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SMOOTHNESS AND APPROXIMATIVE COMPACTNESS IN ORLICZ FUNCTION SPACES

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Abstract. Some criteria for approximative compactness of every weakly* closed convex set of Orlicz function spaces equipped with the Luxemburg norm are given. Although, criteria for approximative compactness of Orlicz function spaces equipped with the Luxemburg norm were known, we can easily deduce them from our main results.

1. Introduction and preliminaries

Let X be a Banach space and X^* the dual space of X. Denote by $B(X)$ and $S(X)$ the closed unit ball and the unit sphere of X. Let $C \subset X$ be a nonempty subset of X. Then the set-valued mapping $P_C : X \to C$

$$P_C(x) = \{z \in C : \|x - z\| = \operatorname{dist}(x, C) = \inf_{y \in C} \|x - y\|\}$$

is called the metric projection operator from X onto C.

A subset C is said to be proximinal if $P_C(x) \neq \emptyset$ for all $x \in X$(see [5]). It is well known that (see [5]) X is reflexive if and only if each closed convex subset of X is proximinal.

Definition 1.1. A nonempty subset C of X is said to be approximatively compact if for any $\{y_n\}_{n=1}^{\infty} \subset C$ and any $x \in X$ satisfying $\|x - y_n\| \to \inf_{y \in C} \|x - y\|$ as $n \to \infty$, there exists a subsequence of $\{y_n\}_{n=1}^{\infty}$ converging to an element in C.

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X is called approximatively compact if every nonempty closed convex subset of X is approximatively compact.

Consider a convex subset A of a Banach space X. A point $x \in A$ is said to be an extreme point of A if $2x = y + z$ and $y, z \in A$ imply $y = z$. The set of all extreme points of A is denoted by $ExtA$. $f_x \in S(X^*)$ is called a supporting functional of $x \in S(X)$ if $f_x(x) = 1$. $x \in S(X)$ is called a smooth point if it has an unique supporting functional f_x. If every $x \in S(X)$ is a smooth point, then X is called a smooth space. A Banach space X is said to be rotund if for any $x, y \in S(X)$ with $\|x + y\| = 2$ we have $x = y$. It is easy to see that $ExtB(X) = S(X)$ if and only if X is rotund. A Banach space X is said to be locally uniformly convex if for any $x \in S(X)$ and $\{x_n\}_{n=1}^{\infty} \subset S(X)$ with $\|x_n + x\| \to 2$ as $n \to \infty$ we have $\|x_n - x\| \to 0$ as $n \to \infty$. The topic of this paper is related to the topic of [1]-[3] and [5]-[12].

Throughout this paper, $B_r(x) = \{y \in X : \|y - x\| < r\}$. $x_n \xrightarrow{w} x$ denotes that $\{x_n\}_{n=1}^{\infty}$ is weakly convergent to x. $x_n^* \xrightarrow{w^*} x^*$ denotes that $\{x_n^*\}_{n=1}^{\infty}$ is weakly* convergent to x^*. The weak topology of X is denoted by $\sigma(X^*, X)$ or (X, w). The weak* topology of X^* is denoted by $\sigma(X, X^*)$ or (X^*, w^*). $\overline{C}(\overline{C^w}, \overline{C^{w^*}})$ denotes the closed hull of C(weak closed hull, weak* closed hull), respectively. $\mathrm{dist}(x, C)$ denotes the distance between x and C.

Definition 1.2. $M : R \to R$ is called an $N-$function if it has the following properties:

(1) M is even, continuous, convex and $M(0) = 0$.
(2) $M(u) > 0$ for all $u \neq 0$.
(3) $\lim\limits_{u \to 0} \frac{M(u)}{u} = 0$ and $\lim\limits_{u \to \infty} \frac{M(u)}{u} = \infty$.

Let M, N be a couple of complementary $N-$functions and (G, Σ, μ) be a finite nonatomic and complete measure space. Denote by p and q the right derivative of M and N, respectively. We define

$$\rho_M(x) = \int_G M(x(t))dt$$

$$L_M = \{x(t) : \int_G M(\lambda x(t))dt < \infty \ \text{ for some } \ \lambda > 0\},$$

$$E_M = \{x(t) : \int_G M(\lambda x(t))dt < \infty \ \text{ for all } \ \lambda > 0\}.$$

It is well known that L_M is a Banach space when it is equipped with the Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \rho_M(\frac{x}{\lambda}) \leq 1\}$$

or with the Orlicz norm

$$\|x\|^0 = \inf_{k>0} \frac{1}{k}(1 + \rho_M(kx)).$$

L_M,E_M denote Orlicz spaces equipped with the Luxemburg norm. L_M^0,E_M^0 denote Orlicz spaces equipped with the Orlicz norm. We know (see [3]) that $(E_N)^* = L_M^0$, $(E_N^0)^* = L_M$.

