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DETERMINANTAL REPRESENTATION OF TRIGONOMETRIC POLYNOMIAL CURVES VIA SYLVESTER METHOD

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Abstract. For any trigonometric polynomial $\phi(\theta)$, we give a constructive algorithm by Sylvester elimination which produces matrices C_1, C_2, C_3 such that $\det(C_1 + \Re(\phi(\theta))C_2 + \Im(\phi(\theta))C_3) = 0$. For a typical trigonometric polynomial, we assert that C_1 is positive definite, and thus the typical polynomial curve admits a determinantal representation.

1. Introduction and preliminaries

Let A be an $n \times n$ matrix. The real ternary form $F_A(t, x, y)$ associated to A is defined as

$$F_A(t, x, y) = \det(tI_n + x\Re(A) + y\Im(A)),$$

where $\Re(A) = (A + A^*)/2$ and $\Im(A) = (A - A^*)/(2i)$. Kippenhahn [8] characterized the numerical range of A, $W(A) = \{\xi^* A\xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}$, as the convex hull of the real affine part of the dual curve of the curve $F_A(t, x, y) = 0$. The form $F_A(t, x, y)$ is hyperbolic with respect to (1,0,0), i.e., $F_A(1, 0, 0) \neq 0$, and for any real pair x, y, $F_A(t, x, y)$ has only real roots in t. The converse part was conjectured by Fiedler [5] and Lax [9], namely, for any real ternary hyperbolic form $f(t, x, y)$, there exist Hermitian(or real symmetric) matrices S_1 and S_2 such that

$$f(t, x, y) = \det(tI_n + xS_1 + yS_2) = F_S(t, x, y),$$

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where $S = S_1 + iS_2$. Helton and Vinnikov [6] gave an affirmative answer to the conjecture (see also [10, 12]). In this case, we call that the form $f(t, x, y)$ admits a determinantal representation by the matrix S.

In [2], the authors of this paper study a typical roulette curve given by

$$\phi(\theta) = \exp(in\theta) + a\exp(-i(n-1)\theta), \tag{1.1}$$

$0 \leq \theta \leq 2\pi, n = 2, 3, \ldots$, and $0 < a < 1$. In particular, they obtain that there exists a $2n \times 2n$ matrix A so that the roulette (1.1) is exactly the algebraic curve defined by $F_A(t, x, y)$. In other words,

$$F_A(1, \Re(\phi(\theta)), \Im(\phi(\theta))) = 0, \ 0 \leq \theta \leq 2\pi. \tag{1.2}$$

A more general form of the roulette curve (1.1) is a class of trigonometric polynomials given by

$$\phi(\theta) = \sum_{j=-n}^{n} c_j \exp(ij\theta). \tag{1.3}$$

The curve C_ϕ in the Gaussian plane associated to the trigonometric polynomial ϕ is defined as

$$C_\phi = \{(\Re(\phi(\theta)), \Im(\phi(\theta))) : 0 \leq \theta \leq 2\pi\}.$$

By using Henrion method [7] based on Bezoutian resultant, it is shown in [3] that there exist $2n \times 2n$ real symmetric matrices A_1, A_2, A_3 so that the curve C_ϕ lies in the curve

$$\det(A_1 + xA_2 + yA_3) = 0.$$

Sufficient conditions are given in [3] that guarantee the matrix A_1 being positive definite. In this case, the curve C_ϕ admits a determinantal representation by the matrix

$$A_0 = A_1^{-1/2}(A_2 + iA_3)A_1^{-1/2},$$

that is $F_{A_0}(1, \Re(\phi(\theta)), \Im(\phi(\theta))) = 0$.

We continue our study to construct another algorithm, based on Sylvester matrix, that produces matrices C_1, C_2, C_3 for trigonometric polynomial $\phi(\theta)$ in (1.3) satisfying

$$\det(C_1 + \Re(\phi(\theta))C_2 + \Im(\phi(\theta))C_3) = 0. \tag{1.4}$$

For a typical trigonometric polynomial $\phi(\theta)$, we assert that C_1 is positive definite, and thus the corresponding curve C_ϕ admits a determinantal representation.

