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ORTHONORMAL BASES AND A STRUCTURE OF FINITE DIMENSIONAL NORMED LINEAR SPACES

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ABSTRACT. In this paper, we consider orthonormal bases for finite dimensional normed linear spaces which are related to Birkhoff orthogonality. We also study the relationship between orthonormal bases and a structure of finite dimensional normed linear spaces.

1. INTRODUCTION

Throughout this paper, the term "normed linear space" always means a real normed linear space. Let X be a normed linear space equipped with the norm $\|\cdot\|$. Then B_X and S_X are, respectively, the closed unit ball and unit sphere of X. If A is a subset of a normed linear space, let $\langle A\rangle$ be the linear span of A.

Orthogonality is one of the most important concepts in the theory of inner product spaces. It appears in many fundamental theorems such as the Pythagorean theorem. However, when switching from the theory of inner product spaces to Banach space geometry, we cannot maintain the notion of orthogonality. Therefore, since 1934 generalized orthogonality types for normed linear spaces have been introduced and studied.

(i) *Roberts* $\perp_R$: $x \perp_R y$ if $\|x+\alpha y\| = \|x-\alpha y\|$ for all $\alpha \in \mathbb{R}$ (cf. Roberts [13]).
(ii) *Isosceles* $\perp_I$: $x \perp_I y$ if $\|x+y\| = \|x-y\|$ (cf. James [9]).
(iii) *Pythagorean* $\perp_P$: $x \perp_P y$ if $\|x-y\|^2 = \|x\|^2 + \|y\|^2$ (cf. James [9]).

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There are many other types of orthogonality in normed linear spaces such as Diminnie [8], Carlsson [5], T-orthogonality [15], and so on. The survey of Alonso-Martini-Wu [2] is a good starting point to study the history of generalized orthogonality.

In this paper, we focus on Birkhoff orthogonality especially. Let X be a normed linear space and let $x, y \in X$. Then, recall that x is said to be Birkhoff orthogonal to y, denoted by $x \perp_B y$, if $\|x + \alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{R}$. If M and N are two subspaces of X, then M is said to be Birkhoff orthogonal to N, denoted by $M \perp_B N$, if $x \perp_B y$ for all $x \in M$ and all $y \in N$. In particular, $\langle \{x\} \rangle \perp_B N$ and $M \perp_B \langle \{y\} \rangle$ are simply denoted by $x \perp_B N$ and $M \perp_B y$, respectively. Recall that Birkhoff orthogonality is always *homogeneous*, that is, $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for any real numbers α and β. However, Birkhoff orthogonality is not *symmetric* in general, that is, $x \perp_B y$ does not imply $y \perp_B x$. Indeed, a real normed linear space of dimension greater than or equal to three is an inner product space if and only if Birkhoff orthogonality is symmetric (Day [7], James [10]). More details about Birkhoff orthogonality can be found in [4, 6, 7, 9, 10, 11].

A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be normal if it satisfies $\|\cdot\|_1 \leq \|\cdot\| \leq \|\cdot\|_\infty$. Let NN_n denote the set of all normal norms on $\mathbb{R}^n$. In the previous paper [16], we showed that every n-dimensional normed linear space is isometrically isomorphic to the space $\mathbb{R}^n$ endowed with a normal norm. If $n = 2$, the result is due to Alonso [1]. This is essentially based on the following result of Day [6]: Every n-dimensional normed linear space X has a basis $\{e_1, e_2, \ldots, e_n\} \subset S_X$ such that

(B_1) $e_k \perp_B M_k$ for all $k = 1, 2, \ldots, n$,

where $M_k = \langle \{e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_n\} \rangle$. However, as mentioned in above, $e_k \perp_B M_k$ is not equivalent to $M_k \perp_B e_k$. So we consider a basis $\{e_1, e_2, \ldots, e_n\} \subset S_X$ satisfying the following condition:

(B_2) $M_k \perp_B e_k$ for all $k = 1, 2, \ldots, n$,

A basis $\{e_1, e_2, \ldots, e_n\}$ for an n-dimensional normed linear space is said to be B_i-*orthogonal* if it satisfies the condition (B_i) $(i = 1, 2)$, and B_i-*orthonormal* if it is B_i-orthogonal and *normal*, that is, $\{e_1, e_2, \ldots, e_n\} \subset S_X$.