Definition 1.3. We say that an N-function M satisfies condition Δ_2 if there exist $K > 2$ and $u_0 \geq 0$ such that

$$M(2u) \leq KM(u) \quad (u \geq u_0).$$

In this case, we write $M \in \Delta_2$ or $N \in \nabla_2$. If $M \in \Delta_2$ and $N \in \Delta_2$, then we write $M \in \Delta_2 \cap \nabla_2$. We know (see [3]) that L_M is reflexive if and only if $M \in \Delta_2 \cap \nabla_2$.

Let M be an $N-$function. An interval $[a, b]$ is called a structural affine interval of M, or simply, SAI of M, provided that M is affine on $[a, b]$ and it is not affine on either in $[a - \varepsilon, b]$ or in $[a, b + \varepsilon]$ for any $\varepsilon > 0$. Let $\{[a_i, b_i]\}_i$ be all the SAIs of M. We call

$$S_M = R\backslash[\bigcup_i (a_i, b_i)]$$

the set of strictly convex points of M. A continuous function $M : R \to R$ is called strictly convex if

$$M(\frac{u+v}{2}) < \frac{1}{2}M(u) + \frac{1}{2}M(v)$$

for all $u \neq v$.

We know (see [3]) that X is rotund if and only if (1) $M \in \Delta_2$ and (2) M is strictly convex.

In 1998, H.Hudzik and B.X. Wang, (see [1]) proved that $(L_M, \|\cdot\|)$ is approximatively compact if and only if $M \in \Delta_2 \cap \nabla_2$ and M is strictly convex, i.e, $(L_M, \|\cdot\|)$ is rotund and reflexive.

In this paper, we discuss approximative compactness of every weakly* closed convex set of Orlicz function spaces equipped with the Luxemburg norm. We prove that every weakly* closed convex set of $(L_M, \|\cdot\|)$ is approximatively compact if and only if E_N^0 is a smooth space and $M \in \Delta_2$. By this result, it is easy to see that the result of [1] mentioned above is true.

First let us recall some results that will be used in the further part of the paper.

Lemma 1.4. *(see [3]) E_N^0 is a smooth space if and only if q is continuous.*

Lemma 1.5. *(see [3]) Let $M \in \Delta_2$ and $u, u_n \in L_M$. Then*

$$\rho_M(u_n) \to \rho_M(u) \text{ and } u_n(t) \xrightarrow{\mu} u(t) \Rightarrow \|u_n - u\| \to 0 (n \to \infty),$$

where $u_n(t) \xrightarrow{\mu} u(t)$ denotes that $(u_n(t))_{n=1}^{\infty}$ is measurable convergent to $u(t)$.

Lemma 1.6. *(see [4]) $x \in S(L_M)$ is an extreme point if and only if $x_n \xrightarrow{E_N^0} x$ imply $x_n \xrightarrow{\mu} x$, where $u_n(t) \xrightarrow{E_N^0} u(t)$ denotes that $\{u_n(t)\}_{n=1}^{\infty}$ is E_N^0-weakly convergent to $u(t)$.*

2. Main results

Theorem 2.1. *The following statements are equivalent:*

(1) Every weak* hyperplane of X^* is approximatively compact.

(2) If $x_n^* \in S(X^*)$, $x \in S(X)$ and $x_n^*(x) \to 1$ as $n \to \infty$, then $\{x_n^*\}_{n=1}^{\infty}$ is relatively compact.

Proof. (1) $\Rightarrow$ (2) If $y_n^* \in S(X^*)$, $x_0 \in S(X)$ and $y_n^*(x_0) \to 1$ as $n \to \infty$, we define the hyperplane

$$H = \{x^* : x^*(x_0) = 1,\ x^* \in X^*\}.$$

We will prove that H is weakly* closed set. In fact, let $z^* \in \overline{H^{w^*}}$, then there exists a net $\{z_\alpha^*\}$ such that $z_\alpha^* \to z^*$. Hence $z_\alpha^*(x_0) \to z^*(x_0)$. Notice that

$$z_\alpha^*(x_0) = 1 \Rightarrow z^*(x_0) = 1 \Rightarrow z^* \in H.$$

It is obvious that H is a convex set. Then H is a proximinal set. Hence there exists $z_n^* \in H$ such that $\|y_n^* - z_n^*\| = \mathrm{dist}(y_n^*, H)$. Pick $x_0^* \in H \cap S(X^*)$. Since

$$\|y_n^* - z_n^*\| = \mathrm{dist}(y_n^*, H) = \mathrm{dist}(y_n^* - x_0^*, H - x_0^*) = x_0(x_0^* - y_n^*) = 1 - x_0(y_n^*) \to 0,$$

we have

$$\|z_n^*\| \le \|y_n^*\| + \|y_n^* - z_n^*\| \to 1 = \mathrm{dist}(0, H) \text{ as } n \to \infty. \tag{2.1}$$

By virtue of $\|z_n^*\| \ge 1$ and (2.1), one get $\|0 - z_n^*\| \to \mathrm{dist}(0, H)$ as $n \to \infty$. This implies that the sequence $\{z_n^*\}_{n=1}^{\infty}$ is relatively compact. Consequently $\{y_n^*\}_{n=1}^{\infty}$ is relatively compact by $y_n^* = z_n^* + (y_n^* - z_n^*)$.