2. Sylvester method

Consider a complex trigonometric polynomial $\phi(\theta)$ as in (1.3). The conjugate of $\phi(\theta)$ is denoted by

$$\psi(\theta) = \sum_{j=-n}^{n} \overline{c_j} \exp(-ij\theta) = \sum_{j=-n}^{n} \overline{c_{-j}} \exp(ij\theta). \tag{2.1}$$

We substitute the variable $u = \exp(i\theta)$. Then (1.3) and (2.1) respectively become

$$\sum_{j=-n}^{n} c_j u^{n+j} - \phi(\theta)u^n = 0, \tag{2.2}$$

$$\sum_{j=-n}^{n} \overline{c_{-j}} u^{n+j} - \psi(\theta) u^n = 0. \tag{2.3}$$

Recall that the $2\ell \times 2\ell$ Sylvester matrix H of two polynomials

$$p(u) = \sum_{j=0}^{\ell} \gamma_{\ell-j} u^j \text{ and } q(u) = \sum_{j=0}^{\ell} \delta_{\ell-j} u^j$$

is defined as

$$H = H_{p,q} = \begin{pmatrix} \gamma_0 & \gamma_1 & \dots & \gamma_\ell & 0 & 0 & \dots & 0 \\ 0 & \gamma_0 & \gamma_1 & \dots & \gamma_\ell & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \gamma_0 & \gamma_1 & \dots & \dots & \gamma_\ell \\ \delta_0 & \delta_1 & \dots & \dots & \delta_\ell & 0 & \dots & 0 \\ 0 & \delta_0 & \delta_1 & \dots & \dots & \delta_\ell & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \delta_0 & \delta_1 & \dots & \dots & \delta_\ell \end{pmatrix}.$$

The determinant of the matrix H is called the *resultant* of $p(u)$ and $q(u)$ with respect to u. It is well known that $p(u)$ and $q(u)$ have a common non-constant factor if and only if $\det(H) = 0$ (cf. [4, 13]).

To construct matrices C_1, C_2, C_3 satisfying (1.4), we introduce a new parameter t in (2.2) and (2.3), and write

$$t \sum_{j=-n}^{n} c_j u^{n+j} - \phi(\theta) u^n = \sum_{j=0}^{2n} \gamma_{2n-j}(t, z) u^j,$$

$$t \sum_{j=-n}^{n} \overline{c_{-j}} u^{n+j} - \psi(\theta) u^n = \sum_{j=0}^{2n} \delta_{2n-j}(t, w) u^j.$$

Now, let H be the $4n \times 4n$ Sylvester matrix of polynomials

$$p(u : t, z) = \sum_{j=0}^{2n} \gamma_{2n-j}(t, z) u^j \text{ and } q(u : t, z) = \sum_{j=0}^{2n} \delta_{2n-j}(t, z) u^j.$$

Denote the matrix H with rows $r_1, r_2, \dots, r_{4n}$ as

$$H = H(r_1, r_2, \dots, r_{4n}). \tag{2.4}$$

More precisely, the j-th row of the matrix H is

$$r_j = (0_{j-1}, c_n t, c_{n-1} t, \dots, c_0 t - \phi, \dots, c_{-n} t, 0_{2n-j})$$

for $1 \le j \le 2n$, and

$$r_j = (0_{j-2n-1}, \overline{c_{-n}} t, \overline{c_{-n+1}} t, \dots, \overline{c_0} t - \psi, \dots, \overline{c_n} t, 0_{4n-j})$$

for $2n+1 \le j \le 4n$, where 0_k stands for k-dimensional zero vector. We will produce a $2n \times 2n$ matrix associated to $\phi(\theta)$ by modifying the matrix H. At first, we define the matrix

$$\tilde{H} = \tilde{H}(r_1, \dots, r_n, \tilde{r}_{n+1}, \dots, \tilde{r}_{3n}, r_{3n+1}, \dots, r_{4n}) \tag{2.5}$$