In this paper, we study B_1-orthonormal bases and B_2-orthonormal bases for finite dimensional normed linear spaces.

2. Orthonormal bases

We first consider a characterization of orthonormal bases in terms of norms.

Proposition 2.1. *Let X be an n-dimensional normed linear space and $\{e_1, \ldots, e_n\}$ be a normal basis for X. Then, the following are equivalent:*

(i) *$\{e_1, e_2, \ldots, e_n\}$ is B_1-orthonormal.*
(ii) *The equation*

$$\left\| \sum_{i=1}^n a_i e_i \right\| \geq \max_{1 \leq i \leq n} |a_i|$$

holds for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$.

Proof. We only note that for each $k = 1, 2, \ldots, n$, $e_k \perp_B [\{e_i\}_{i \neq k}]$ if and only if

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| = \left\| a_k e_k + \sum_{i \neq k} a_i e_i \right\| \geq |a_k|$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Then one can easily check that (i) is equivalent to (ii). □

Remark 2.2. Bases satisfying (ii) of Proposition 2.1 was considered by Khalil [12]. He mainly studied Schauder bases which satisfies (ii).

Proposition 2.3. *Let X be an n-dimensional normed linear space and $\{e_1, \ldots, e_n\}$ be a normal basis for X. Then, the following are equivalent:*

(i) *$\{e_1, e_2, \ldots, e_n\}$ is B_1-orthonormal.*
(ii) *The inequality*

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| \geq \max_{1 \leq k \leq n} \left\| \sum_{i \neq k} a_i e_i \right\|$$

holds for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$.

Proof. Let $i = 1, 2, \ldots, n$. Since $[\{e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_n\}] \perp_B e_k$ if and only if

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| \geq \left\| \sum_{i \neq k} a_i e_i \right\|$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, one has the equivalence of (i) and (ii). □

We next consider the relationship between the conditions (B_1) and (B_2). Then, the following lemma is needed, which is an important characterization of the condition (B_2). The proof is easy, and so is omitted.

Lemma 2.4. *Let $\{e_1, e_2, \ldots, e_n\}$ be a B_2-orthonormal basis for an n-dimensional normed linear space. Then,*

$$\left\| \sum_{i=1}^{n} a_i e_i \right\| = \max \left\{ \left\| \sum_{i \in A} a_i e_i \right\| : A \subset \{1, 2, \ldots, n\} \right\}$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$.

This means that if $\{e_1, e_2, \ldots, e_n\}$ is B_2-orthonormal, any natural projection on X has norm 1. Hence, B_2-orthonormal bases can be viewed as finite dimensional version of 1-unconditional bases.

From the preceding lemma, we immediately have the following result.

Theorem 2.5. *Let X be an n-dimensional real normed linear space. Suppose that $\{e_1, e_2, \ldots, e_n\}$ is a normal basis for X. Then, the following hold:*

(i) *If $\{e_1, e_2, \ldots, e_n\}$ is B_2-orthonormal, then it is B_1-orthonormal.*
(ii) *If $n = 2$, then $\{e_1, e_2\}$ is B_2-orthonormal if and only if it is B_1-orthonormal.*

In symbols, $(B_2) \Rightarrow (B_1)$, and $(B_2) \Leftrightarrow (B_1)$ if $n = 2$.

As seen in the following example, (B_1) does not imply (B_2) if $n \geq 3$.