(2) $\Rightarrow$ (1) (a) First we will prove that every weakly* closed convex set is a proximinal set. Let C^* be a weakly* closed convex set and $x^* \in X^* \backslash C^*$. Then there exists $\{y_n^*\}_{n=1}^{\infty} \subset C^*$ such that

$$\|x^* - y_n^*\| \to \mathrm{dist}(x^*, C^*) \text{ as } n \to \infty.$$

We may assume without loss of generality that $y_n^* \ne y_m^*$ for any $m \ne n$. Since $B(X^*)$ is weak* compact, there exists $y_0^* \in B(X^*)$ such that y_0^* is weak* accumulation point of $\{y_n^*\}_{n=1}^{\infty}$. Put

$$\Delta = \{U_{y_0^*} : U_{y_0^*} \text{ is weak}^* \text{ neighbourhood of } y_0^*\}.$$

We define a order by the containing relations. i.e, $U_{y_0^*} \subset V_{y_0^*}$ if and only if $V_{y_0^*} > U_{y_0^*}$ Thus we define a order set Δ. Hence for any weak* neighbourhood U_α of 0, there exists y_n^* such that $y_n^* \in y_0^* + U_\alpha$. Let $y_\alpha^* = y_n^*$. Hence we now define a net $\{y_\alpha^*\}_{\alpha\in\Delta} \subset \{y_n^*\}_{n=1}^{\infty}$ and $y_\alpha^* \stackrel{w^*}{\to} y_0^*$. Since C^* is a weakly* closed convex set, we have $y_0^* \in C^*$. For any $\varepsilon > 0$, there exists a $x_0 \in S(X)$ such that

$$(x^* - y_0^*)x_0 > \|x^* - y_0^*\| - \varepsilon.$$

Since $y_\alpha^* \xrightarrow{w^*} y_0^*$, we have $x^* - y_\alpha^* \xrightarrow{w^*} x^* - y_0^*$. Then there exists α_0 for which if $\alpha > \alpha_0$, then we have $(x^* - y_\alpha^*)x_0 > (x^* - y_0^*)x_0 - \varepsilon$. Hence

$$(x^* - y_\alpha^*)x_0 > (x^* - y_0^*)x_0 - \varepsilon > \|x^* - y_0^*\| - \varepsilon - \varepsilon = \|x^* - y_0^*\| - 2\varepsilon$$

and

$$\|x^* - y_\alpha^*\| \geq (x^* - y_\alpha^*)x_0 > (x^* - y_0^*)x_0 - \varepsilon > \|x^* - y_0^*\| - \varepsilon - \varepsilon = \|x^* - y_0^*\| - 2\varepsilon.$$

In view of the fact that weak* topology is a Hausdorff topology, it follows that $\{y_\alpha^*\}_{\alpha>\alpha_0}$ is infinite set. Then there exists a subsequence $\{m\}$ of $\{n\}$ such that $\{y_m^*\}_{m=1}^\infty \subset \{y_\alpha^*\}_{\alpha>\alpha_0}$. So we have

$$\|x^* - y_m^*\| \geq (x^* - y_m^*)x_0 > (x^* - y_0^*)x_0 - \varepsilon > \|x^* - y_0^*\| - \varepsilon - \varepsilon = \|x^* - y_0^*\| - 2\varepsilon$$

and

$$\operatorname{dist}(x^*, C^*) = \lim_{m\to\infty} \|x^* - y_m^*\| \geq \|x^* - y_0^*\| - 2\varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we have

$$\left.\begin{array}{r} \operatorname{dist}(x^*, C^*) = \lim\limits_{m\to\infty} \|x^* - y_m^*\| \geq \|x^* - y_0^*\| \\ \|x^* - y_0^*\| \geq \operatorname{dist}(x^*, C^*). \end{array}\right\} \Rightarrow \|x^* - y_0^*\| = \operatorname{dist}(x^*, C^*).$$

Hence C^* is a proximinal set.

(b) Let $H^* = \{x^* \in X^* : x^*(x) = k\}$ be weak* hyperplane. For any $x^* \notin H^*$ we may assume without loss of generality that $x^* = 0$. We will prove that if $\|0 - y_n^*\| \to \operatorname{dist}(x^*, H^*)$ as $n \to \infty$, then the sequence $\{y_n^*\}_{n=1}^\infty$ has a subsequence converging to a element in H^*, where $\{y_n^*\}_{n=1}^\infty \subset H^*$. Notice that H^* is weakly* closed set, so we have $\operatorname{dist}(0, H^*) = r > 0$. Pick $y_0^* \in P_{H^*}(0)$. Then

$$r = \operatorname{dist}(0, P_{H^*}(0)) = \|y_0^*\|, \qquad B_r(0) \cap H^* = \emptyset, \qquad \overline{B_r(0)} \cap H^* = P_{H^*}(0).$$

Without loss of generality, we may assume and we do so that $k \leq 0$. Hence we have

$$k = \sup\{x(y^*) : y^* \in H^*\} \leq \inf\{x(y^*) : y^* \in B_r(0)\} = -\|x\| \cdot \|y_0^*\|.$$