which is obtained from H (2.4) by replacing the $n+1, n+2, \ldots, 3n$ rows with the following new rows

$$\begin{aligned}
\tilde{r}_{n+1} &= r_{n+1} - c_{-n}/\overline{c_n}\; r_{3n+1}, \\
\tilde{r}_{n+2} &= r_{n+2} - c_{-n}/\overline{c_n}\; r_{3n+2} - (c_{-n+1}\overline{c_n} - c_{-n}\overline{c_{n-1}})/\overline{c_n}^2\; r_{3n+1} \\
\tilde{r}_{n+3} &= r_{n+3} - c_{-n}/\overline{c_n}\; r_{3n+3} - (c_{-n+1}\overline{c_n} - c_{-n}\overline{c_{n-1}})/\overline{c_n}^2\; r_{3n+2} \\
&\quad -[c_{-n+2}\overline{c_n}^2 - c_{-n+1}\overline{c_{n-1}}\,\overline{c_n} + c_{-n}(\overline{c_{n-1}}^2 - \overline{c_{n-2}}\,\overline{c_n})]/\overline{c_n}^3\; r_{3n+1}, \\
&\quad \ldots\ldots,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{r}_{3n} &= r_{3n} - \overline{c_{-n}}/c_n\; r_n, \\
\tilde{r}_{3n-1} &= r_{3n-1} - \overline{c_{-n}}/c_n\; r_{n-1} - (c_n\overline{c_{-n+1}} - c_{n-1}\overline{c_{-n}})/c_n^2\; r_n, \\
\tilde{r}_{3n-2} &= r_{3n-2} - \overline{c_{-n}}/c_n\; r_{n-2} - (c_n\overline{c_{-n+1}} - c_{n-1}\overline{c_{-n}})/c_n^2\; r_{n-1} \\
&\quad -[(c_n^2\overline{c_{-n+2}} - c_n c_{n-1}\overline{c_{-n+1}}) + \overline{c_{-n}}(c_{n-1}^2 - c_{n-2}c_n)]/c_n^3\; r_n, \\
&\quad \ldots\ldots\ldots
\end{aligned}$$

The general rows $\tilde{r}_{n+k}, k = 1, 2, \ldots, n$, are formulated by

$$\tilde{r}_{n+k} = r_{n+k} + \sum_{j=1}^{k} \alpha_j r_{3n+k+1-j},$$

where the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_k$ are uniquely determined so that the $(3n+1)$-th, $(3n+2)$-th, $\ldots$, $(3n+k)$-th entries of the row $\tilde{r}_{n+k}$ all equal 0, while the coefficients $\beta_1, \beta_2, \ldots, \beta_k$ of the general rows

$$\tilde{r}_{3n+1-k} = r_{3n+1-k} + \sum_{j=1}^{k} \beta_j r_{n+j-k},\ \ k = 1, \ldots, n$$

are uniquely determined so that the n-th, $(n-1)$-th, ..., $(n-k+1)$-th entries of the row $\tilde{r}_{3n+1-k}$ equal 0.

The following result is a key observation for the properties of the matrix $\tilde{H}$ in (2.5).

Theorem 2.1. *Let $\tilde{H}$ be the matrix defined in (2.5) corresponding to the trigonometric polynomial $\phi(\theta)$ in (1.3). Then the following hold:*

- (*i*) *The upper left $n \times n$ principal submatrix of $\tilde{H}$ is an upper triangular matrix with diagonals $(c_n t, c_n t, \ldots, c_n t)$.*
- (*ii*) *The lower right $n \times n$ principal submatrix of $\tilde{H}$ is a lower triangular matrix with diagonals $(\overline{c_n} t, \overline{c_n} t, \ldots, \overline{c_n} t)$.*
- (*iii*) *The first n entries and the last n entries of the new rows $\tilde{r_{n+1}}, \ldots, \tilde{r_{2n}}$, $\tilde{r_{2n+1}}, \ldots, \tilde{r_{3n}}$ are all 0.*
- (*iv*) *The form associated to $\phi(\theta)$ in (1.3) is given by*

$$R(t, x, y) \equiv \det(H) = \det(\tilde{H}) = |c_n|^{2n} t^{2n} \times \det(H_0), \tag{2.6}$$

where H_0 is the $2n \times 2n$ principal submatrix of $\tilde{H}$ by deleting the first n and last n rows and columns.