Example 2.6. Suppose that $n \geq 3$. Our purpose is to show that (B_1) does not imply (B_2) if $n \geq 3$. Let $\{e_1, e_2, \dots, e_n\}$ be the standard unit vector basis for $\mathbb{R}^n$ and let $\|\cdot\|$ be the normalized norm on $\mathbb{R}^n$ defined by

$$\|(a_1, a_2, \dots, a_n)\| = \max\left\{|a_1|, \left|\sum_{k=1}^n a_k\right|, \max_{2\leq i\leq n}\left|\sum_{k=1}^n \varepsilon_{ik} a_k\right|\right\}$$

for all $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, where

$$\varepsilon_{ik} = \begin{cases} 1 & \text{if } k \neq i, \\ -1 & \text{if } k = i. \end{cases}$$

Then we have

$$\begin{aligned} |a_i| &= \left|\frac{1}{2}\left(\sum_{k=1}^n a_k - \sum_{k=1}^n \varepsilon_{ik} a_k\right)\right| \\ &\leq \frac{1}{2}\left(\left|\sum_{k=1}^n a_k\right| + \left|\sum_{k=1}^n \varepsilon_{ik} a_k\right|\right) \\ &\leq \|(a_1, a_2, \dots, a_n)\| \end{aligned}$$

for each $i \geq 2$, and so

$$\|(a_1, a_2, \dots, a_n)\| \geq \max_{1\leq i\leq n} |a_i|$$

for all $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$. Thus $\{e_1, e_2, \dots, e_n\}$ satisfies (B_1). However,

$$\|(-1, 1, \dots, 1)\| = n - 2 < n - 1 = \|(0, 1, \dots, 1)\|.$$

Therefore $\langle \{e_k\}_{k\geq 2}\rangle \not\perp_B e_1$, and so $\{e_1, e_2, \dots, e_n\}$ does not satisfy (B_2).

3. The norm structure

In this section, we characterize the existence of orthonormal bases. The existence of B_1-orthonormal bases is characterized using normal norms.

Theorem 3.1. *Let X be an n-dimensional normed linear space. Then, X has a B_1-orthonormal basis if and only if it is isometrically isomorphic to the space $\mathbb{R}^n$ endowed with a normal norm.*

Proof. Suppose that X has a B_1-orthonormal basis $\{e_1, e_2, \dots, e_n\}$. Define a norm $\|\cdot\|_0$ on $\mathbb{R}^n$ by

$$\|(a_1, a_2, \dots, a_n)\|_0 = \left\|\sum_{k=1}^n a_k e_k\right\|$$

for all $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$. Then, $\|\cdot\|_0 \in NN_n$ by Proposition 2.1, and X is isometrically isomorphic to the space $(\mathbb{R}^n, \|\cdot\|_0)$.

Conversely, we assume that X is isometrically isomorphic to the space $\mathbb{R}^n$ endowed with a regular norm $\|\cdot\|_0$. For each $i = 1, 2, \dots, n$, let e_i be the element of X which identified with the i-th coordinate vector of $\mathbb{R}^n$. Then, we have

$$\left\|\sum_{k=1}^n \alpha_k e_k\right\| = \|(\alpha_1, \alpha_2, \dots, \alpha_n)\|_0$$

for all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, and so

$$\begin{aligned}\left\| \sum_{k=1}^{n} \alpha_k e_k \right\| &= \|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_0 \\ &\geq \max_{1\leq i\leq n} |a_i|\end{aligned}$$

holds for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Hence, $\{e_1, e_2, \ldots, e_n\}$ is a B_1-orthonormal basis for X by Proposition 2.1. The proof is complete. □

Remark 3.2. As was mentioned in introduction, Day [6] showed that every finite dimensional normed linear space has a B_1-orthonormal basis.

We next consider a characterization of B_2-orthonormal bases. For this purpose, we introduce the notion of regular norm. We recall some about norms on $\mathbb{R}^n$.

Definition 3.3. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be absolute if

$$\|(a_1, a_2, \ldots, a_n)\| = \|(|a_1|, |a_2|, \ldots, |a_n|)\|$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, and normalized if

$$\|(1, 0, \ldots, 0)\| = \|(0, 1, 0, \ldots, 0)\| = \|(0, \ldots, 0, 1)\| = 1.$$

Let AN_n denote the family of all absolute normalized norms on $\mathbb{R}^n$.