In fact, suppose that there exists $y_0^* \in B_r(0)$ such that $x(y_0^*) < k$. Then there exists $\lambda \in (0,1)$ such that $x(\lambda y_0^*) = k$. It is easy to see that $\lambda y_0^* \in B_r(0)$ and $\lambda y_0^* \in H^*$, a contradiction. Since $y_0^* \in P_{H^*}(0) \subset H^*$, we have

$$\begin{aligned} -\|x\| \cdot \|y_0^*\| \leq x(y_0^*) &\leq \sup\{x(y^*) : y^* \in H^*\} \\ &\leq \inf\{x(y^*) : y^* \in B_r(0)\} \\ &= -\|x\| \cdot \|y_0^*\| \end{aligned}$$

and

$$-\|x\| \cdot \|y_0^*\| = x(y_0^*) = \sup\{x(y^*) : y^* \in H^*\}.$$

This means that the inequality $x(y_0^*) \geq x(y_n^*)$ holds. Therefore

$$\|0 - y_0^*\| = x(0 - y_0^*) \leq x(0 - y_n^*) \leq \|0 - y_n^*\| \to \operatorname{dist}(0, H^*) = \|0 - y_0^*\| = \|y_0^*\|.$$

Hence

$$\|y_n^*\| \to \|y_0^*\| \text{ as } n \to \infty$$

and

$$x(0 - y_n^*) \to \|0 - y_0^*\| \text{ as } n \to \infty.$$

Furthermore, we have

$$x\left(-\frac{y_n^*}{\|y_n^*\|} + \frac{y_n^*}{\|y_0^*\|}\right) = \left(\frac{1}{\|y_0^*\|} - \frac{1}{\|y_n^*\|}\right) \cdot x(y_n^*) \to 0 \text{ as } n \to \infty,$$

which shows that

$$x(-\frac{y_n^*}{\|y_n^*\|}) \to 1 \text{ as } n \to \infty.$$

By (a), we have that $\{-\frac{y_n^*}{\|y_n^*\|}\}_{n=1}^{\infty}$ is a relatively compact set. Notice that $\|y_n^*\| \to \|y_0^*\|$ as $n \to \infty$, so we obtain that $\{y_n^*\}_{n=1}^{\infty}$ is relatively compact. This implies that $\{y_n^*\}_{n=1}^{\infty}$ has a subsequence converging to an element in H^*. Hence we have that weak* hyperplane H^* is approximatively compact. □

Theorem 2.2. *Let $(L_M, \|\cdot\|)$ be an Orlicz function space. Then the following statements are equivalent:*

(1) Every weak* hyperplane of $(L_M, \|\cdot\|)$ is approximatively compact.

(2) (a) $M \in \Delta_2$; and (b) if x is norm attainable on $S(E_N^0)$, then x is an extreme point of $B(L_M)$.

(3) E_N^0 is a smooth space and $M \in \Delta_2$.

(4) q is continuous and $M \in \Delta_2$.

In order to prove the theorem, we will give some auxiliary lemmas.

Lemma 2.3. *X is a smooth space if and only if any $x^* \in S(X^*)$ is an extreme point of $B(X^*)$ provided that x^* is norm attainable on $S(X)$.*

Proof. Sufficiency. Suppose that there exist $x \in S(X)$ and two different $x_1^*, x_2^* \in S(X^*)$ such that $x_1^*(x) = x_2^*(x) = 1$. Then $\frac{1}{2}(x_1^* + x_2^*)(x) = 1$. This implies that $\frac{1}{2}(x_1^* + x_2^*) \in S(X^*)$ is norm attainable on $S(X)$. It is easy to see that $\frac{1}{2}(x_1^* + x_2^*)$ is not an extreme point of $B(X^*)$, a contradiction!

Necessity. Let $x^* \in S(X^*)$ be norm attainable on $S(X)$. Then there exists $x \in S(X)$ such that $x^*(x) = 1$. Suppose that x^* is not an extreme point. Then there exists $x_1^*, x_2^* \in S(X^*)$ such that $x^* = \frac{1}{2}(x_1^* + x_2^*)$. Hence we have

$$x^*(x) = \frac{1}{2}(x_1^* + x_2^*)(x) = 1 \Rightarrow x_1^*(x) = x_2^*(x) = 1.$$

This implies that x is not a smooth point, a contradiction! □

Lemma 2.4. *The following statements are equivalent:*

(a) $x \in (L_M, \|\cdot\|)$ is an extreme point of $B(L_M)$;

(b) If $x = \sum_{i=1}^{\infty} t_i x_i$ where $x_i \in B(X)$, $t_i \in (0,1)$ and $\sum_{i=1}^{\infty} t_i = 1$ belongs to $S(L_M)$, then the sequence $\{x_i\}_{i=1}^{\infty}$ is relatively compact.