(*v*) *If we denote the matrix* H_0 *by*

$$H_0 = H_0(t, \phi, \psi) = H_0(t, x+iy, x-iy) = tC_1 + xC_2 + yC_3, \tag{2.7}$$

then we have

$$\det(C_1 + \Re(\phi(\theta))C_2 + \Im(\phi(\theta))C_3) = 0.$$

The matrix C_1 obtained in Theorem 2.1 is not necessarily Hermitian and is therefore not positive definite; see, for example, the remark at the end of this section. It is shown in [2] that a special trigonometric polynomial (1.1) admits a determinantal representation. We apply Theorem 2.1 to more general typical trigonometric polynomials of the form $\phi(\theta) = \exp(in\theta) + a\exp(-im\theta)$ which guarantee the positive definiteness of C_1.

Theorem 2.2. *Let* $\phi(\theta)$ *be a trigonometric polynomial defined by*

$$\phi(\theta) = \exp(in\theta) + a\exp(-im\theta),$$

$0 \le \theta \le 2\pi$, *where* $0 < m < n$ *are positive integers and* $0 < a < 1$ *is a positive real number. Then the matrix* $H_0 = tC_1 + xC_2 + yC_3$ *in (2.7) satisfies the following conditions:*

(*i*) *The* $2n \times 2n$ *matrices* C_1, C_2, C_3 *are Hermitian and* C_1 *is positive definite.*
(*ii*) *The matrix* $C_0 = C_1^{-1/2}(C_2 + iC_3)C_1^{-1/2}$ *satisfies*

$$F_{C_0}(t, x, y)\det(C_1) = \det(H_0).$$

(*iii*) *For* $0 \le \theta \le 2\pi$,

$$F_{C_0}(1, \cos(n\theta) + a\cos(m\theta), \sin(n\theta) - a\sin(m\theta)) = 0.$$

Proof. From (2.7), the matrix $H_0(0, x, y) = xC_2 + yC_3$ is the following form

$$\begin{pmatrix} 0 & P(x,y) \\ Q(x,y) & 0 \end{pmatrix},$$

where $P(x, y)$ is a lower triangular Toeplitz matrix

$$P(x,y) = \begin{pmatrix} p_1(x,y) & 0 & 0 & \ldots \\ rp_2(x,y) & p_1(x,y) & 0 & \ldots \\ rp_3(x,y) & p_2(x,y) & p_1(x,y) & \ldots \\ r\ldots & \ldots & \ldots & \ldots \end{pmatrix} \in M_n$$

with

$$\begin{aligned} p_1(x,y) &= [(-\overline{c_n} + c_{-n})x + i(-\overline{c_n} - c_{-n})y]/\overline{c_n}, \\ p_2(x,y) &= (c_{-n+1}\overline{c_n} - c_{-n}\overline{c_{n-1}})(x - iy)/\overline{c_n}^2, \\ p_3(x,y) &= \{c_{-n+2}\overline{c_n}^2 - c_{-n+1}\overline{c_{n-1}}\,\overline{c_n} + c_{-n}(\overline{c_{n-1}}^2 - \overline{c_{n-2}}\,\overline{c_n})\}(x - iy)/\overline{c_n}^3, \\ &\ldots\ldots, \end{aligned}$$

and $Q(x,y)$ is an upper triangular Toeplitz matrix

$$Q(x,y)=\begin{pmatrix} q_1(x,y) & q_2(x,y) & q_3(x,y) & \dots \\ 0 & q_1(x,y) & q_2(x,y) & \dots \\ 0 & 0 & q_1(x,y) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \in M_n$$

