2$ as well as a functional $x^* \in S_{X^*}$ with $x^*(x_n) \to 1$. Choose a sequence $(y_n^*)_{n\in\mathbb{N}}$ in S_{X^*}

such that $y_n^*(x_n + x) = \|x_n + x\|$ for every $n \in \mathbb{N}$. It follows that $y_n^*(x_n) \to 1$ and $y_n^*(x) \to 1$.

Because of $\|y_n^* + x^*\| \geq y_n^*(x_n) + x^*(x_n)$ for every n it follows that $\|y_n^* + x^*\| \to 2$. If $y^* \in S_{X^*}$ is the Fréchet-derivative of $\|\,.\,\|$ at x then $y_n^*(x) \to 1$ implies $\|y_n^* - y^*\| \to 0$. Hence we get $\|x^* + y^*\| = 2$ and $y^*(x) = 1$.

Since X^* is acs we can conclude that $x^*(x) = 1$. □

We conclude this section with a simple lemma that will be frequently used in the sequel. It is the generalisation of [1, Lemma 2.1] to sequences, while the proof remains virtually the same.

Lemma 2.30. *Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in the (real or complex) normed space X such that $\|x_n + y_n\| - \|x_n\| - \|y_n\| \to 0$.*

Then for any two bounded sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ of non-negative real numbers we also have $\|\alpha_n x_n + \beta_n y_n\| - \alpha_n \|x_n\| - \beta_n \|y_n\| \to 0$.

Proof. Let $n \in \mathbb{N}$ be arbitrary. If $\alpha_n \geq \beta_n$ then

$$\begin{aligned}
&\|\alpha_n x_n + \beta_n y_n\| \geq \alpha_n \|x_n + y_n\| - (\alpha_n - \beta_n)\|y_n\| \\
&= \alpha_n (\|x_n + y_n\| - \|x_n\| - \|y_n\|) + \alpha_n \|x_n\| + \beta_n \|y_n\|
\end{aligned}$$

and hence

$$\|\alpha_n x_n + \beta_n y_n\| - \alpha_n \|x_n\| - \beta_n \|y_n\| \geq \alpha_n (\|x_n + y_n\| - \|x_n\| - \|y_n\|).$$

Analogously one can show that

$$\|\alpha_n x_n + \beta_n y_n\| - \alpha_n \|x_n\| - \beta_n \|y_n\| \geq \beta_n (\|x_n + y_n\| - \|x_n\| - \|y_n\|)$$

if $\alpha_n < \beta_n$. Since $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are bounded we obtain the desired conclusion. □

3. Absolute sums

We begin by recalling some preliminaries on absolute sums. Let I be a non-empty set, E a subspace of $\mathbb{R}^I$ with $e_i \in E$ for all $i \in I$ and $\|\,.\,\|_E$ a complete norm on E (here e_i denotes the characteristic function of $\{i\}$).
The norm $\|\,.\,\|_E$ is called *absolute* if the following holds

$$\begin{aligned}
&(a_i)_{i\in I} \in E,\ (b_i)_{i\in I} \in \mathbb{R}^I \text{ and } |a_i| = |b_i|\ \forall i \in I \\
&\Rightarrow\ (b_i)_{i\in I} \in E \text{ and } \|(a_i)_{i\in I}\|_E = \|(b_i)_{i\in I}\|_E.
\end{aligned}$$

The norm is called *normalised* if $\|e_i\| = 1$ for every $i \in I$.

Standard examples of subspaces of $\mathbb{R}^I$ with absolute normalised norm are the spaces $\ell^p(I)$ for $1 \leq p \leq \infty$.

We have the following important lemma on absolute normalised norms, whose proof can be found for example in [21, Remark 2.1].

Lemma 3.1. *Let $(E, \|\,.\,\|_E)$ be a subspace of $\mathbb{R}^I$ with an absolute normalised norm. Then the following is true.*

$$\begin{aligned}
&(a_i)_{i\in I} \in E,\ (b_i)_{i\in I} \in \mathbb{R}^I \text{ and } |b_i| \leq |a_i|\ \forall i \in I \\
&\Rightarrow\ (b_i)_{i\in I} \in E \text{ and } \|(b_i)_{i\in I}\|_E \leq \|(a_i)_{i\in I}\|_E.
\end{aligned}$$

Furthermore, the inclusions $\ell^1(I) \subseteq E \subseteq \ell^\infty(I)$ hold and the respective inclusion mappings are contractive.

For a given subspace $(E, \|\,.\,\|_E)$ of $\mathbb{R}^I$ endowed with an absolute normalised norm we put

$$E' := \left\{ (a_i)_{i\in I} \in \mathbb{R}^I : \sup_{(b_i)_{i\in I}\in B_E} \sum_{i\in I} |a_i b_i| < \infty \right\}.$$

It is easy to check that E' is a subspace of $\mathbb{R}^I$ and that

$$\|(a_i)_{i\in I}\|_{E'} := \sup_{(b_i)_{i\in I}\in B_E} \sum_{i\in I} |a_i b_i| \quad \forall (a_i)_{i\in I} \in E'$$

defines an absolute normalised norm on E'.

The map $T : E' \to E^*$ defined by

$$T((a_i)_{i\in I})((b_i)_{i\in I}) := \sum_{i\in I} a_i b_i \quad \forall (a_i)_{i\in I} \in E', \forall (b_i)_{i\in I} \in E$$

is easily seen to be an isometric embedding. Moreover, if $\operatorname{span}\{e_i : i \in I\}$ is dense in E then T is onto, so in this case we can identify E^* and E'.

Now if $(X_i)_{i\in I}$ is a family of (real or complex) Banach spaces we put

$$\Big[\bigoplus_{i\in I} X_i\Big]_E := \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i : (\|x_i\|)_{i\in I} \in E \right\}.$$

It is not hard to see that this defines a subspace of the product space $\prod_{i\in I} X_i$ which becomes a Banach space when endowed with the norm

$$\|(x_i)_{i\in I}\|_E := \|(\|x_i\|)_{i\in I}\|_E \quad \forall (x_i)_{i\in I} \in \Big[\bigoplus_{i\in I} X_i\Big]_E.$$

We call this Banach space the absolute sum of the family $(X_i)_{i\in I}$ with respect to E. Again, the map

$$S : \Big[\bigoplus_{i\in I} X_i^*\Big]_{E'} \to \Big[\bigoplus_{i\in I} X_i\Big]_E^*$$
$$S((x_i^*)_{i\in I})((x_i)_{i\in I}) := \sum_{i\in I} x_i^*(x_i)$$

is an isometric embedding and it is onto if $\operatorname{span}\{e_i : i \in I\}$ is dense in E.

We also mention the following well-known fact, which will be needed later.

Lemma 3.2. *If E is a subspace of $\mathbb{R}^I$ endowed with an absolute normalised norm and* $\operatorname{span}\{e_i : i \in I\}$ *is dense in E then E contains no isomorphic copy of ℓ^1 iff* $\operatorname{span}\{e_i : i \in I\}$ *is dense in E'.*

If not otherwise stated, we shall henceforth assume E to be a subspace of $\mathbb{R}^I$ with an absolute normalised norm such that $\operatorname{span}\{e_i : i \in I\}$ is dense in E.

Now let us first have a look at absolute sums of acs spaces.

