Banach J. Math. Anal. 7 (2013), no. 2, 1–14

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

MATRIX TRANSFORMATIONS AND SEQUENCE SPACES EQUATIONS

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Communicated by M. A. Ragusa

ABSTRACT. In this paper we study sequence spaces equations (SSE) with operators, which are determined by an identity whose each term is a sum or a sum of products of sets of the form $\chi_a(T)$ *and* $\chi_{f(x)}(T)$ where f maps U^+ to itself, χ is either of the symbols s, s^0, or $s^{(c)}$. Then we solve five (SSE) of the form $\chi_a + \chi'_x = \chi'_b$, where χ, χ' are either s^0, $s^{(c)}$, or s. We apply the previous results to the solvability of the systems $s_a^0 + s_x(\Delta) = s_b$, $s_x \supset s_b$ and $s_a + s_x^{(c)}(\Delta) = s_b^{(c)}$, $s_x^{(c)} \supset s_b^{(c)}$. Finally we solve the (SSE) with operators defined by $\chi_a(C(\lambda)D_\tau) + s_x^{(c)}(C(\mu)D_\tau) = s_b^{(c)}$ where χ is either s^0, or s.

1. INTRODUCTION

In the book entitled *Summability through Functional Analysis* [16] Wilansky introduced sets of the form $a^{-1} * E$ where E is a BK space, and $a = (a_n)_{n\geq 1}$ is a nonzero sequence. Recall that $\xi = (\xi_n)_{n\geq 1}$ belongs to $a^{-1} * E$ if $a\xi \in E$. In [5], the sets s_a, s_a^0 and $s_a^{(c)}$ were defined for positive sequences a by $(1/a)^{-1} * \chi$ and $\chi = \ell_\infty, c_0, c$, respectively. In [6, 10] the sum $\chi_a + \chi'_b$ and the product $\chi_a * \chi'_b$ were defined where χ, χ' are any of the symbols s, s^0, or $s^{(c)}$, among other things characterizations of matrix transformations mapping in the sets $s_a + s_b^0(\Delta^q)$ and $s_a + s_b^{(c)}(\Delta^q)$ were given, where Δ is the *operator of the first difference.* In [11] de Malafosse and Malkowsky gave among other things properties of the

Date: Received: 16 October 2012; Accepted: 30 October 2012.
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2010 *Mathematics Subject Classification.* Primary 40H05; Secondary 46A45.
Key words and phrases. Matrix transformations, BK space, multiplier of sets of sequences, sequence spaces inclusion equations, sequence spaces equations with operator.

matrix of weighted means considered as an operator in the set s_a. In [12] the characterizations of the sets $(s_a(\Delta^q), F)$ can be found where F is any of the sets c_0, c and ℓ_∞. We also cite Hardy's results [2] extended by Móricz and Rhoades [14, 15], de Malafosse and Rakočević [9] and formulated as follows, in [2] it is said that a series $\sum_{m=1}^{\infty} y_m$ is *summable* $(C,1)$ if $\varkappa_n = n^{-1}\sum_{m=1}^{n} s_m \to l$, where $s_m = \sum_{k=1}^{m} y_k$, it was shown by Hardy that if a series $\sum_{m=1}^{\infty} y_m$ is *summable* $(C,1)$ then $\sum_{m=1}^{\infty}\left(\sum_{i=m}^{\infty} y_i/i\right)$ is convergent. On the other hand Hardy's *Tauberian* theorem for *Cesàro means* states that if $(y_n)_n \in s_{(1/n)}(\Delta)$, then $n^{-1}s_n \to l$ implies $y_n \to l$ for some $l \in \mathbb{C}$. This problem is a consequence of the following one: What are the sequences x such that $c(C_1) \cap s_x(\Delta) \subset c$ where C_1 is the Cesàro operator?