Proof. (b)⇒(a). We know (see [3]) that $x \in (L_M, \|\cdot\|)$ is an extreme point if and only if (1) $\rho_M(x) = 1$ and (2) $\mu\{t \in G : x(t) \notin S_M\} = 0$. If (1) is not satisfied, then $\varepsilon := 1 - \rho_M(x) > 0$. Certainly we can choose $E \in \Sigma$ and $\delta > 0$ such that

$$0 < \int_E M(2x(t))dt < \frac{\varepsilon}{4},$$

and

$$\eta = \int_E M(2\delta)dt \leq \frac{1}{4}\varepsilon.$$

Decompose E into E_1^1, E_2^1 such that $E_1^1 \cap E_2^1 = \emptyset$, $E_1^1 \cup E_2^1 = E$, $\mu E_1^1 = \mu E_2^1$. Next, decompose E_1^1 into E_1^2, E_2^2 such that $E_1^2 \cap E_2^2 = \emptyset$, $E_1^2 \cup E_2^2 = E_1^1$, $\mu E_1^2 = \mu E_2^2$. Finally, decompose E_2^1 into E_3^2, E_4^2 such that $E_3^2 \cap E_4^2 = \emptyset$, $E_4^2 \cup E_4^2 = E_2^1$, $\mu E_3^2 = \mu E_4^2$. Generally, decompose E_i^{n-1} into E_{2i-1}^n, E_{2i}^n such that

$$E_{2i-1}^n \cap E_{2i}^n = \emptyset, \quad E_{2i-1}^n \cup E_{2i}^n = E_i^n - 1, \quad \mu E_{2i-1}^n = \mu E_{2i}^n,$$

$$(n = 1, 2, \cdots \; i = 1, 2, \cdots, 2^{n-1}).$$

Define

$$u_n(t) = \begin{cases} x(t), & t \in G \backslash E \\ x(t) - \delta, & t \in E_1^n \\ x(t) + \delta, & t \in E_2^n \\ \cdots & \cdots \\ x(t) - \delta, & t \in E_{2^n-1}^n \\ x(t) + \delta, & t \in E_{2^n}^n, \end{cases} \qquad u_n^{'}(t) = \begin{cases} x(t), & t \in G \backslash E \\ x(t) + \delta, & t \in E_1^n \\ x(t) - \delta, & t \in E_2^n \\ \cdots & \cdots \\ x(t) + \delta, & t \in E_{2^n-1}^n \\ x(t) - \delta, & t \in E_{2^n}^n, \end{cases}$$

and

$$\{y_n(t)\}_{n=1}^{\infty} = (u_1(t), u_1^{'}(t), u_2(t), u_2^{'}(t), \cdots, u_n(t), u_n^{'}(t), \cdots).$$

Then we have

$$\begin{aligned} \rho_M(u_n) &= \int\limits_{G\backslash E} M(x(t))dt + \int\limits_{E_1^n} M(x(t) - \delta)dt + \int\limits_{E_2^n} M(x(t) + \delta)dt \\ &\quad + \cdots + \int\limits_{E_{2^n-1}^n} M(x(t) - \delta)dt + \int\limits_{E_{2^n}^n} M(x(t) + \delta)dt \\ &\leq \int\limits_{G\backslash E} M(x(t))dt + \frac{1}{2}\int\limits_{E_1^n} M(2x(t))dt + \frac{1}{2}\int\limits_{E_1^n} M(2\delta)dt + \frac{1}{2}\int\limits_{E_2^n} M(2x(t))dt \\ &\quad + \frac{1}{2}\int\limits_{E_2^n} M(2\delta)dt + \cdots + \frac{1}{2}\int\limits_{E_{2^n-1}^n} M(2x(t))dt + \frac{1}{2}\int\limits_{E_{2n-1}^n} M(2\delta)dt \\ &\quad + \frac{1}{2}\int\limits_{E_{2n}^n} M(2x(t))dt + \frac{1}{2}\int\limits_{E_{2n}^n} M(2\delta)dt \\ &= \int\limits_{G\backslash E} M(x(t))dt + \frac{1}{2}\int\limits_{E} M(2x(t))dt + \frac{1}{2}\int\limits_{E} M(2\delta)dt \\ &\leq \int\limits_{G} M(x(t))dt + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < 1. \end{aligned}$$

Similarly, we have $\rho_M(u_n^{'}) \leq 1$. Hence $\rho_M(y_n) \leq 1$. This implies that $y_n(t) \in B(L_M)$. On the other hand, we have

$$\sum_{n=1}^{\infty} (\frac{1}{2} \cdot \frac{1}{2^n} u_n(t) + \frac{1}{2} \cdot \frac{1}{2^n} u_n^{'}(t)) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} (u_n(t) + u_n^{'}(t)) = \sum_{n=1}^{\infty} \frac{2}{2^{n+1}} x(t) = x(t),$$

and

$$\sum_{n=1}^{\infty}(\frac{1}{2}\cdot\frac{1}{2^n}+\frac{1}{2}\cdot\frac{1}{2^n})=\sum_{n=1}^{\infty}(\frac{1}{2^n})=1.$$

But $\{y_n(t)\}_{n=1}^{\infty}$ is not relatively compact. In fact, taking any subsequence $\{u_n(t)\}_{n=1}^{\infty}$ of $\{y_n(t)\}_{n=1}^{\infty}$, we have for $m\neq n$

$$\rho_M(u_n-u_m)=\frac{1}{2}\int\limits_E M(2\delta)dt>0.$$

This implies that $\|u_n-u_m\|\geq\min\{1,\frac{1}{2}\int\limits_E M(2\delta)dt\}$. Hence we obtain that $\{y_n(t)\}_{n=1}^{\infty}$ is not relatively compact, a contradiction!