Proposition 3.3. *If $(X_i)_{i\in I}$ is a family of acs spaces and E is acs then $\big[\bigoplus_{i\in I} X\big]_E$ is also acs.*

Proof. Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be elements of the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ and $x^* = (x_i^*)_{i\in I}$ an element of the dual unit sphere such that $\|x + y\|_E = 2$ and $x^*(x) = 1$. We then have

$$1 = x^*(x) = \sum_{i\in I} x_i^*(x_i) \leq \sum_{i\in I} \|x_i^*\|\|x_i\| \leq \|x^*\|_{E'}\|x\|_E = 1$$

and hence

$$x_i^*(x_i) = \|x_i^*\|\|x_i\| \ \forall i \in I \text{ and } \sum_{i\in I}\|x_i^*\|\|x_i\| = 1. \tag{3.1}$$

Moreover, by Lemma 3.1 we have

$$\begin{aligned} 2 = \|x + y\|_E &= \|(\|x_i + y_i\|)_{i\in I}\|_E \leq \|(\|x_i\| + \|y_i\|)_{i\in I}\|_E \\ &\leq \|x\|_E + \|y\|_E = 2 \end{aligned}$$

and thus

$$\|(\|x_i\| + \|y_i\|)_{i\in I}\|_E = 2. \tag{3.2}$$

Since E is acs (3.2) and the second part of (3.1) imply that

$$\sum_{i\in I}\|x_i^*\|\|y_i\| = 1. \tag{3.3}$$

Another application of Lemma 3.1 shows

$$\|(\|x_i + y_i\| + \|x_i\| + \|y_i\|)_{i\in I}\|_E = 4. \tag{3.4}$$

Again, since E is acs we get from (3.4), (3.3) and the second part of (3.1) that

$$\sum_{i\in I}\|x_i^*\|\|x_i + y_i\| = 2$$

which together with (3.1) and (3.3) implies

$$\|x_i^*\| \left(\|x_i\| + \|y_i\| - \|x_i + y_i\|\right) = 0 \ \ \forall i \in I. \tag{3.5}$$

Next we claim that

$$x_i^*(y_i) = \|x_i^*\|\|y_i\| \ \ \forall i \in I. \tag{3.6}$$

To see this, fix any $i_0 \in I$ with $x_{i_0}^* \neq 0$ and $y_{i_0} \neq 0$. Define $a_i = \|x_i^*\|$ for all $i \in I \setminus \{i_0\}$ and $a_{i_0} = 0$. Then $(a_i)_{i\in I} \in B_{E'}$, because of Lemma 3.1.

If $x_{i_0} = 0$ it would follow that $\sum_{i\in I} a_i\|x_i\| = \sum_{i\in I}\|x_i^*\|\|x_i\| = 1$ and hence (because of (3.2) and since E is acs) we would also have $\sum_{i\in I} a_i \|y_i\| = 1$. But by (3.3) this would imply $\|y_{i_0}\|\|x_{i_0}^*\| = \sum_{i\in I}\|y_i\| \left(\|x_i^*\| - a_i\right) = 0$, a contradiction.

Thus $x_{i_0} \neq 0$. From (3.5) and Lemma 2.30 we get that

$$\left\|\frac{x_{i_0}}{\|x_{i_0}\|} + \frac{y_{i_0}}{\|y_{i_0}\|}\right\| = 2.$$

Taking into account the first part of (3.1) and the fact that X_{i_0} is acs we get $x_{i_0}^*(y_{i_0}) = \|x_{i_0}^*\|\|y_{i_0}\|$, as desired.

Now from (3.6) and (3.3) it follows that $x^*(y) = 1$ and we are done. □

We remark that the special case of finitely many summands in the above proposition has already been treated in [11] (for two summand) and [23] (for finitely many summands) in the context of u-spaces and the so called ψ-direct sums.

Before we can get on, we have to introduce another technical definition.

Definition 3.4. The space E is said to have the property (P) if for every sequence $(a_n)_{n\in\mathbb{N}}$ in S_E and every $a \in S_E$ we have

$$\|a_n + a\|_E \to 2 \ \Rightarrow \ a_n \to a \text{ pointwise.}$$

If E is WLUR then it obviously has property (P). The converse is true if E contains no isomorphic copy of ℓ^1 by Lemma 3.2.

With this notion we can formulate the following proposition.

Proposition 3.5. *If $(X_i)_{i\in I}$ is a family of sluacs (resp. luacs) spaces and E is sluacs (resp. luacs) and has the property (P) then $\left[\bigoplus_{i\in I} X_i\right]_E$ is sluacs (resp. luacs) as well.*

Proof. We only prove the sluacs case. The argument for luacs spaces is analogous.

So let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ and $x = (x_i)_{i\in I}$ another element of norm one such that $\|x_n + x\|_E \to 2$ and let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in the dual unit sphere such that $x_n^*(x_n) \to 1$.

Write $x_n = (x_{n,i})_{i\in I}$ and $x_n^* = (x_{n,i}^*)_{i\in I}$ for each n. We then have

$$x_n^*(x_n) = \sum_{i\in I} x_{n,i}^*(x_{n,i}) \le \sum_{i\in I} \|x_{n,i}^*\| \|x_{n,i}\| \le \|x_n^*\|_{E'} \|x_n\|_E = 1$$

which gives us

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\| \|x_{n,i}\| = 1 \tag{3.7}$$

and

$$\lim_{n\to\infty} \left(x_{n,i}^*(x_{n,i}) - \|x_{n,i}^*\| \|x_{n,i}\|\right) = 0 \quad \forall i \in I. \tag{3.8}$$

Applying Lemma 3.1 we also get

$$\|x_n + x\|_E \le \|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E \le \|x_n\|_E + \|x\|_E = 2$$

and hence

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E = 2. \tag{3.9}$$

Since E has property (P) this implies

$$\lim_{n\to\infty} \|x_{n,i}\| = \|x_i\| \quad \forall i \in I. \tag{3.10}$$

Because E is sluacs we get from (3.7) and (3.9) that

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\| \|x_i\| = 1. \tag{3.11}$$

If we apply Lemma 3.1 again we arrive at

$$\lim_{n\to\infty} \|(\|x_{n,i} + x_i\| + \|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E = 4. \tag{3.12}$$

We further have

$$\begin{aligned}\|x_n + x\|_E + 1 &\geq \|(\|x_{n,i} + x_i\| + \|x_{n,i}\|)_{i\in I}\|_E \\ &\geq \|(\|x_{n,i} + x_i\| + \|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E - 1\end{aligned}$$

and thus it follows from (3.12) that

$$\lim_{n\to\infty} \|(\|x_{n,i} + x_i\| + \|x_{n,i}\|)_{i\in I}\|_E = 3. \tag{3.13}$$

Analogously one can shown

$$\lim_{n\to\infty} \|(\|x_{n,i} + x_i\| + \|x_i\|)_{i\in I}\|_E = 3. \tag{3.14}$$

But because of Lemma 3.1 we also have

$$\begin{aligned}&\|(\|x_{n,i} + x_i\| + \|x_{n,i}\|)_{i\in I}\|_E + 3 \\ &\geq \|(\|x_{n,i} + x_i\| + \|x_{n,i}\| + 3\|x_i\|)_{i\in I}\|_E \\ &\geq 2\,\|(\|x_{n,i} + x_i\| + \|x_i\|)_{i\in I}\|_E\end{aligned}$$

and thus (3.14) and (3.13) imply

$$\lim_{n\to\infty} \|(\|x_{n,i} + x_i\| + \|x_{n,i}\| + 3\|x_i\|)_{i\in I}\|_E = 6. \tag{3.15}$$

Since E has property (P) it follows from (3.13) and (3.15) (and some standard normalisation arguments) that

$$\lim_{n\to\infty} (\|x_{n,i} + x_i\| + \|x_{n,i}\|) = 3\|x_i\| \quad \forall i \in I$$

which together with (3.10) gives us

$$\lim_{n\to\infty} \|x_{n,i} + x_i\| = 2\|x_i\| \quad \forall i \in I. \tag{3.16}$$