In this paper we extend some results given in [1, 8, 7]. In [1] were given solvability of the equations $s_a + s_x = s_b$ and $s_{\varphi(x)} = s_b$ where φ maps U^+ to itself. In [8] it is shown that the solutions of the equations $\chi_a + s_x^0 = s_b^0$ where χ is any of the symbols s, or $s^{(c)}$ if $a/b \in c_0$ are given by $s_x = s_b$ and if $a/b \notin c_0$ each of these equations has no solution. In this paper for given sequences a and b, we determine the set of all sequences $x \in U^+$ such that for every sequence y, we have $y_n/b_n \to l$ if and only if there are sequences u and v such that $y = u + v$ and $u_n/a_n \to 0$, $v_n/x_n \to l'$ as n tends to infinity for some scalars l, l'. This statement means $s_a^0 + s_x^{(c)} = s_b^{(c)}$. So we are led to deal with special *sequence spaces inclusion equations (SSIE), (resp. sequence spaces equations (SSE)),* which are determined by an inclusion, (resp. identity), where each term is a *sum or a sum of products of sets of the form* $\chi_a(T)$ *and* $\chi_{f(x)}(T)$ where f maps U^+ to itself, χ is any of the symbols s, s^0, or $s^{(c)}$, x is the unknown and T is a triangle. For instance the solutions of the elementary (SSE) defined by $s_x = s_a$ with $a \in U^+$ are given by $K_1 a_n \le x_n \le K_2 a_n$ for some K_1, $K_2 > 0$ and for all n. In [1] we dealt with the equation $s_a + s_x = s_b$ whose the solutions are given by $s_x = s_b$ if $a/b \in c_0$, if $s_a = s_b$ the solutions of this equation are given by $x \in s_a$ and if $a/b \notin \ell_\infty$ this one has no solution. Except for these cases until now we don't know the behaviour of this equation. In [7] are determined the solutions of (SSE) with operators of the form $(\chi_a * \chi_x + \chi_b)(\Delta) = \chi_\eta$ and $\left[\chi_a * (\chi_x)^2 + \chi_b * \chi_x\right](\Delta) = \chi_\eta$. and $\chi_a + \chi_x(\Delta) = \chi_x$ where χ is any of the symbols s, or s^0.

This paper is organized as follows. In Section 2 we recall some definitions and results on sequence spaces and matrix transformations. In Section 3 we recall some properties of the multiplier of two sequence spaces. Then we state some results on the sum and the product of sequence spaces. In section 4 we deal with the solvability of the five (SSE) $s_a^0 + s_x = s_b$, $s_a^{(c)} + s_x^0 = s_b^0$, $s_a + s_x^0 = s_b^0$, $s_a + s_x^{(c)} = s_b^{(c)}$ and $s_a^0 + s_x^{(c)} = s_b^{(c)}$. In Section 5 we deal with the solvability of the system constituted with an (SSE) and an (SSIE) defined by $s_a^0 + s_x(\Delta) = s_b$ and $s_x \supset s_b$. Finally we solve the (SSE) with operators defined by $\chi_a(C(\lambda)D_\tau) + s_x^{(c)}(C(\mu)D_\tau) = s_b^{(c)}$ where χ is either s^0, or s. These results extend some recent results of Farés and de Malafosse [1] and de Malafosse [6, 10, 7, 8].

2. Notations and preliminary results.

For a given infinite matrix $\Lambda = (\lambda_{nm})_{n,m\geq 1}$ we define the operators Λ_n for any integer $n \geq 1$, by

$$\Lambda_n(\xi) = \sum_{m=1}^{\infty} \lambda_{nm}\xi_m$$

where $\xi = (\xi_m)_{m\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator A defined by $\Lambda\xi = (\Lambda_n(\xi))_{n\geq 1}$ mapping between sequence spaces.

A Banach space E of complex sequences with the norm $\|\|_E$ is a *BK space if each projection* $P_n{:}E \to \mathbb{C}$ with $P_n\xi = \xi_n$ *is continuous.* A BK space E is *said to have AK if every sequence* $\xi = (\xi_m)_{m\geq 1} \in E$ has a *unique representation* $\xi = \sum_{n=1}^{\infty} \xi_n e^{(n)}$ where $e^{(n)}$ is the sequence with 1 in the n-th position and 0 otherwise.