Suppose that (2) is not satisfied. Then $\mu\{t\in G:x(t)\notin S_M\}>0$. Since $R\setminus S_M$ is the union of at most countably many open intervals, there exists an $\varepsilon>0$ and an interval (a,b) such that $-\infty<a<b<+\infty$,

$$\mu\{t\in G:x(t)\in(a+\varepsilon,b-\varepsilon)\}>0$$

and that M is affine on $[a,b]$, that is, $M(u)=ku+\beta$, for $u\in[a,b]$. Let $e=\{t\in G:x(t)\in(a+\epsilon,b-\epsilon)\}$. Decompose e into e_1^1,e_2^1 such that $e_1^1\cap e_2^1=\emptyset$, $e_1^1\cup e_2^1=e$, $\mu e_1^1=\mu e_2^1$. Next, decompose e_1^1 into e_1^2, e_2^2 such that $e_1^2\cap e_2^2=\emptyset$, $e_1^2\cup e_2^2=e_1^1$, $\mu e_1^2=\mu e_2^2$. Finally, decompose e_2^1 into e_3^2, e_4^2 such that $e_3^2\cap e_4^2=\emptyset$, $e_4^2\cup e_4^2=e_2^1$, $\mu e_3^2=\mu e_4^2$. Generally, decompose e_i^{n-1} into e_{2i-1}^n, e_{2i}^n such that

$$e_{2i-1}^n\cap e_{2i}^n=\emptyset,\ e_{2i-1}^n\cup e_{2i}^n=e_i^n-1,\quad \mu e_{2i-1}^n=\mu e_{2i}^n$$

$$n=1,2,\cdots i=1,2,\cdots,2^{n-1}.$$

Define

$$u_n(t)=\begin{cases} x(t), & t\in G\setminus e\\ x(t)-\varepsilon, & t\in e_1^n\\ x(t)+\varepsilon, & t\in e_2^n\\ \cdots & \cdots\\ x(t)-\varepsilon, & t\in e_{2^n-1}^n\\ x(t)+\varepsilon, & t\in e_{2^n}^n,\end{cases}\qquad u_n^{'}(t)=\begin{cases} x(t), & t\in G\setminus e\\ x(t)+\varepsilon, & t\in e_1^n\\ x(t)-\varepsilon, & t\in e_2^n\\ \cdots & \cdots\\ x(t)+\varepsilon, & t\in e_{2^n-1}^n\\ x(t)-\varepsilon, & t\in e_{2^n}^n,\end{cases}$$

and

$$\{y_n(t)\}_{n=1}^{\infty}=(u_1(t),u_1^{'}(t),u_2(t),u_2^{'}(t),\cdots,u_n(t),u_n^{'}(t),\cdots).$$

Then we have

$$\begin{aligned}
\rho_M(u_n) = & \int\limits_{G\setminus e} M(x(t))dt + \int\limits_{e_1^n} M(x(t)-\varepsilon)dt + \int\limits_{e_2^n} M(x(t)+\varepsilon)dt \\
& + \cdots + \int\limits_{e_{2^n-1}^n} M(x(t)-\varepsilon)dt + \int\limits_{e_{2^n}^n} M(x(t)+\varepsilon)dt \\
& = \int\limits_{G\setminus e} M(x(t))dt + \int\limits_{e_1^n} (k(x(t)-\varepsilon)+\beta)dt + \int\limits_{e_2^n} (k(x(t)+\varepsilon)+\beta)dt \\
& + \cdots + \int\limits_{e_{2^n-1}^n} (k(x(t)-\varepsilon)+\beta)dt + \int\limits_{e_{2^n}^n} (k(x(t)+\varepsilon)+\beta)dt \\
& = \int\limits_{G\setminus e} M(x(t))dt + \int\limits_{e_1^n} (kx(t)+\beta)dt - \int\limits_{e_1^n} k\varepsilon dt + \int\limits_{e_2^n} (kx(t)+\beta)dt + \int\limits_{e_2^n} k\varepsilon dt \\
& + \cdots + \int\limits_{e_{2^n-1}^n} (kx(t)+\beta)dt - \int\limits_{e_{2^n-1}^n} k\varepsilon dt + \int\limits_{e_{2^n}^n} (kx(t)+\beta)dt + \int\limits_{e_{2^n}^n} k\varepsilon dt \\
& = \int\limits_{G\setminus e} M(x(t))dt + \int\limits_{e_1^n} (k(x(t))+\beta)dt + \int\limits_{e_2^n} (k(x(t))+\beta)dt \\
& + \cdots + \int\limits_{e_{2^n-1}^n} (k(x(t))+\beta)dt + \int\limits_{e_{2^n}^n} (k(x(t))+\beta)dt \\
& = \int\limits_{G} M(x(t))dt = 1.
\end{aligned}$$