Because each X_i is sluacs it follows from (3.10), (3.16) and (3.8) (and again some standard normalisation arguments) that

$$\lim_{n\to\infty} \left(x^*_{n,i}(x_i) - \|x^*_{n,i}\|\|x_i\|\right) = 0 \quad \forall i \in I. \tag{3.17}$$

Now take any $\varepsilon > 0$. Then there is a finite subset $J \subseteq I$ such that

$$\left\|\sum_{i\in J} \|x_i\| e_i - (\|x_i\|)_{i\in I}\right\|_E \leq \varepsilon. \tag{3.18}$$

By (3.17) we can find an index $n_0 \in \mathbb{N}$ such that

$$\left|\sum_{i\in J} \left(x^*_{n,i}(x_i) - \|x^*_{n,i}\|\|x_i\|\right)\right| \leq \varepsilon \quad \forall n \geq n_0. \tag{3.19}$$

Then for all $n \geq n_0$ we have

$$\left| x_n^*(x) - \sum_{i\in I} \|x_{n,i}^*\| \|x_i\| \right| = \left| \sum_{i\in I} \left(x_{n,i}^*(x_i) - \|x_{n,i}^*\| \|x_i\| \right) \right|$$

$$\leq \left| \sum_{i\in J} \left(x_{n,i}^*(x_i) - \|x_{n,i}^*\| \|x_i\| \right) \right| + \left| \sum_{i\in I\setminus J} \left(x_{n,i}^*(x_i) - \|x_{n,i}^*\| \|x_i\| \right) \right|$$

$$\overset{(3.19)}{\leq} \varepsilon + 2 \sum_{i\in I\setminus J} \|x_{n,i}^*\| \|x_i\| \leq \varepsilon + 2 \left\| \sum_{i\in J} \|x_i\| e_i - (\|x_i\|)_{i\in I} \right\|_E \overset{(3.18)}{\leq} 3\varepsilon.$$

Thus we have shown $x_n^*(x) - \sum_{i\in I} \|x_{n,i}^*\| \|x_i\| \to 0$ which together with (3.11) leads to $x_n^*(x) \to 1$ finishing the proof. □

In our next result we shall see that instead of supposing that E possesses the property (P) we can also assume that E is sluacs$^+$ (resp. luacs$^+$) to come to the same conclusion.

Proposition 3.6. *If $(X_i)_{i\in I}$ is a family of sluacs (resp. luacs) spaces and E is sluacs$^+$ (resp. luacs$^+$) then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also sluacs (resp. luacs).*

Proof. Again we only show the sluacs case, the luacs case being analogous.

So fix a sequence $(x_n)_{n\in\mathbb{N}}$, a point x and a sequence $(x_n^*)_{n\in\mathbb{N}}$ of functionals just like in the proof of the preceding proposition.

As in this very proof we can show

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\| \|x_{n,i}\| = 1 \tag{3.20}$$

and

$$\lim_{n\to\infty} \left(x_{n,i}^*(x_{n,i}) - \|x_{n,i}^*\| \|x_{n,i}\| \right) = 0 \quad \forall i \in I \tag{3.21}$$

as well as

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E = 2 \tag{3.22}$$

and

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\| \|x_i\| = 1. \tag{3.23}$$

Also as in the proof of Proposition 3.5 we can see

$$\lim_{n\to\infty} \|(\|x_{n,i} + x_i\| + \|x_{n,i}\|)_{i\in I}\|_E = 3 \tag{3.24}$$

and

$$\lim_{n\to\infty} \|(\|x_{n,i} + x_i\| + \|x_{n,i}\| + 3\|x_i\|)_{i\in I}\|_E = 6. \tag{3.25}$$

Since E is sluacs$^+$ it follows from (3.24), (3.25) and (3.23) (with the usual normalisation arguments) that

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\| \left(\|x_{n,i} + x_i\| + \|x_{n,i}\| \right) = 3.$$

Together with (3.20) we get

$$\lim_{n\to\infty}\sum_{i\in I}\|x^*_{n,i}\|\left(\|x_{n,i}+x_i\|-\|x_{n,i}\|-\|x_i\|\right)=0$$

and hence

$$\lim_{n\to\infty}\|x^*_{n,i}\|\left(\|x_{n,i}+x_i\|-\|x_{n,i}\|-\|x_i\|\right)=0 \ \ \forall i\in I. \tag{3.26}$$

Next we show that

$$\lim_{n\to\infty}\left(x^*_{n,i}(x_i)-\|x^*_{n,i}\|\|x_i\|\right)=0 \ \ \forall i\in I. \tag{3.27}$$

To see this we fix $i_0\in I$ with $x_{i_0}\neq 0$. If $\|x^*_{n,i_0}\|\to 0$ the statement is clear. Otherwise there is some $\varepsilon>0$ such that $\|x^*_{n,i_0}\|\geq\varepsilon$ for infinitely many n. Without loss of generality we may assume that this inequality holds for every $n\in\mathbb{N}$.

For each $n\in\mathbb{N}$ we put $a_{n,i}=\|x^*_{n,i}\|$ for $i\in I\setminus\{i_0\}$ and $a_{n,i_0}=0$. Then $(a_{n,i})_{i\in I}\in B_{E'}$ for every n.

It is $|\sum_{i\in I}(a_{n,i}-\|x^*_{n,i}\|)\|x_{n,i}\||=\|x^*_{n,i_0}\|\|x_{n,i_0}\|\leq\|x_{n,i_0}\|$.

So if $\|x_{n,i_0}\|\to 0$ then by (3.20) we would also have $\lim_{n\to\infty}\sum_{i\in I}a_{n,i}\|x_{n,i}\|=1$.

But since E is a sluacs$^+$ space this together with (3.22) would also imply $\lim_{n\to\infty}\sum_{i\in I}a_{n,i}\|x_i\|=1$, which in turn implies (because of (3.23)) $\|x^*_{n,i_0}\|\|x_{i_0}\|=|\sum_{i\in I}(a_{n,i}-\|x^*_{n,i}\|)\|x_i\||\to 0$, where on the other hand $\|x^*_{n,i_0}\|\|x_{i_0}\|\geq\varepsilon\|x_{i_0}\|>0$ for all $n\in\mathbb{N}$, a contradiction.

So we must have $\|x_{n,i_0}\|\not\to 0$ and hence there is some $\delta>0$ such that $\|x_{n,i_0}\|\geq\delta$ for infinitely many (say for all) $n\in\mathbb{N}$.

Now since $(\|x^*_{n,i_0}\|)_{n\in\mathbb{N}}$ is bounded away from zero (3.26) gives us that

$$\lim_{n\to\infty}\left(\|x_{n,i_0}+x_{i_0}\|-\|x_{n,i_0}\|-\|x_{i_0}\|\right)=0.$$

Because $(\|x_{n,i_0}\|)_{n\in\mathbb{N}}$ is bounded away from zero as well, this together with Lemma 2.30 tells us that

$$\lim_{n\to\infty}\left\|\frac{x_{n,i_0}}{\|x_{n,i_0}\|}+\frac{x_{i_0}}{\|x_{i_0}\|}\right\|=2.$$

Using (3.21) and the fact that X_{i_0} is sluacs we now get the desired conclusion.

Now that we have established (3.27), the rest of the proof can be carried out exactly as in Proposition 3.5. □

The next two propositions deal with sums of luacs$^+$ and sluacs$^+$ spaces.