We will denote by s, c_0, c, ℓ_∞ the sets of all sequences, the sets of sequences that converge to zero, that are convergent and that are bounded, respectively. If ξ and η are sequences and E and F are two subsets of s, then we write $\xi\eta = (\xi_n\eta_n)_n$ and

$$M(E,F) = \{\xi = (\xi_n)_{n\geq 1} : \ \xi\eta \in F \text{ for all } \eta \in E\},$$

$M(E,F)$ is called the *multiplier space of* E *and* F. We shall use the set $U^+ = \left\{(u_n)_{n\geq 1} \in s \ : u_n > 0 \text{ for all } n\right\}$. Using Wilansky's notations [16], we define for any sequence $a = (a_n)_{n\geq 1} \in U^+$ and for any set of sequences E, the set $(1/a)^{-1} * E = \{(\xi_n)_{n\geq 1} \in s \ : (\xi_n/a_n)_n \in E\}$. To simplify, we use the diagonal matrix D_a defined by $[D_a]_{nn} = a_n$ for all n and write $D_a * E = (1/a)^{-1} * E$ and define $s_a = D_a * \ell_\infty$, $s_a^0 = D_a * c_0$ and $s_a^{(c)} = D_a * c$, see for instance [4, 6, 5, 13]. Each of the spaces $D_\alpha * \chi$, where $\chi \in \{\ell_\infty, c_0, c\}$, is a BK space normed by $\|\xi\|_{s_a} = \sup_{n\geq 1}(|\xi_n|/a_n)$ and s_a^0 has AK.

Now let $a = (a_n)_{n\geq 1}$, $b = (b_n)_{n\geq 1} \in U^+$. By $S_{a,b}$ we denote the set of infinite matrices $\Lambda = (\lambda_{nm})_{n,m\geq 1}$ such that $\|\Lambda\|_{S_{a,b}} = \sup_{n\geq 1}\left[(1/b_n)\sum_{m=1}^{\infty}|\lambda_{nm}|a_m\right] < \infty$. The set $S_{a,b}$ is a Banach space with the norm $\|\|_{S_{a,b}}$. Let E and F be any subsets of s. When Λ maps E into F we write $\Lambda \in (E,F)$, see [3]. So we have $\Lambda\xi \in F$ for all $\xi \in E$, ($\Lambda\xi \in F$ means that for each $n \geq 1$ the series $\Lambda_n(\xi) = \sum_{m=1}^{\infty} \lambda_{nm}\xi_m$ is convergent and $(\Lambda_n(\xi))_{n\geq 1} \in F$). It was proved in [11] that $A \in (s_a, s_b)$ if and only if $\Lambda \in S_{a,b}$. So we can write that $(s_a, s_b) = S_{a,b}$.

When $s_a = s_b$ we obtain the *Banach algebra with identity* $S_{a,b} = S_a$ normed by $\|\Lambda\|_{S_a} = \|\Lambda\|_{S_{a,a}}$, see [5]. We also have $\Lambda \in (s_a, s_a)$ if and only if $\Lambda \in S_a$.

If $a = (r^n)_{n\geq 1}$, the sets S_a, s_a, s_a^0 and $s_a^{(c)}$ are denoted by S_r, s_r, s_r^0 and $s_r^{(c)}$, respectively; see [4]. When $r = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$, and putting $e = (1, 1, \cdots)$ we have $S_1 = S_e$. It is well known, see [3] that $(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1$. We also have $\Lambda \in (c_0, c_0)$ if and only if $\Lambda \in S_1$ and $\lim_{n\to\infty} \lambda_{nm} = 0$ for all $m \geq 1$.

In the sequel we will frequently use the fact that $\Lambda \in (\chi_a, \chi'_b)$ if and only if $D_{1/b}\Lambda D_a \in (\chi_e, \chi'_e)$ where χ, χ' are any of the symbols s^0, $s^{(c)}$, or s; see [11].

For any subset E of s, we put $\Lambda E = \{\eta \in s \ : \eta = \Lambda\xi \text{ for some } \xi \in E\}$. If F is a subset of s, we write $F(\Lambda) = F_\Lambda = \{\xi \in s \ : \ \Lambda\xi \in F\}$.

3. The multiplier, the sum and the product of certain sets of sequences

3.1. **The multiplier of certain sets of sequences.** First we need to recall some well known results. By [6, Lemma 3.1, p. 648] and [6, Example 1.28, p. 157], we obtain the next result.

Lemma 3.1. *We have*
i) $M(c,c_0)=M(\ell_\infty,c)=M(\ell_\infty,c_0)=c_0$ *and* $M(c,c)=c$*;*
ii) $M(\chi,\ell_\infty)=M(c_0,\chi')=\ell_\infty$ *for* χ, $\chi'=c_0$, c, *or* ℓ_∞.