with

$$\begin{aligned} q_1(x,y) &= [(-c_n+\overline{c_{-n}})x+i(c_n+\overline{c_{-n}})y]/c_n, \\ q_2(x,y) &= [(c_n\overline{c_{-n+1}}-c_{n-1}\overline{c_{-n}})(x+iy)]/c_n^2, \\ q_3(x,y) &= [\{c_n^2\overline{c_{-n+2}}-c_{n-1}c_n\overline{c_{-n+1}})+\overline{c_{-n}}(c_{n-1}^2-c_{n-2}c_n)\}(x+iy)]/c_n^3, \\ & \dots\dots\dots \end{aligned}$$

Hence the matrices C_2, C_3 are Hermitian, and

$$\begin{aligned} \det(H_0(0,x,y)) &= \det(xC_2+yC_3) \\ &= (-1)^n\, p_1(x,y)^n q_1(x,y)^n \\ &= (-1)^n\{-\overline{c_n}(x+iy)+c_{-n}(x-iy)\}^n \\ &\quad \times\{\overline{c_{-n}}(x+iy)-c_n(x-iy)\}^n/|c_n|^{2n}, \end{aligned}$$

Let $\ell=n-m$. Then the matrix C_1 is given by

$$\begin{pmatrix} I_\ell & 0_{\ell,2n-2\ell} & aI_\ell \\ 0_{2n-2\ell,\ell} & (1-a^2)I_{2n-2\ell} & 0_{2n-2\ell,\ell} \\ aI_\ell & 0_{\ell,2n-2\ell} & I_\ell \end{pmatrix},$$

which is a real symmetric positive definite matrix. The matrix

$$C_0=C_1^{-1/2}(C_2+iC_3)C_1^{-1/2}$$

gives a homogeneous polynomial

$$F_{C_0}(t,x,y)=\det(tI_n+xC_1^{-1/2}C_2C_1^{-1/2}+yC_1^{-1/2}C_3C_1^{-1/2})$$

satisfying

$$F_{C_0}(t,x,y)\det(C_1)=\det(H_0)=\det(tC_1+xC_2+yC_3).$$

The assertion (iii) follows from the Sylvester construction (2.6) and (2.7) for the trigonometric polynomial $\phi(\theta)$, i.e.,

$$F_{C_0}(1,\Re(\phi(\theta)),\Im(\phi(\theta)))=0,\ 0\le\theta\le 2\pi.$$

□

Remark 2.3. Although the matrix C_1 in Theorem 2.2 is positive definite for $\phi(\theta)=\exp(in\theta)+a\exp(-im\theta)$, in general, C_1 is not Hermitian for an arbitrary trigonometric polynomial $\phi(\theta)$ given in (1.3). For example, let $n=2$ and

$$\phi(\theta)=\exp(2i\theta)-\frac{1}{4}\exp(i\theta)-\frac{17}{72}+\frac{1}{36}\exp(-i\theta)+\frac{1}{72}\exp(-2i\theta).$$

Then

$$\phi(\theta)\exp(2i\theta)=(\exp(i\theta)+\frac{1}{3})(\exp(i\theta)+\frac{1}{4})(\exp(i\theta)-\frac{1}{3})(\exp(i\theta)-\frac{1}{2}).$$

The matrices constructed by Theorem 2.2 are

$$C_1 = \begin{pmatrix} 20732 & -5192 & -4828 & 648 \\ -9 & 20714 & -5039 & -4666 \\ -4666 & -5039 & 20714 & -9 \\ r648 & -4828 & -5192 & 20732 \end{pmatrix},$$

and

$$xC_2 + yC_3 = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & \beta & \alpha \\ \bar{\alpha} & \bar{\beta} & 0 & 0 \\ 0 & \bar{\alpha} & 0 & 0 \end{pmatrix},$$

where $\alpha = -20448x - 21024y\, i$, $\beta = 648x - 648y\, i$. The matrix C_1 is not Hermitian.