Proposition 3.7. *If $(X_i)_{i\in I}$ is a family of luacs$^+$ spaces and E is luacs$^+$ and has the property (P) then $\left[\bigoplus_{i\in I}X_i\right]_E$ is also a luacs$^+$ space.*

Proof. By Proposition 3.6 (or Proposition 3.5) we already know that the space $\left[\bigoplus_{i\in I}X_i\right]_E$ is luacs.

Now take a sequence $(x_n)_{n\in\mathbb{N}}$ and an element $x=(x_i)_{i\in I}$ in the unit sphere of $\left[\bigoplus_{i\in I}X_i\right]_E$ such that $\|x_n+x\|\to 2$ and a functional $x^*=(x^*_i)_{i\in I}$ of norm one with $x^*(x)=1$. Write $x_n=(x_{n,i})_{i\in I}$ for al $n\in\mathbb{N}$.

As in the proof of Proposition 3.3 it follows from $x^*(x) = 1$ that

$$x_i^*(x_i) = \|x_i^*\|\|x_i\| \ \forall i \in I \text{ and } \sum_{i\in I}\|x_i^*\|\|x_i\| = 1 \tag{3.28}$$

and as in the proof of Proposition 3.5 one can show that

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E = 2. \tag{3.29}$$

Since E is luacs$^+$ it follows from (3.29) and the second part of (3.28) that we also have

$$\lim_{n\to\infty} \sum_{i\in I}\|x_i^*\|\|x_{n,i}\| = 1. \tag{3.30}$$

Because E has property (P) it also follows from (3.29) that

$$\lim_{n\to\infty} \|x_{n,i}\| = \|x_i\| \quad \forall i \in I. \tag{3.31}$$

Exactly as in the proof of Proposition 3.5 we can see

$$\lim_{n\to\infty} \|x_{n,i} + x_i\| = 2\|x_i\| \quad \forall i \in I. \tag{3.32}$$

Since each X_i is luacs$^+$ we infer from (3.32), (3.31) and the first part of (3.28) that

$$\lim_{n\to\infty} x_i^*(x_{n,i}) = \|x_i^*\|\|x_i\| \quad \forall i \in I. \tag{3.33}$$

Now take an arbitrary $\varepsilon > 0$ and fix a finite subset $J \subseteq I$ such that

$$\left\| \sum_{i\in J}\|x_i\|e_i - (\|x_i\|)_{i\in I} \right\|_E \le \varepsilon. \tag{3.34}$$

From (3.28), (3.30) and (3.31) it follows that

$$\lim_{n\to\infty} \sum_{i\in I\setminus J} \|x_i^*\|\|x_{n,i}\| = \sum_{i\in I\setminus J} \|x_i^*\|\|x_i\|$$

and by (3.33) we also have

$$\lim_{n\to\infty} \sum_{i\in J} x_i^*(x_{n,i}) = \sum_{i\in J}\|x_i^*\|\|x_i\|.$$

Hence there is some $n_0 \in \mathbb{N}$ such that

$$\left| \sum_{i\in J} (x_i^*(x_{n,i}) - \|x_i^*\|\|x_i\|) \right| \le \varepsilon \quad \text{and} \tag{3.35}$$

$$\left| \sum_{i\in I\setminus J} \|x_i^*\| \left(\|x_{n,i}\| - \|x_i\|\right) \right| \le \varepsilon \quad \forall n \ge n_0. \tag{3.36}$$

But then we have for every $n \geq n_0$

$$\begin{aligned}
|x^*(x_n)-1| &\overset{(3.28)}{=} \left|\sum_{i\in I}(x_i^*(x_{n,i})-\|x_i^*\|\|x_i\|)\right| \\
&\overset{(3.35)}{\leq} \varepsilon + \left|\sum_{i\in I\setminus J}(x_i^*(x_{n,i})-\|x_i^*\|\|x_i\|)\right| \\
&\leq \varepsilon + \sum_{i\in I\setminus J}\|x_i^*\|\left(\|x_{n,i}\|+\|x_i\|\right) \\
&\overset{(3.36)}{\leq} 2\varepsilon + 2\sum_{i\in I\setminus J}\|x_i^*\|\|x_i\| \overset{(3.34)}{\leq} 4\varepsilon.
\end{aligned}$$

Thus we have $x^*(x_n) \to 1$ and the proof is finished. □

Proposition 3.8. *If $(X_i)_{i\in I}$ is a family of sluacs$^+$ (resp. luacs$^+$) spaces and E is sluacs$^+$ then $\left[\bigoplus_{i\in I} X_i\right]_E$ is sluacs$^+$ (resp. luacs$^+$) as well.*

Proof. Suppose all the X_i and E are sluacs$^+$. Then by Proposition 3.6 $\left[\bigoplus_{i\in I} X_i\right]_E$ is sluacs.

Now take sequences $(x_n)_{n\in\mathbb{N}}$ and $(x_n^*)_{n\in\mathbb{N}}$ in the unit sphere and in the dual unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ respectively, as well as another element $x=(x_i)_{i\in I}$ in $\left[\bigoplus_{i\in I} X_i\right]_E$ of norm one such that $\|x_n+x\|_E \to 2$ and $x_n^*(x) \to 1$.

As usual we write $x_n=(x_{n,i})_{i\in I}$ and $x_n^*=(x_{n,i}^*)_{i\in I}$ for every $n\in\mathbb{N}$.

Much as we have done before we can show that

$$\lim_{n\to\infty}\left(x_{n,i}^*(x_i)-\|x_{n,i}^*\|\|x_i\|\right)=0\ \forall i\in I \text{ and } \lim_{n\to\infty}\sum_{i\in I}\|x_{n,i}^*\|\|x_i\|=1 \tag{3.37}$$

as well as

$$\lim_{n\to\infty}\|(\|x_{n,i}\|+\|x_i\|)_{i\in I}\|_E=2. \tag{3.38}$$

It follows from (3.38), the second part of (3.37), and the fact that E is sluacs$^+$ that

$$\lim_{n\to\infty}\sum_{i\in I}\|x_{n,i}^*\|\|x_{n,i}\|=1. \tag{3.39}$$

As in the proof of Proposition 3.6 we see that

$$\lim_{n\to\infty}\|x_{n,i}^*\|\left(\|x_{n,i}+x_i\|-\|x_{n,i}\|-\|x_i\|\right)=0 \quad \forall i\in I. \tag{3.40}$$

Now using an argument analogous to that in the proof of Proposition 3.6 shows

$$\lim_{n\to\infty}\left(x_{n,i}^*(x_{n,i})-\|x_{n,i}^*\|\|x_{n,i}\|\right)=0 \quad \forall i\in I. \tag{3.41}$$

Put $b_J = (\|x_i\|)_{i\in I} - \sum_{i\in J}\|x_i\|e_i$ and $c_{n,J} = \sum_{i\in J}\|x^*_{n,i}\|e_i$ for every $n \in \mathbb{N}$ and every finite subset $J \subseteq I$. Then for every n and J we have

$$
\begin{aligned}
|c_{n,J}((\|x_i\|)_{i\in I}) - 1| &= \left|\sum_{i\in J}\|x^*_{n,i}\|\|x_i\| - 1\right| \\
&\leq \left|\sum_{i\in I\setminus J}\|x^*_{n,i}\|\|x_i\|\right| + \left|\sum_{i\in I}\|x^*_{n,i}\|\|x_i\| - 1\right| \\
&\leq \|b_J\|_E + \left|\sum_{i\in I}\|x^*_{n,i}\|\|x_i\| - 1\right|. \qquad (3.42)
\end{aligned}
$$

Now take any $\varepsilon > 0$. Because E is sluacs$^+$ there is some $\delta > 0$ such that

$$
\begin{aligned}
&a \in S_E,\ g \in B_{E^*} \text{ with } \|a + (\|x_i\|)_{i\in I}\|_E \geq 2 - \delta \\
&\text{and } g((\|x_i\|)_{i\in I}) \geq 1 - \delta \ \Rightarrow\ g(a) \geq 1 - \varepsilon. \qquad (3.43)
\end{aligned}
$$

Fix a finite subset $J_0 \subseteq I$ such that $\|b_{J_0}\|_E \leq \delta/2$ and also fix an index n_0 such that $\left|\sum_{i\in I}\|x^*_{n,i}\|\|x_i\| - 1\right| \leq \delta/2$ and $\|(\|x_{n,i}\| + \|x_i\|)_{i\in I}\|_E \geq 2 - \delta$ for all $n \geq n_0$ (which is possible because of (3.37) and (3.38)).