We deduce from the preceding the next corollary.

Corollary 3.2. *(i)* $M(s_a^0,\chi_b')=s_{b/a}$ *where* χ' *is any of the symbols* s^0, $s^{(c)}$, *or* s*;*
(ii) $M(\chi_a,s_b)=s_{b/a}$ *where* χ *is any of the symbols* $s^{(c)}$, *or* s*;*
(iii) $M\left(s_a,s_b^{(c)}\right)=s_{b/a}^0$ *and* $M\left(s_a^{(c)},s_b^{(c)}\right)=s_{b/a}^{(c)}$.

In the sequel we will use the next lemma.

Lemma 3.3. *Let* a, $b\in U^+$. *Then*

$$a/b\in M(\chi_1,\chi_1')\text{ if and only if }\chi_a\subset\chi_b',$$

where χ, χ' *are any of the symbols* s^0, $s^{(c)}$, *or* s.

Proof. The proof comes from the fact that $a/b\in M(\chi_1,\chi_1')$ is equivalent to $D_{a/b}\in(\chi_1,\chi_1')$ and to $I\in(D_a*\chi_1,D_b*\chi_1')=(\chi_a,\chi_b')$. □

3.2. **Sum and product of sets of the form χ_a, where χ is any of the symbols s^0, $s^{(c)}$, or s.** In this section we recall some properties of the *sum* $\chi_a+\chi_b'$ where χ and χ' are any of the symbols s^0, $s^{(c)}$, or s.

3.2.1. *Sum $E+F$ of sets of sequences..* We can state some results concerning the *sum* of particular interesting sequence spaces.

Let E, $F\subset s$ be two linear spaces. The set $E+F$ is defined by

$$E+F=\{\xi\in s:\xi=u+v\text{ for some }u\in E\text{ and }v\in F\}.$$

It can easily be seen that $E+F=F$ if and only if $E\subset F$. This permits us to show some of the next results that extend some results given in [6].

Theorem 3.4. *Let* a, $b\in U^+$.
(i) a) $s_a\subset s_b$ *if and only if* $a/b\in\ell_\infty$*;*
b) $s_a=s_b$ *if and only if there are* K_1, $K_2>0$ *such that* $K_1\le b_n/a_n\le K_2$ *for all n;*
c) $s_a+s_b=s_{a+b}=s_{\max(a,b)}$, *where* $[\max(a,b)]_n=\max(a_n,b_n)$*;*
d) $s_a+s_b=s_a$ *if and only if* $b/a\in\ell_\infty$.
(ii) a) $s_a^0\subset s_b^0$ *if and only if* $a/b\in\ell_\infty$*;*
b) $s_a^0=s_b^0$ *if and only if* $s_a=s_b$*;*
c) $s_a^0+s_b^0=s_{a+b}^0$*;*
d) $s_a^0+s_b^0=s_a^0$ *if and only if* $b/a\in l_\infty$*;*
e) $s_a^{(c)}\subset s_b^{(c)}$ *if and only if* $a/b\in c$.

f) The condition $a_n/b_n \to l \neq 0$ *for some scalar* l*, is equivalent to* $s_a^{(c)} = s_b^{(c)}$*; and if* $a_n/b_n \to l \neq 0$*, then* $s_a = s_b$*,* $s_a^0 = s_b^0$ *and* $s_a^{(c)} = s_b^{(c)}$*.*

(iii) a) $s_{a+b}^{(c)} \subset s_a^{(c)} + s_b^{(c)}$*.*

b) The conditions $a_n/b_n \to L \in \mathbb{R}^+ \cup \{\infty\}$ *is equivalent to*

$$s_a^{(c)} + s_b^{(c)} = s_{a+b}^{(c)}. \tag{3.1}$$

c) The condition $b/a \in c$ *is equivalent to* $s_a^{(c)} + s_b^{(c)} = s_{a+b}^{(c)} = s_a^{(c)}$*.*

Proof. The proof of this theorem was given in [6, 10] except for iii) b). Proof of iii) b). First show $a_n/b_n \to L \in \mathbb{R}^+ \cup \{\infty\}$ implies (3.1). Let $y \in s_a^{(c)} + s_b^{(c)}$. Then there are φ and $\psi \in c$ such that $y = a\varphi + b\psi$. Then

$$\frac{y}{a+b} = \frac{a}{a+b}\varphi + \frac{b}{a+b}\psi = \frac{\frac{a}{b}}{1+\frac{a}{b}}\varphi + \frac{1}{1+\frac{a}{b}}\psi.$$