The following is an important characterization of absolute norm. The proof can be found in [3, Proposition IV.1.1] (see, also [14, Lemma 4.1]).

Lemma 3.4. *A norm $\|\cdot\|$ on $\mathbb{R}^n$ is absolute if and only if it is monotone, that is, if $|a_k| \leq |b_k|$ for all $k = 1, 2, \ldots, n$ then $\|(a_1, a_2, \ldots, a_n)\| \leq \|(b_1, b_2, \ldots, b_n)\|$.*

We now introduce regular norms on $\mathbb{R}^n$.

Definition 3.5. A norm $\|\cdot\|$ on $\mathbb{R}^n$ is said to be regular if it is normalized and

$$\|(a_1, a_2, \ldots, a_n)\| \geq \max_{1\leq k\leq n} \|(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)\|$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Let RN_n denote the family of all regular norms on $\mathbb{R}^n$.

Regular norms have a characterization analogous to Lemma 3.4. To see this, we introduce the notion of the Ω_i-quadrant. For each $n \geq 2$, define a $2^{n-1} \times n$ matrix R_n^+ by the formula

$$R_2^+ = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad R_n^+ = \left(\begin{array}{c|c} 1 & \\ \vdots & R_{n-1}^+ \\ 1 & \\ \hline 1 & \\ \vdots & -R_{n-1}^+ \\ 1 & \end{array}\right) \quad (n \geq 3).$$

Fix a positive integer $n \geq 2$. Let

$$R_n^+ = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{2^{n-1}} \end{pmatrix}.$$

Then, the Ω_i-quadrant of $\mathbb{R}^n$ is given by

$$\Omega_i = \theta_i \cdot \mathbb{R}_+^n = \{\theta_i \cdot x : x \in \mathbb{R}_+^n\},$$

where $\mathbb{R}_+$ is the set of all nonnegative real numbers, and $\theta_i \cdot x$ denotes the pointwise product of θ_i and x. For example, if $n = 3$,

$$R_3^+ = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \Omega_1 &= \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0\}, \\ \Omega_2 &= \{(x, y, -z) \in \mathbb{R}^3 : x, y, z \geq 0\}, \\ \Omega_3 &= \{(x, -y, -z) \in \mathbb{R}^3 : x, y, z \geq 0\}, \\ \Omega_4 &= \{(x, -y, z) \in \mathbb{R}^3 : x, y, z \geq 0\}. \end{aligned}$$

The following is an analog of Lemma 3.4 for regular norms.

Theorem 3.6. *A normalized norm $\|\cdot\|$ on $\mathbb{R}^n$ is regular if and only if it is semi-monotone, that is,*

$$\|(a_1, a_2, \ldots, a_n)\| \leq \|(b_1, b_2, \ldots, b_n)\|,$$

whenever $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in \Omega_i$ for some i and $|a_k| \leq |b_k|$ for all $k = 1, 2, \ldots, n$, or equivalently, whenever $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)$ satisfies $a_k b_k \geq 0$ and $|a_k| \leq |b_k|$ for all $k = 1, 2, \ldots, n$.

Proof. If $\|\cdot\|$ is semi-monotone, it is clearly regular.

For the converse, suppose that $\|\cdot\|$ is regular. Then

$$\|(a_1, a_2, \ldots, a_n)\| \geq \max_{1 \leq k \leq n} \|(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)\|$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Let k be a positive integer such that $1 \leq k \leq n$, let $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, and let $t \geq 1$. Then we have

$$\begin{aligned} &\|(a_1, a_2, \ldots, a_n)\| \\ &\leq (1 - t^{-1})\|(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)\| \\ &\qquad\qquad + t^{-1}\|(a_1, \ldots, a_{k-1}, ta_k, a_{k+1}, \ldots, a_n)\| \\ &\leq (1 - t^{-1})\|(a_1, \ldots, a_{k-1}, ta_k, a_{k+1}, \ldots, a_n)\| \\ &\qquad\qquad + t^{-1}\|(a_1, \ldots, a_{k-1}, ta_k, a_{k+1}, \ldots, a_n)\| \\ &= \|(a_1, \ldots, a_{k-1}, ta_k, a_{k+1}, \ldots, a_n)\|. \end{aligned}$$