Then (3.42) and (3.43) give us

$$
c_{n,J_0}((\|x_{n,i}\|)_{i\in I}) = \sum_{i\in J_0}\|x^*_{n,i}\|\|x_{n,i}\| \geq 1 - \varepsilon \quad \forall n \geq n_0. \qquad (3.44)
$$

By (3.41) we may also assume that

$$
\left|\sum_{i\in J_0}\left(x^*_{n,i}(x_{n,i}) - \|x^*_{n,i}\|\|x_{n,i}\|\right)\right| \leq \varepsilon \quad \forall n \geq n_0. \qquad (3.45)
$$

Then for every $n \geq n_0$ we have

$$
\begin{aligned}
\left|x^*_n(x_n) - \sum_{i\in I}\|x^*_{n,i}\|\|x_{n,i}\|\right| &= \left|\sum_{i\in I}\left(x^*_{n,i}(x_{n,i}) - \|x^*_{n,i}\|\|x_{n,i}\|\right)\right| \\
&\overset{(3.45)}{\leq} \varepsilon + \left|\sum_{i\in I\setminus J_0}\left(x^*_{n,i}(x_{n,i}) - \|x^*_{n,i}\|\|x_{n,i}\|\right)\right| \\
&\leq \varepsilon + 2\sum_{i\in I\setminus J_0}\|x^*_{n,i}\|\|x_{n,i}\| \overset{(3.44)}{\leq} 3\varepsilon.
\end{aligned}
$$

Thus $x^*_n(x_n) - \sum_{i\in I}\|x^*_{n,i}\|\|x_{n,i}\| \to 0$ which together with (3.39) implies $x^*_n(x_n) \to 1$.

The proof for the luacs$^+$ case can be done in a very similar fashion. $\square$

In our next result we consider sums of wuacs and luacs$^+$ spaces for the case that E does not contain ℓ^1.

Proposition 3.9. *If $(X_i)_{i\in I}$ is a family of wuacs (resp. luacs$^+$) spaces and if E is wuacs (resp. luacs$^+$) and does not contain an isomorphic copy of ℓ^1 then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also wuacs (resp. luacs$^+$).*

Proof. Let us suppose that E and all the X_i are wuacs and fix two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ as well as a norm one functional $x^* = (x_i^*)_{i\in I}$ on $\left[\bigoplus_{i\in I} X_i\right]_E$ such that $\|x_n+y_n\|_E \to 2$ and $x^*(x_n) \to 1$. Write $x_n = (x_{n,i})_{i\in I}$ and $y_n = (y_{n,i})_{i\in I}$ for each n.

As we have often done before we deduce

$$\lim_{n\to\infty} (x_i^*(x_{n,i}) - \|x_i^*\|\|x_{n,i}\|) = 0 \ \forall i \in I \text{ and } \lim_{n\to\infty} \sum_{i\in I} \|x_i^*\|\|x_{n,i}\| = 1 \tag{3.46}$$

and

$$\lim_{n\to\infty} \|(\|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E = 2 \tag{3.47}$$

as well as

$$\lim_{n\to\infty} \|(\|x_{n,i} + y_{n,i}\| + \|x_{n,i}\| + \|y_{n,i}\|)_{i\in I}\|_E = 4. \tag{3.48}$$

Since E is wuacs (3.47) and the second part of (3.46) imply

$$\lim_{n\to\infty} \sum_{i\in I} \|x_i^*\|\|y_{n,i}\| = 1. \tag{3.49}$$

Applying again the fact that E is wuacs together with (3.48), (3.49) and the second part of (3.46) gives us

$$\lim_{n\to\infty} \sum_{i\in I} \|x_i^*\| \left(\|x_{n,i}\| + \|y_{n,i}\| - \|x_{n,i} + y_{n,i}\|\right) = 0$$

and hence

$$\lim_{n\to\infty} \|x_i^*\| \left(\|x_{n,i}\| + \|y_{n,i}\| - \|x_{n,i} + y_{n,i}\|\right) = 0 \quad \forall i \in I. \tag{3.50}$$

Now we can show

$$\lim_{n\to\infty} (x_i^*(y_{n,i}) - \|x_i^*\|\|y_{n,i}\|) = 0 \quad \forall i \in I. \tag{3.51}$$

The argument for this is similar to what we have done before but we state it here for the sake of completeness. Fix $i_0 \in I$ with $x_{i_0}^* \neq 0$ and $y_{n,i_0} \not\to 0$. Then there is $\tau > 0$ such that $\|y_{n,i_0}\| \geq \tau$ for infinitely many (without loss of generality for all) $n \in \mathbb{N}$.

Put $a_{i_0} = 0$ and $a_i = \|x_i^*\|$ for every $i \in I \setminus \{i_0\}$. If $\|x_{n,i_0}\| \to 0$ then because of the second part of (3.46) it would follow that $\lim_{n\to\infty} \sum_{i\in I} a_i\|x_{n,i}\| = 1$.

Since E is wuacs this together with (3.47) would imply that we also have $\lim_{n\to\infty} \sum_{i\in I} a_i\|y_{n,i}\| = 1$ which because (3.49) would give us $\|x_{i_0}^*\|\|y_{n,i_0}\| \to 0$, a contradiction.

Hence there must be some $\delta > 0$ such that $\|x_{n,i_0}\| \geq \delta$ for infinitely many (say for every) $n \in \mathbb{N}$.

Now since the sequences $(\|x_{n,i_0}\|)_{n\in\mathbb{N}}$ and $(\|y_{n,i_0}\|)_{n\in\mathbb{N}}$ are bounded away from zero it follows from (3.46), (3.50) and Lemma 2.30 that

$$\lim_{n\to\infty} \left\| \frac{x_{n,i_0}}{\|x_{n,i_0}\|} + \frac{y_{n,i_0}}{\|y_{n,i_0}\|} \right\| = 2 \text{ and } \lim_{n\to\infty} \frac{x_{i_0}^*}{\|x_{i_0}^*\|} \left(\frac{x_{n,i_0}}{\|x_{n,i_0}\|} \right) = 1.$$

Since X_{i_0} is wuacs this implies our desired conclusion.