We have

$$\lim_{n\to\infty} \frac{\frac{a_n}{b_n}}{1+\frac{a_n}{b_n}} = \begin{cases} \dfrac{L}{1+L} & \text{if } L < \infty, \\ 1 & \text{if } L = \infty; \end{cases}$$

and

$$\lim_{n\to\infty} \frac{1}{1+\frac{a_n}{b_n}} = \begin{cases} \dfrac{1}{1+L} & \text{if } L < \infty, \\ 0 & \text{if } L = \infty. \end{cases}$$

We conclude $s_a^{(c)} + s_b^{(c)} \subset s_{a+b}^{(c)}$.

Now let $y \in s_{a+b}^{(c)}$. Then there is $\zeta \in c$ such that $y/(a+b) = \zeta$, $y = a\zeta + b\zeta$ and $y \in s_a^{(c)} + s_b^{(c)}$. This shows $s_{a+b}^{(c)} \subset s_a^{(c)} + s_b^{(c)}$. We conclude that $a_n/b_n \to L \in \mathbb{R}^+ \cup \{\infty\}$ implies (3.1).

Conversely, show (3.1) implies $a_n/b_n \to L$ $(n \to \infty)$ for some $L \in \mathbb{R}^+ \cup \{\infty\}$. Put $\nu = a/b$. Since trivially we have $a \in s_a^{(c)} + s_b^{(c)} = s_{a+b}^{(c)}$, then

$$\frac{a}{a+b} = \frac{\nu}{1+\nu} = \tau \in c.$$

Put $L' = \lim_{n\to\infty} \tau_n$. We then have $\nu = \tau(1+\nu)$, $\nu = \tau/(1-\tau)$ and

$$\lim_{n\to\infty} \nu_n = \begin{cases} \dfrac{L'}{1-L'} & \text{if } L' \neq 1, \\ \infty & \text{if } L' = 1. \end{cases}$$

So we have shown $a_n/b_n \to L' \in \mathbb{R}^+ \cup \{\infty\}$ $(n \to \infty)$. This concludes the proof. □

3.2.2. *Product of sets of the form χ_ξ where χ is any of the symbols s^0, $s^{(c)}$, or s.* In this part we recall some properties of the *product* $E * F$ of particular subsets E and F of s. These results can be found in [4, 6, 12].

For any given sets of sequences E and F, we write

$$E * F = \bigcup_{\xi \in E} D_\xi * F = \{\xi\eta \in s \; : \; \xi \in E \text{ and } \eta \in F\}.$$

We immediately have the following results,

Proposition 3.5. *Let a, b, $\gamma \in U^+$. Then*

(i) $s_a * s_b = s_a * s_b^{(c)} = s_a^{(c)} * s_b = s_{ab}$,

(ii) $s_a * s_b^0 = s_a^0 * s_b^0 = s_a^{(c)} * s_b^0 = s_{ab}^0$,

(iii) $s_a^{(c)} * s_b^{(c)} = s_{ab}^{(c)}$,

(iv) Let χ be any of the symbols s^0, $s^{(c)}$, or s. Then the solutions of the (SSE) $\chi_a * s_x^0 = s_\gamma^0$ are determined by $K_1\gamma_n/a_n \leq x_n \leq K_2\gamma_n/a_n$ for all n and for some K_1, $K_2 > 0$.

4. Solvability of five (SSE)

In [1] we saw that we don't know the solvability of $s_a + s_x = s_b$, when $a/b \in \ell_\infty \backslash c_0$ and $s_a \neq s_b$. In this section we consider (SSE) of the form $\chi_a + \chi'_x = \chi'_b$ where χ, χ' are distinct and are either s^0, $s^{(c)}$, or s. We show that the next five equations $s_a^0 + s_x = s_b$, $s_a^{(c)} + s_x^0 = s_b^0$, $s_a + s_x^0 = s_b^0$, $s_a + s_x^{(c)} = s_b^{(c)}$ and $s_a^0 + s_x^{(c)} = s_b^{(c)}$ can be totally solved. It remains the (SSE) $s_a^{(c)} + s_x = s_b$, that is not solved until now for $a/b \in \ell_\infty \backslash c_0$.