3. Discussion

Let $0 < m < n$ be two positive integers and $0 < a < 1$ be a real number. Consider a trigonometric polynomial $\phi(\theta) = \exp(in\theta) + a\exp(-im\theta), 0 \leq \theta \leq 2\pi$ which defines a real affine curve by the relation

$$x = x(\theta) = \Re(\phi(\theta)), y = y(\theta) = \Im(\phi(\theta)),$$

$0 \leq \theta \leq 2\pi$. Based on Bezoutian, the authors of this paper [3] gave a constructive proof by providing real symmetric matrices A_1, A_2, A_3 so that the curve $(x(\theta), y(\theta))$ lies on $\det(A_1 + xA_2 + yA_3) = 0$.

We compare the two construction matrices obtained in [3] and Theorem 2.2 by investigating the following example. The relation between Bezoutian and Sylvester resultants can be found in [11]. Let $n = 2, m = 1, a = 4/5$,

$$\phi(\theta) = \exp(2i\theta) + \frac{4}{5}\exp(-i\theta),$$

Then the matrix $H_0(t, x, y) = tC_1 + xC_2 + yC_3$ in (2.7) is computed by

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 4/5 \\ 0 & 9/25 & 0 & 0 \\ 0 & 0 & 9/25 & 0 \\ 4/5 & 0 & 0 & 1 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 4/5 & -1 \\ -1 & 4/5 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; C_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & -4i/5 & -i \\ i & 4i/5 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$

We have that

$$(C_1)^{-1/2}C_2(C_1)^{-1/2} = \frac{5}{9}\begin{pmatrix} 0 & \sqrt{5} & -2\sqrt{5} & 0 \\ \sqrt{5} & 0 & 4 & -2\sqrt{5} \\ -2\sqrt{5} & 4 & 0 & \sqrt{5} \\ 0 & -2\sqrt{5} & \sqrt{5} & 0 \end{pmatrix}$$

and

$$(C_1)^{-1/2}C_3(C_1)^{-1/2} = \frac{5}{9}\begin{pmatrix} 0 & -i\sqrt{5} & -2i\sqrt{5} & 0 \\ i\sqrt{5} & 0 & -4i & -2i\sqrt{5} \\ 2i\sqrt{5} & 4i & 0 & -i\sqrt{5} \\ 0 & 2i\sqrt{5} & i\sqrt{5} & 0 \end{pmatrix}.$$

Thus the matrix $C_0 = C_1^{-1/2}(C_2 + iC_3)C_1^{-1/2}$ in Theorem 2.2 is given by

$$C_0 = \frac{10}{9}\begin{pmatrix} 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 4 & 0 \\ -2\sqrt{5} & 0 & 0 & \sqrt{5} \\ 0 & -2\sqrt{5} & 0 & 0 \end{pmatrix}. \tag{3.1}$$

On the other hand, the matrices constructed by Bezoutian in [3] satisfying

$$6250000 \det(tC_1 + xC_2 + yC_3) = \det(tA_1 + xA_2 + yA_3)$$

are given by

$$A_1 = \begin{pmatrix} 27 & 0 & -63 & 0 \\ 0 & 27 & 0 & -3 \\ -63 & 0 & 207 & 0 \\ 0 & -3 & 0 & 7 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} -15 & 0 & 35 & 0 \\ 0 & 65 & 0 & 15 \\ 35 & 0 & 85 & 0 \\ 0 & 15 & 0 & -35 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -60 & 0 & -10 \\ -60 & 0 & -10 & 0 \\ 0 & -10 & 0 & 40 \\ -10 & 0 & 40 & 0 \end{pmatrix},$$

The matrix $A_1^{-1/2}$ is a scalar multiple of the matrix

$$S = \begin{pmatrix} p & 0 & q & 0 \\ 0 & u & 0 & v \\ q & 0 & r & 0 \\ 0 & v & 0 & w \end{pmatrix},$$

where

$$p = \sqrt{218(6217 + 98\sqrt{5})},\ q = 7\sqrt{218(13 - 2\sqrt{5})},\ r = \sqrt{218(257 + 98\sqrt{5})},$$

$$u = \sqrt{298(1373 + 54\sqrt{5})},\ v = 3\sqrt{298(17 - 6\sqrt{5})},\ w = 3\sqrt{298(637 + 6\sqrt{5})}.$$