Now we fix any $\varepsilon > 0$. Because $\ell^1 \not\subseteq E$ by Lemma 3.2 there must be some finite set $J \subseteq I$ such that

$$\left\| (\|x_i^*\|)_{i\in I} - \sum_{i\in J} \|x_i^*\| e_i \right\|_{E'} \leq \varepsilon. \tag{3.52}$$

By (3.51) we can find some $n_0 \in \mathbb{N}$ such that

$$\left| \sum_{i\in J} (x_i^*(y_{n,i}) - \|x_i^*\| \|y_{n,i}\|) \right| \leq \varepsilon \quad \forall n \geq n_0. \tag{3.53}$$

We then have for every $n \geq n_0$

$$\left| x^*(y_n) - \sum_{i\in I} \|x_i^*\| \|y_{n,i}\| \right| \overset{(3.53)}{\leq} \varepsilon + \left| \sum_{i\in I\setminus J} (x_i^*(y_{n,i}) - \|x_i^*\| \|y_{n,i}\|) \right|$$
$$\leq \varepsilon + 2 \sum_{i\in I\setminus J} \|x_i^*\| \|y_{n,i}\| \overset{(3.52)}{\leq} 3\varepsilon.$$

So we have $x^*(y_n) - \sum_{i\in I} \|x_i^*\| \|y_{n,i}^*\| \to 0$. From (3.49) it now follows that $x^*(y_n) \to 1$.

The luacs$^+$ case is proved analogously. □

Note that the above Proposition especially applies to the case that E is WUR because a WUR space cannot contain an isomorphic copy of ℓ^1 (cf. [29, Remark 4]). Frankly, the author does not know whether a wuacs space can contain an isomorphic copy of ℓ^1 at all, but at least it cannot contain particularly "good" copies of ℓ^1 in the following sense (introduced in [12]).

Definition 3.10. A Banach space X is said to contain an *asymptotically isometric copy of* ℓ^1 if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in B_X and a decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $[0, 1[$ with $\varepsilon_n \to 0$ such that for each $m \in \mathbb{N}$ and all scalars $a_1, \ldots, a_m$ we have

$$\sum_{i=1}^{m} (1-\varepsilon_i)|a_i| \leq \left\| \sum_{i=1}^{m} a_i x_i \right\| \leq \sum_{i=1}^{m} |a_i|.$$

Likewise, X is said to contain an *asymptotically isomorphic copy of* c_0 if there are two such sequences $(x_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ which fulfil

$$\max_{i=1,\ldots,m} (1-\varepsilon_i)|a_i| \leq \left\| \sum_{i=1}^{m} a_i x_i \right\| \leq \max_{i=1,\ldots,m} |a_i|$$

for each $m \in \mathbb{N}$ and all scalars $a_1, \ldots, a_m$.

We then have the following observation.

Proposition 3.11. *If the Banach space X is wuacs then it does not contain an asymptotically isometric copy of ℓ^1.*

Proof. Suppose that X contains an asymptotically isometric copy of ℓ^1. Then fix two sequences $(x_n)_{n\in\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ as in the above definition.

We can find $\alpha > 1$ such that $\alpha\varepsilon_n < 1$ for every $n \in \mathbb{N}$. Put $\tilde{x}_n = (1-\alpha\varepsilon_n)^{-1}x_n$ for each n. Then for every finite sequence $(a_i)_{i=1}^m$ of scalars we have

$$\left\|\sum_{i=1}^m a_i\tilde{x}_i\right\| = \left\|\sum_{i=1}^m \frac{a_i}{1-\alpha\varepsilon_i}x_i\right\| \geq \sum_{i=1}^m \frac{1-\varepsilon_i}{1-\alpha\varepsilon_i}|a_i| \geq \sum_{i=1}^m |a_i|. \tag{3.54}$$

In other words, the operator $T : \ell^1 \to X$ defined by $T((a_n)_{n\in\mathbb{N}}) = \sum_{n=1}^\infty a_n\tilde{x}_n$ is an isomorphism onto its range $U = \operatorname{ran} T$ with $\|T^{-1}\| \leq 1$.

Define $(b_n)_{n\in\mathbb{N}} \in \ell^\infty = (\ell^1)^*$ by $b_n = 1$ if n is even and $b_n = 0$ if n is odd. Then $u^* = (T^{-1})^*((b_n)_{n\in\mathbb{N}}) \in B_{U^*}$. Take a Hahn-Banach extension x^* of u^* to X.

Note that because of (3.54) we have in particular $\|\tilde{x}_n\| \geq 1$ for every n and on the other hand $\|\tilde{x}_n\| \leq (1-\alpha\varepsilon_n)^{-1}$ and $\varepsilon_n \to 0$, hence $\|\tilde{x}_n\| \to 1$. Again because of (3.54) we have $\|\tilde{x}_n + \tilde{x}_{n+1}\| \geq 2$ for every n. It follows that $\|\tilde{x}_n + \tilde{x}_{n+1}\| \to 2$ and thus in particular $\|\tilde{x}_{2n} + \tilde{x}_{2n+1}\| \to 2$.

But we also have $x^*(\tilde{x}_{2n}) = u^*(\tilde{x}_{2n}) = b_{2n} = 1$ and likewise $x^*(\tilde{x}_{2n+1}) = b_{2n+1} = 0$ for every n and hence X cannot be a wuacs space. $\square$

If the space X contains an asymptotically isometric copy of c_0 then by [12, Theorem 2] X^* contains an asymptotically isometric copy of ℓ^1 and thus we get the following corollary.

Corollary 3.12. *If X is a Banach space whose dual X^* is wuacs then X does not contain an asymptotically isometric copy of c_0.*

We also remark that since $\ell^p(I)$ is UR for every $1 < p < \infty$ we can obtain the following corollary from our above results.

Corollary 3.13. *If $(X_i)_{i\in I}$ is a family of Banach space such that each X_i is acs resp. luacs resp. luacs$^+$ resp. sluacs resp. sluacs$^+$ resp. wuacs then $\left[\bigoplus_{i\in I} X_i\right]_p$ is also acs resp. luacs resp. luacs$^+$ resp. sluacs resp. sluacs$^+$ resp. wuacs for every $1 < p < \infty$.*

Now we turn to sums of uacs spaces. We first consider sums of finitely many spaces. In fact, this has been done before in [11] (for two summands) and in [23] (for finitely many summands) in the context of U-spaces and the so called ψ-direct sums. However, we include a sketch of our own slightly different proof here, for the sake of completeness.

Proposition 3.14. *If I is a finite set, $(X_i)_{i\in I}$ a family of uacs Banach spaces and $\|\,.\,\|_E$ is an absolute normalized norm on $\mathbb{R}^I$ such that $E := (\mathbb{R}^I, \|\,.\,\|_E)$ is acs then $\left[\bigoplus_{i\in I} X_i\right]_E$ is also a uacs space.*

Proof. First note that since E is finite-dimensional it is actually uacs. Now if we take two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in the unit sphere of $\left[\bigoplus_{i\in I} X_i\right]_E$ and a sequence $(x_n^*)_{n\in\mathbb{N}}$ in the dual unit sphere such that $\|x_n + y_n\|_E \to 2$ and $x_n^*(x_n) \to 1$ then we can show just as we have done before that

$$\lim_{n\to\infty} \sum_{i\in I} \|x_{n,i}^*\|\|x_{n,i}\| = 1 \tag{3.55}$$

and

$$\lim_{n\to\infty}\left(x^*_{n,i}(x_{n,i})-\|x^*_{n,i}\|\|x_{n,i}\|\right)=0\ \ \forall i\in I \tag{3.56}$$

as well as

$$\lim_{n\to\infty}\|(\|x_{n,i}\|+\|y_{n,i}\|)_{i\in I}\|_E=2 \tag{3.57}$$

and

$$\lim_{n\to\infty}\|(\|x_{n,i}+y_{n,i}\|+\|x_{n,i}\|+\|y_{n,i}\|)_{i\in I}\|_E=4. \tag{3.58}$$