Similarly, we have $\rho_M(u_n^{'}) = 1$. Hence $\rho_M(y_n) = 1$. This implies that $y_n(t) \in B(L_M)$. On the other hand, we have

$$\sum_{n=1}^{\infty}(\frac{1}{2}\cdot\frac{1}{2^n}u_n(t)+\frac{1}{2}\cdot\frac{1}{2^n}u_n^{'}(t)) = \sum_{n=1}^{\infty}\frac{1}{2^{n+1}}(u_n(t)+u_n^{'}(t)) = \sum_{n=1}^{\infty}\frac{2}{2^{n+1}}x(t) = x(t),$$

and

$$\sum_{n=1}^{\infty}(\frac{1}{2}\cdot\frac{1}{2^n}+\frac{1}{2}\cdot\frac{1}{2^n}) = \sum_{n=1}^{\infty}(\frac{1}{2^n}) = 1.$$

But $\{y_n(t)\}_{n=1}^{\infty}$ is not relatively compact. In fact, taking any subsequence $\{u_n(t)\}_{n=1}^{\infty}$ of $\{y_n(t)\}_{n=1}^{\infty}$, we have for $m \neq n$

$$\rho_M(u_n - u_m) = \frac{1}{2}\int\limits_{e} M(2\varepsilon)dt > 0.$$

This implies that $\|u_n - u_m\| \geq \min\{1, \frac{1}{2}\int\limits_{e} M(2\varepsilon)dt\}$. Hence we obtain that $\{y_n(t)\}_{n=1}^{\infty}$ is not relatively compact, a contradiction!

The implication (a)⇒(b) is obvious. □

Proof of Theorem 2.2. By virtue of Lemmas 1.4 and 2.3, we have (2) ⇔ (3) ⇔ (4). (1) ⇒ (2) (a) Let $v_0(t) \in S(E_N^0)$, $v_0(t) \leq 0$. By $(E_N^0)^* = L_M$ and the Hahn–Banach theorem, there exists $x(t) \in S(L_M)$ such that $\int_G x(t)v_0(t)dt = 1$. It is easy to see that $x(t) \leq 0$. Pick a nonnull set $E \in \Sigma$ such that $|x(t)| \leq C$ on E. If $M \notin \Delta_2$, then there exist a sequence $\{u_n\}_{n=1}^{\infty} \uparrow$ and $E_n \subseteq E$ such that

$$M((1+\frac{1}{n})u_n) \geq 3^n M(u_n), \quad M(u_n)\mu E_n = 2^{-n}.$$

Define

$$x_n(t) = x(t) \cdot \chi_{G \setminus E_n} + u_n \chi_{E_n}.$$

Then $\liminf_{n\to\infty} \|x_n\| \geq \|x\|$. On the other hand from

$$\rho_M(x_n) \leq \rho_M(x) + M(u_n)\mu E_n \leq 1 + 2^{-n} \to 1 \text{ as } n \to \infty.$$

we deduce that $\|x_n\| \to \|x\| = 1$ as $n \to \infty$.

Next, we will show that $x_n \xrightarrow{w^*} x$ as $n \to \infty$. In fact, for any $v(t) \in E_N^0$, we have

$$\begin{aligned}
\left|\int_G (x_n(t) - x(t))v(t)dt\right| &= \left|\int_{E_n} (x_n(t) - x(t))v(t)dt\right| \\
&= \left|\int_{E_n} u_n v(t)dt - \int_{E_n} x(t)v(t)dt\right| \\
&\leq \left|\int_{E_n} u_n v(t)dt\right| + \left|\int_{E_n} x(t) \cdot v(t)dt\right| \\
&\leq M(u_n)\mu E_n + 2\rho_N(v \cdot \chi_{E_n}) \\
&\quad + \rho_M(x \cdot \chi_{E_n}) \to \ 0 \text{ as } n \to \infty.
\end{aligned}$$

By $\int_G x(t)v_0(t)dt = 1$, we have $\int_G x_n(t)v_0(t)dt \to 1$ as $n \to \infty$. By Theorem 2.1, we obtain that $\{x_n\}_{n=1}^{\infty}$ is relatively compact. By $x_n \xrightarrow{w^*} x$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \to x$ as $k \to \infty$. Moreover,

$$\begin{aligned}
\rho_M((1+\frac{1}{n})(x_n - x)) &\geq \int_{E_n} M((1+\frac{1}{n})(u_n - x(t)))dt \\
&\geq \int_{E_n} M((1+\frac{1}{n})(u_n))dt \\
&\geq 3^n M(u_n)\mu E_n \\
&\geq 3^n \cdot 2^{-n} \\
&> 1, \ (n \in N).
\end{aligned}$$

This implies that $\|x_n - x\| \geq \frac{1}{2}$, a contradiction!