4.1. **Solvability of five (SSE) of the form $\chi_a + \chi'_x = \chi'_b$ where χ, χ' are any of the symbols s^0, $s^{(c)}$, or s.** The solvability of the equation $s_a + s_x^{(c)} = s_b^{(c)}$ for a, $b \in U^+$ consists in determining the set of all $x \in U^+$ such that for every $y \in s$ we have $y_n/b_n \to l$ $(n \to \infty)$ if and only if there are two sequences u, v such that $y = u + v$ and

$$\frac{u_n}{a_n} = O(1) \text{ and } \frac{v_n}{x_n} \to l' \; (n \to \infty).$$

For χ, $\chi' \in \{s^0, s^{(c)}, s\}$ we put

$$\mathcal{I}(\chi, \chi') = \left\{x \in U^+ : \chi'_b \subset \chi_a + \chi'_x\right\},$$

$$\mathcal{I}'(\chi, \chi') = \left\{x \in U^+ : \chi_a + \chi'_x \subset \chi'_b\right\},$$

and

$$\mathcal{S}(\chi, \chi') = \left\{x \in U^+ : \chi_a + \chi'_x = \chi'_b\right\}.$$

In the following we consider the equivalence relation on U^+ defined by $x\mathcal{R}y$ if $s_x = s_y$ and we denote by $cl(b)$ the equivalence class of $b \in U^+$. Similarly let $cl^{(c)}(b)$ be the equivalence class of b for the equivalence relation $\mathcal{R}_c$ defined on U^+ by $x\mathcal{R}_c y$ if $s_x^{(c)} = s_y^{(c)}$. By Theorem 3.4 ii) f) we have

$$cl^{(c)}(b) = \left\{x \in U^+ : x_n \sim kb_n \; (n \to \infty) \text{ for some } k > 0\right\}.$$

In the following we will use the next elementary result, if E, F and G are linear subspaces of s, then $E+F\subset G$ if and only if $E\subset G$ and $F\subset G$. Now state the following.

Theorem 4.1. [8] *Suppose $a/b\in c_0$. Then the solutions of the equation*

$$\chi_a+\chi'_x=\chi'_b$$

where χ is any of the symbols s^0, $s^{(c)}$, or s, and χ' is any of the symbols s^0, or s are given by $x\in cl\,(b)$.

Corollary 4.2. *If $a/b\in c_0$, then $\mathcal{S}_1(\chi,s)=\mathcal{S}_1(\chi,s^0)=cl\,(b)$ where χ is any of the symbols s^0, $s^{(c)}$, or s.*

Proposition 4.3. [8] *Let χ, χ', χ'' be any of the symbols s^0, $s^{(c)}$, or s. If $a/b\notin M(\chi_1,\chi''_1)$, then the equation*

$$\chi_a+\chi'_x=\chi''_b$$

has no solution.

Now put

$$s_b^*=\left\{x\in U^+: x_n\geq Kb_n \text{ for some } K>0 \text{ and for all } n\right\},$$

and

$$s_b^{*(c)}=\left\{x\in U^+:\lim_{n\to\infty}\frac{x_n}{b_n}=l \text{ for some } l\in\,]0,+\infty]\right\}.$$

Notice that $cl^{(c)}(b)=s_b^{(c)}\backslash s_b^0$. It can easily be seen that

$$s_b^{*(c)}=\left\{x\in U^+: s_b^{(c)}\subset s_x^{(c)}\right\}.$$

Indeed $\lim_{n\to\infty}x_n/b_n=l\in\,]0,+\infty]$ means that $\lim_{n\to\infty}b_n/x_n=1/l\in[0,+\infty[$ and $b\in s_x^{(c)}$. Since $b/x\in M(c,c)=c$ we have $s_b^{(c)}\subset s_x^{(c)}$.

State the next results.