Since E is uacs it follows from (3.55) and (3.57) that

$$\lim_{n\to\infty}\sum_{i\in I}\|x^*_{n,i}\|\|y_{n,i}\|=1. \tag{3.59}$$

Again, since E is uacs it follows from (3.55), (3.59) and (3.58) that

$$\lim_{n\to\infty}\|x^*_{n,i}\|\left(\|x_{n,i}\|+\|y_{n,i}\|-\|x_{n,i}+y_{n,i}\|\right)=0\ \ \forall i\in I. \tag{3.60}$$

Now using (3.60), Lemma 2.30, (3.55), (3.59), (3.56), the fact that each X_i is uacs and an argument similar the one used in the proof of Proposition 3.9 we can infer that

$$\lim_{n\to\infty}\left(x^*_{n,i}(y_{n,i})-\|x^*_{n,i}\|\|y_{n,i}\|\right)=0\ \ \forall i\in I.$$

Since I is finite it follows that $x^*_n(y_n)-\sum_{i\in I}\|x^*_{n,i}\|\|y_{n,i}\|\to 0$ which together with (3.59) gives us $x^*_n(y_n)\to 1$ and the proof is over. □

Before we can come to the study of absolute sums of infinitely many uacs spaces we have to introduce one more definition.

Definition 3.15. The space E is said to have the property (u^+) if for every $\varepsilon>0$ there is some $\delta>0$ such that for all $(a_i)_{i\in I},(b_i)_{i\in I}\in S_E$ and each $(c_i)_{i\in I}\in S_{E'}=S_{E^*}$ we have

$$\sum_{i\in I}a_ic_i=1 \text{ and } \|(a_i+b_i)_{i\in I}\|_E\geq 2(1-\delta)\ \Rightarrow\ \sum_{i\in I}|c_i||a_i-b_i|\leq\varepsilon.$$

Clearly, if E is UR then it has property (u^+) and the property (u^+) in turn implies that E is uacs. Unfortunately, the author does not know whether these impliactions are strict.

Now we can formulate and prove the following theorem, which is an analogue of Day's results on sums of UR spaces from [5, Theorem 3] (for the ℓ^p-case) and [6, Theorem 3] (for the general case). Also, its proof is just a slight modification of Day's technique.

Theorem 3.16. *If $(X_i)_{i\in I}$ is a family of Banach spaces such that for every $0<\varepsilon\leq 2$ we have $\delta(\varepsilon):=\inf_{i\in I}\delta^{X_i}_{\mathrm{uacs}}(\varepsilon)>0$ and if the space E has the property (u^+) then $\left[\bigoplus_{i\in I}X_i\right]_E$ is also uacs.*

Proof. As in [5] and [6] the proof is divided into two steps. In the first step we show that for every $0<\varepsilon\leq 2$ there is some $\eta>0$ such that for any two elements $x=(x_i)_{i\in I}$ and $y=(y_i)_{i\in I}$ of the unit sphere of $\left[\bigoplus_{i\in I}X_i\right]_E$ with $\|x_i\|=\|y_i\|$ for every $i\in I$ and each functional $x^*=(x^*_i)_{i\in I}$ with $\|x^*\|_{E'}=x^*(x)=1$ and $x^*(y)<1-\varepsilon$ we have $\|x+y\|_E\leq 2(1-\eta)$.

So let $0 < \varepsilon \le 2$ be arbitrary. Since E is uacs there exists some $\eta > 0$ such that

$$a, b \in B_E, l \in B_{E^*}, l(a) = 1 \text{ and } l(b) < 1 - \frac{\varepsilon}{4}\delta\left(\frac{\varepsilon}{2}\right) \Rightarrow \|a+b\|_E \le 2(1-\eta). \tag{3.61}$$

We claim that this η fulfils our requirement. To show this, fix x, y and x^* as above and put $\beta_i = \|x_i\| = \|y_i\|$, $\nu_i = \|x_i^*\|$ and $\gamma_i = \nu_i\beta_i - x_i^*(y_i)$ for each $i \in I$. Then we have

$$0 \le \gamma_i \le 2\beta_i\nu_i \quad \forall i \in I. \tag{3.62}$$

From $x^*(x) = 1 = \|x^*\|_{E'} = \|x\|_E$ we get

$$\sum_{i\in I} \nu_i\beta_i = 1 \text{ and } x_i^*(x_i) = \nu_i\beta_i \ \forall i \in I. \tag{3.63}$$

Next we define

$$\alpha_i = \begin{cases} \frac{1}{2}\delta\left(\frac{\gamma_i}{\nu_i\beta_i}\right) & \text{if } \gamma_i > 0 \\ 0 & \text{if } \gamma_i = 0. \end{cases} \tag{3.64}$$

From the definition of the $\delta_{\text{uacs}}^{X_i}$ and the second part of (3.63) it easily follows that

$$\|x_i + y_i\| \le 2(1-\alpha_i)\beta_i \quad \forall i \in I. \tag{3.65}$$

By (3.62) and the first part of (3.63) we have $\sum_{i\in I}\gamma_i \le 2$ and further it is

$$\varepsilon < 1 - x^*(y) = x^*(x-y) = \sum_{i\in I} x_i^*(x_i - y_i) \le \sum_{i\in I}\gamma_i$$

thus

$$\varepsilon < \sum_{i\in I}\gamma_i \le 2. \tag{3.66}$$

Now put $A = \{i \in I : 2\gamma_i > \varepsilon\nu_i\beta_i\}$ and $B = I \setminus A$. Then we get

$$\sum_{i\in B}\gamma_i \le \frac{\varepsilon}{2}\sum_{i\in B}\nu_i\beta_i \le \frac{\varepsilon}{2}\sum_{i\in I}\nu_i\beta_i \overset{(3.63)}{=} \frac{\varepsilon}{2}. \tag{3.67}$$

From (3.66) and (3.67) it follows that

$$\sum_{i\in A}\gamma_i = \sum_{i\in I}\gamma_i - \sum_{i\in B}\gamma_i > \frac{\varepsilon}{2}. \tag{3.68}$$

Using (3.62) and (3.68) we now get

$$\sum_{i\in A}\nu_i\beta_i > \frac{\varepsilon}{4}. \tag{3.69}$$

Write $t = (\beta_i\chi_B(i))_{i\in I}$ and $t' = (\beta_i\chi_A(i))_{i\in I}$, where χ_B and χ_A denote the characteristic function of B and A respectively. Then $t, t' \in B_E$ (by Lemma 3.1) and $t + t' = (\beta_i)_{i\in I}$. We also put $t'' = (1 - \delta(\varepsilon/2))t'$. Again by Lemma 3.1 we have $\|t + t''\|_E \le \|t + t'\|_E = 1$. Further, $l = (\nu_i)_{i\in I}$ defines an element of S_{E^*} such that $l(t + t') = \sum_{i\in I}\nu_i\beta_i = 1$ (by (3.63)) and $l(t + t'') = 1 - \delta(\varepsilon/2)l(t') = 1 - \delta(\varepsilon/2)\sum_{i\in A}\nu_i\beta_i$ and hence (by (3.69))

$$l(t + t'') < 1 - \frac{\varepsilon}{4}\delta\left(\frac{\varepsilon}{2}\right).$$

Thus we can apply (3.61) to deduce

$$\frac{1}{2}\left\|2t+t'+t''\right\|_E=\left\|t+\left(1-\frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right)t'\right\|_E\leq 1-\eta. \tag{3.70}$$

Since δ is obviously an increasing function we also have

$$\alpha_i\geq\frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\quad\forall i\in A. \tag{3.71}$$

Now we can conclude (with the aid of Lemma 3.1)

$$\begin{aligned}\|x+y\|_E&=\|(\|x_i+y_i\|)_{i\in I}\|_E\overset{(3.65)}{\leq}2\left\|((1-\alpha_i)\beta_i)_{i\in I}\right\|_E\\&\overset{(3.71)}{\leq}2\left\|\left(\left(1-\frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right)\beta_i\chi_A(i)+\beta_i\chi_B(i)\right)_{i\in I}\right\|_E\\&=2\left\|\left(1-\frac{1}{2}\delta\left(\frac{\varepsilon}{2}\right)\right)t'+t\right\|_E\overset{(3.70)}{\leq}2(1-\eta),\end{aligned}$$

finishing the first step of the proof. Note that so far we have only used the fact that E is uacs and not the property (u^+).