More precisely $S = 2\sqrt{108}\sqrt{149}A_1^{-1/2}$.
The matrices $A_1^{-1/2}A_2A_1^{-1/2}$ and $A_1^{-1/2}A_3A_1^{-1/2}$ are respectively real symmetric matrices of the form

$$\begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ 0 & a_{22} & 0 & a_{24} \\ a_{13} & 0 & a_{33} & 0 \\ 0 & a_{24} & 0 & a_{44} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & a_{12} & 0 & a_{14} \\ a_{12} & 0 & a_{23} & 0 \\ 0 & a_{23} & 0 & a_{34} \\ a_{14} & 0 & a_{34} & 0 \end{pmatrix},$$

where a_{ij}'s are distinct non-zero real numbers. Therefore none of entries of the matrix $A_0 = A_1^{-1/2}(A_2 + iA_3)A_1^{-1/2}$ is 0, while the matrix C_0 in (3.1) obtained by

Theorem 2.2 is rather sparse. The sparsity of A_0 and C_0, obtained by the two methods, is an interesting subject for further study.

We have proposed two constructive algorithms for determinantal representations of the trigonometric polynomial $\phi(\theta) = \exp(in\theta) + a\exp(-im\theta)$ by matrices $A_0 = A_1^{-1/2}(A_2 + iA_3)A_1^{-1/2}$ and $C_0 = C_1^{-1/2}(C_2 + iC_3)C_1^{-1/2}$ satisfying (1.2). It is interesting to ask whether the two matrices A_0 and C_0 are unitarily similar. At this time, we cannot answer this question. Nevertheless, we give a positive answer for the case when

$$\phi(\theta) = \exp(2i\theta) + 4/5\exp(-i\theta).$$

According to [2], there constructs a matrix

$$B = \frac{10}{9}\begin{pmatrix} 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & -3 \\ -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfying

$$729\det(tI_4 + x\Re(B) + y\Im(B)) = 15625\det(tC_1 + xC_2 + yC_3).$$

At first, we show that the matrices A_0 and B are unitarily similar by a unitary intertwining matrix W:

$$WA_1^{-1/2}(A_2 + iA_3)A_1^{-1/2} = BW.$$

Setting $WA_1^{1/2} = V$, the matrix V satisfies

$$VA_1^{-1}(A_2+iA_3) = WA_1^{1/2}A_1^{-1}(A_2+iA_3) = WA_1^{-1/2}(A_2+iA_3) = BWA_1^{1/2} = BV, \tag{3.2}$$

and

$$VA_1^{-1}V^* = WA_1^{1/2}A_1^{-1}A_1^{1/2}W^* = WW^* = I_4. \tag{3.3}$$

Conversely, if V satisfies (3.2) and (3.3) then the unitary matrix $W = VA_1^{-1/2}$ satisfies $WA_1^{-1/2}(A_2 + iA_3)A_1^{-1/2}W^* = B$. Such a matrix V is given by

$$V = \begin{pmatrix} -3i/2 & 3/2 & -3i/2 & 3/2 \\ 3i/2 & 3/2 & 3i/2 & 3/2 \\ -3i/2 & -9/2 & 9i/2 & 3/2 \\ 9i/2 & -3/2 & -27i/2 & 1/2 \end{pmatrix}.$$

This shows that A_0 and B are unitarily similar.

On the other hand, the matrix C_0 is unitarily similar to B, and $UC_0U^* = B$ for the unitary matrix

$$U = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & 0 & -2/\sqrt{5} \\ 2/\sqrt{5} & 0 & 0 & 1/\sqrt{5} \end{pmatrix}.$$

Thus, both A_0 and C_0 are unitarily similar to B.

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