(b) Let x be norm attainable on $S(E_N^0)$, i.e, $\int\limits_G x(t)v_0(t)dt = 1$ for some $v_0(t) \in S(E_N^0)$. If $x = \sum\limits_{i=1}^{\infty} t_i x_i$, then

$$1 = \int\limits_G x(t) \cdot v_0(t)dt = \int\limits_G (\sum_{i=1}^{\infty} t_i x_i(t)) \cdot v_0(t)dt = \sum_{i=1}^{\infty} t_i \int\limits_G x_i(t) \cdot v_0(t)dt,$$

where $x_i \in B(L_M)$, $t_i \in (0,1)$ and $\sum\limits_{i=1}^{\infty} t_i = 1$. Hence for any $i \in N$, we have $\int\limits_G x_i(t) \cdot v_0(t)dt = 1$. Which implies that $\{x_i\}_{i=1}^{\infty}$ is relatively compact by Theorem 2.1. By virtue of Lemma 2.4, we obtain that x is an extreme point.

(2) $\Rightarrow$ (1), Let $v_0(t) \in S(E_N^0)$, $x_n(t) \in S(L_M)$ and $\int\limits_G x_n(t)v_0(t)dt \to 1$ as $n \to \infty$. Since the unit ball of a separable dual space is w^*−sequentially compact, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \xrightarrow{w^*} x$ as $k \to \infty$. This means that $x_{n_k} \xrightarrow{E_N^0} x$ as $k \to \infty$. Hence we have

$$\int\limits_G x_{n_k}(t)v_0(t)dt \to \int\limits_G x(t)v_0(t)dt = 1 \text{ as } k \to \infty,$$

which implies that x is norm attainable on $S(E_N^0)$. Hence we obtain that x is an extreme point of $B(L_M)$. By Lemma 1.6, we have that $x_{n_k}(t) \xrightarrow{\mu} x(t)$. From $M \in \Delta_2$ we know that $\rho_M(x_{n_k}) = \rho_M(x) = 1$ for any $k \in N$. By Lemma 1.5, we have that $x_{n_k} \to x$ as $k \to \infty$, i.e, $\{x_n\}_{n=1}^{\infty}$ is relatively compact. This completes the proof. □

Theorem 2.5. *Let $(L_M, \|\cdot\|)$ be an Orlicz function space. Then the following statements are equivalent:*
(1) Every weak hyperplane of $(L_M, \|\cdot\|)$ is approximatively compact;*
(2) Every weakly closed convex set of $(L_M, \|\cdot\|)$ is approximatively compact;*
(3) Every proximinal convex set of $(L_M, \|\cdot\|)$ is approximatively compact;
(4) E_N^0 is smooth space and $M \in \Delta_2$;
(5) q is continuous and $M \in \Delta_2$.

In order to prove the theorem, we will give some auxiliary lemmas.

Lemma 2.6. *Let X be a locally uniformly convex space. Then every proximinal convex set of X is approximatively compact.*

Proof. Let $C \subset X$ be a proximinal convex set. For any $x \notin C$, we may assume without loss of generality that $x = 0$ and $d = \mathrm{dist}(0, C) = 1$. Let $x_n \in C$ and $\|x_n\| \to 1$. Pick $y \in P_C(0)$. It is easy to see that $\|y\| = 1$ and $\frac{1}{2}(x_n + y) \in C$. Then

$$1 \le \left\|0 - (\frac{1}{2}(x_n + y))\right\| = \left\|\frac{x_n + y}{2}\right\| \le \left\|\frac{x_n}{2}\right\| + \left\|\frac{y}{2}\right\| \to 1 \text{ as } n \to \infty.$$

This implies that $\|x_n + y\| \to 2$ as $n \to \infty$. Moreover,

$$\|x_n + y\| - \left\| \frac{x_n}{\|x_n\|} - x_n \right\| \leq \left\| \frac{x_n}{\|x_n\|} + y \right\| \leq \|x_n + y\| + \left\| \frac{x_n}{\|x_n\|} - x_n \right\|$$

for any $n \in N$. Then $\left\| \frac{x_n}{\|x_n\|} + y \right\| \to 2$ as $n \to \infty$. Since X is a locally uniformly convex space, we obtain that $\frac{x_n}{\|x_n\|} \to y$ as $n \to \infty$. This means that $\{x_n\}_{n=1}^{\infty}$ has a subsequence converging to an element in C. This completes the proof. $\square$

Lemma 2.7. *(see [3]) L_M is locally uniformly convex if and only if $M \in \Delta_2$ and M is strictly convex.*

Proof of Theorem 2.5. (5)$\Rightarrow$(3) Since q is continuous, we obtain that p is strictly increasing. This implies that M is strictly convex. Since M is strictly convex and $M \in \Delta_2$, by Lemma 2.7, we obtain that $(L_M, \|\cdot\|)$ is locally uniformly convex. By Lemma 2.6, (3) is true. By the proof of Theorem 2.1, (3)$\Rightarrow$(2) is true. The implication (2)$\Rightarrow$(1) is obvious. By Theorem 2.2, (1)$\Leftrightarrow$(4)$\Leftrightarrow$(5) is true. This completes the proof. $\square$

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