Now, let $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ be elements of $\mathbb{R}^n$ such that $|a_i| \leq |b_i|$ and $a_i b_i \geq 0$ for all $i = 1, 2, \ldots, n$. Then we obtain

$$\begin{aligned}\|(a_1, a_2, \ldots, a_n)\| &\leq \|(b_1, a_2, \ldots, a_n)\| \\ &\leq \|(b_1, b_2, a_3, \ldots, a_n)\| \\ &\vdots \\ &\leq \|(b_1, b_2, \ldots, b_n)\|\end{aligned}$$

Thus, the norm $\|\cdot\|$ is semi-monotone. □

Therefore, the regularity of norms can be viewed as monotonicity in each quadrants. We now characterize the existence of B_2-orthonormal bases in terms of regular norm.

Theorem 3.7. *Let X be an n-dimensional normed linear space. Then, X has a B_2-orthonormal basis if and only if it is isometrically isomorphic to the space $\mathbb{R}^n$ endowed with a regular norm.*

Proof. Assume that X has a B_2-orthonormal basis $\{e_1, e_2, \ldots, e_n\}$. Then, X is identified with the space $(\mathbb{R}^n, \|\cdot\|_0)$ under the identification

$$X \ni \sum_{k=1}^{n} \alpha_k e_k \longleftrightarrow (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n,$$

where the norm $\|\cdot\|_0$ is given by

$$\|(a_1, a_2, \ldots, a_n)\|_0 = \left\| \sum_{k=1}^{n} a_k e_k \right\|$$

for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Moreover, one can obtain that $\|\cdot\|_0 \in RN_n$ since $\{e_1, e_2, \ldots, e_n\}$ is B_2-orthonormal.

For the converse, suppose that X is isometrically isomorphic to the space $\mathbb{R}^n$ endowed with a regular norm $\|\cdot\|_0$. Let T be an isometric isomorphism from X onto $(\mathbb{R}^n, \|\cdot\|_0)$. For each $i = 1, 2, \ldots, n$, let e_i be the element of X such that

$$Te_i = (0, \ldots, 0, \overset{(i)}{1}, 0, \ldots, 0).$$

Then, it follows that

$$\left\| \sum_{k=1}^{n} \alpha_k e_k \right\| = \|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_0$$

for all $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$. Thus,

$$\begin{aligned}\left\|\sum_{k=1}^{n} \alpha_k e_k\right\| &= \|(\alpha_1, \alpha_2, \ldots, \alpha_n)\|_0 \\ &\geq \max_{1\leq k\leq n} \|(a_1, \ldots, a_{k-1}, 0, a_{k+1}, \ldots, a_n)\|_0 \\ &= \max_{1\leq k\leq n} \left\|\sum_{i\neq k} a_i e_i\right\|\end{aligned}$$

holds for all $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, which shows that $\{e_1, e_2, \ldots, e_n\}$ satisfies $(B2)$ by Proposition 2.1. The proof is complete. $\square$

Remark 3.8. A normal basis $\{e_1, e_2, \ldots, e_n\}$ for an n-dimensional normed linear space X is said to be R-orthonormal if it satisfies the following condition:

(R) $e_k \perp_R M_k$ for all $k = 1, 2, \ldots, n$.

Then, it is easy to check that an n-dimensional normed linear space X has an R-orthonormal basis if and only if it is isometrically isomorphic to the space $\mathbb{R}^n$ endowed with an absolute normalized norm. Thus, we finally have the following relationship.

$$\begin{array}{ccccc} (R) & \Rightarrow & (B_2) & \Rightarrow & (B_1) \\ \updownarrow & & \updownarrow & & \updownarrow \\ AN_n & \subset & RN_n & \subset & NN_n \end{array}$$

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