Now for the second step we fix $0<\varepsilon\leq 2$ and choose an $\eta>0$ to the value $\varepsilon/2$ according to step one. Then we take $0<\nu<2\eta/3$. Since E is uacs we can find $\tau>0$ such that

$$\begin{aligned}&a,b\in B_E,l\in B_{E^*},l(a)\geq 1-\tau\text{ and }\|a+b\|_E\geq 2(1-\tau)\\&\Rightarrow\ l(b)\geq 1-\nu.\end{aligned} \tag{3.72}$$

Next we fix $0<\alpha<\min\{\varepsilon/2,2\tau,\nu\}$. Now we can find a number $\tilde{\tau}>0$ to the value α according to the definition of the property (u^+) (Definition 3.15). Finally, we take $0<\xi<\min\{\tau,\tilde{\tau}\}$.

Now suppose $x=(x_i)_{i\in I}$ and $y=(y_i)_{i\in I}$ are elements of the unit sphere of $\left[\bigoplus_{i\in I}X_i\right]_E$ and $x^*=(x_i^*)_{i\in I}$ is an element of the dual unit sphere such that $\|x+y\|_E\geq 2(1-\xi)$ and $x^*(x)=1$. We will show that $x^*(y)>1-\varepsilon$.

To do so, we define

$$z_i=\begin{cases}\frac{\|x_i\|}{\|y_i\|}y_i & \text{if } y_i\neq 0\\ x_i & \text{if } y_i=0.\end{cases} \tag{3.73}$$

Then we have

$$\|z_i\|=\|x_i\|\text{ and }\|z_i-y_i\|=|\|x_i\|-\|y_i\||\ \ \forall i\in I. \tag{3.74}$$

As before we can see that $\sum_{i\in I}\|x_i^*\|\|x_i\|=1$ and further we have $2(1-\tilde{\tau})\leq 2(1-\xi)\leq\|x+y\|_E\leq\|(\|x_i\|+\|y_i\|)_{i\in I}\|_E$.

Thus we get from the choice of $\tilde{\tau}$ that

$$\sum_{i\in I}\|x_i^*\|\|z_i-y_i\|\overset{(3.74)}{=}\sum_{i\in I}\|x_i^*\|\,|\|x_i\|-\|y_i\||\leq\alpha. \tag{3.75}$$

Further, we have

$$\|(\|x_i\|+\|y_i\|+\|x_i+y_i\|)_{i\in I}\|_E\geq 2\|x+y\|_E\geq 4(1-\xi)\geq 4(1-\tau)$$

and

$$\sum_{i\in I}\|x_i^*\|\left(\|x_i\|+\|y_i\|\right)=1+\sum_{i\in I}\|x_i^*\|\|y_i\|$$
$$\geq 1+\sum_{i\in I}\|x_i^*\|\|x_i\|-\sum_{i\in I}\|x_i^*\|\,|\|x_i\|-\|y_i\||$$
$$=2-\sum_{i\in I}\|x_i^*\|\,|\|x_i\|-\|y_i\||\overset{(3.75)}{\geq} 2-\alpha\geq 2(1-\tau).$$

Hence we can conclude from (3.72) that

$$\sum_{i\in I}\|x_i^*\|\|x_i+y_i\|\geq 2(1-\nu). \tag{3.76}$$

Using (3.75) and (3.76) we get

$$\|x+z\|_E\geq\sum_{i\in I}\|x_i^*\|\|x_i+z_i\|\geq\sum_{i\in I}\|x_i^*\|\|x_i+y_i\|-\sum_{i\in I}\|x_i^*\|\|y_i-z_i\|$$
$$\geq 2(1-\nu)-\alpha>2(1-\eta)$$

and thus the choice of η implies $x^*(z)\geq 1-\varepsilon/2$. But from (3.75) it also follows that $|x^*(y)-x^*(z)|\leq\alpha$ and hence $x^*(y)\geq 1-\varepsilon/2-\alpha>1-\varepsilon$. □

Because of the uniform rotundity of $\ell^p(I)$ for $1<p<\infty$ we have the following corollary.

Corollary 3.17. *If $(X_i)_{i\in I}$ is a family of Banach spaces such that for every $0<\varepsilon\leq 2$ we have $\inf_{i\in I}\delta_{\mathrm{uacs}}^{X_i}(\varepsilon)>0$ then $\left[\bigoplus_{i\in I}X_i\right]_p$ is also uacs for every $1<p<\infty$.*

We can also get a more general corollary for a US space E.

Corollary 3.18. *If $(X_i)_{i\in I}$ is a family of Banach spaces such that for every $0<\varepsilon\leq 2$ we have $\delta(\varepsilon):=\inf_{i\in I}\delta_{\mathrm{uacs}}^{X_i}(\varepsilon)>0$ and if E is US then $\left[\bigoplus_{i\in I}X_i\right]_E$ is also a uacs space.*

Proof. Since E is US it is reflexive and hence it cannot contain an isomorpic copy of ℓ^1. Thus by Lemma 3.2 span$\{e_i : i\in I\}$ is dense in E'.

Further, since E is US the dual space $E^*=E'$ is UR, as already mentioned in the introduction. Because the spaces X_i are uacs they are also reflexive and hence Proposition 2.12 and the monotonicity of the functions $\delta_{\mathrm{uacs}}^{X_i}$ gives us $\inf_{i\in I}\delta_{\mathrm{uacs}}^{X_i^*}(\varepsilon)\geq\delta(\delta(\varepsilon))>0$ for every $0<\varepsilon\leq 2$. So by Theorem 3.16 the space $\left[\bigoplus_{i\in I}X_i^*\right]_{E'}=\left[\bigoplus_{i\in I}X_i\right]_E^*$ is uacs and hence $\left[\bigoplus_{i\in I}X_i\right]_E$ is also uacs by Proposition 2.12. □

Finally, we summarise all the results on absolute sums we have obtained in this section in the following table.

Table 1. *Summary of the results*

E	X_i	$\left[\bigoplus_{i\in I} X_i\right]_E$
acs	acs	acs
luacs + (P)	luacs	luacs
luacs$^+$	luacs	luacs
luacs$^+$ + (P)	luacs$^+$	luacs$^+$
luacs$^+$ + $\ell^1 \not\subseteq E$	luacs$^+$	luacs$^+$
sluacs + (P)	sluacs	sluacs
sluacs$^+$	sluacs	sluacs
sluacs$^+$	luacs$^+$	luacs$^+$
sluacs$^+$	sluacs$^+$	sluacs$^+$
wuacs + $\ell^1 \not\subseteq E$	wuacs	wuacs
acs + I finite	uacs	uacs
(u^+)	$\inf_{i\in I} \delta_{\mathrm{uacs}}^{X_i} > 0$	uacs
US	$\inf_{i\in I} \delta_{\mathrm{uacs}}^{X_i} > 0$	uacs

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