Theorem 4.4. *Let $a,\,b\in U^+$. Then*

i) a) We have

$$\mathcal{S}\left(s^0,s\right)=\begin{cases} cl\,(b) & \text{if } a/b\in\ell_\infty,\\ \varnothing & \text{if } a/b\notin\ell_\infty;\end{cases}$$

b) $\mathcal{S}\left(s^{(c)},s^0\right)=\mathcal{S}\left(s,s^0\right)$ and

$$\mathcal{S}\left(s,s^0\right)=\begin{cases} cl\,(b) & \text{if } a/b\in c_0,\\ \varnothing & \text{if } a/b\notin c_0;\end{cases}$$

ii) a)

$$\mathcal{S}\left(s^0,s^{(c)}\right)=\begin{cases} cl^{(c)}(b) & \text{if } a/b\in\ell_\infty,\\ \varnothing & \text{if } a/b\notin\ell_\infty;\end{cases}$$

b)

$$\mathcal{S}\left(s,s^{(c)}\right)=\begin{cases} cl^{(c)}(b) & \text{if } a/b\in c_0,\\ \varnothing & \text{if } a/b\notin c_0.\end{cases}$$

Proof. i) a) Show $\mathcal{I}(s^0, s) = s_b^*$ if $a/b \in \ell_\infty$. Let $x \in \mathcal{I}(s^0, s)$. Since $s_b \subset s_a^0 + s_x$ and $b \in s_b$, there are $\varepsilon \in c_0$ and $h \in \ell_\infty$ such that $b = a\varepsilon + xh$. We then have $1 = (a/b)\varepsilon + (x/b)h$ and since $a/b \in \ell_\infty$ we deduce $\varepsilon' = (a/b)\varepsilon \in c_0$. Since $1 - \varepsilon'_n \to 1$ $(n \to \infty)$ we conclude

$$\frac{b}{x} = \frac{h}{1 - \varepsilon'} \in \ell_\infty$$

and $\mathcal{I}(s^0, s) \subset s_b^*$ if $a/b \in \ell_\infty$.

Conversely let $x \in s_b^*$. We then have $s_b \subset s_x$ and trivially $s_b \subset s_x \subset s_a^0 + s_x$, for all $x \in U^+$. So we have shown that if $a/b \in \ell_\infty$, then $\mathcal{I}(s^0, s) = s_b^*$.

Now determine $\mathcal{I}'(s^0, s)$. We see that $x \in \mathcal{I}'(s^0, s)$ is equivalent to $s_a^0 \subset s_b$ and $s_x \subset s_b$. We then have $a/b \in M(c_0, \ell_\infty) = \ell_\infty$ and $x/b \in M(\ell_\infty, \ell_\infty) = \ell_\infty$. So

$$\mathcal{I}'\left(s^0, s\right) = \begin{cases} s_b & \text{if } a/b \in \ell_\infty, \\ \varnothing & \text{if } a/b \notin \ell_\infty. \end{cases}$$

We conclude $\mathcal{S}(s^0, s) = \mathcal{I}(s^0, s) \cap \mathcal{I}'(s^0, s) = s_b \cap s_b^* = cl(b)$ if $a/b \in \ell_\infty$ and $\mathcal{S}(s^0, s) = \varnothing$ if $a/b \notin \ell_\infty$. So we have shown i) a).

i) b) Let χ be any of the symbols s^0, or $s^{(c)}$. By Theorem 4.1, we have $\mathcal{S}(\chi, s^0) = cl(b)$ if $a/b \in c_0$. Then by Proposition 4.3 since we have

$$M\left(s_1^{(c)}, s_1^0\right) = M(c, c_0) = c_0 \text{ and } M\left(s_1, s_1^0\right) = M(\ell_\infty, c_0) = c_0,$$

we deduce that if $a/b \notin c_0$, then $\mathcal{S}(\chi, s^0) = \varnothing$. This concludes the proof of i).

ii) a) Show $\mathcal{I}\left(s^0, s^{(c)}\right) = s_b^{*(c)}$ if $a/b \in \ell_\infty$. Let $x \in \mathcal{I}\left(s^0, s^{(c)}\right)$. Since $s_b^{(c)} \subset s_a^0 + s_x^{(c)}$ and $b \in s_b^{(c)}$ there are $\varepsilon \in c_0$ and $\varphi \in c$ such that $b = a\varepsilon + x\varphi$. Then we have $1 = (a/b)\varepsilon + (x/b)\varphi$ and since $a/b \in \ell_\infty$ we deduce $\varepsilon' = (a/b)\varepsilon \in c_0$. Since $1 - \varepsilon'_n \to 1$ $(n \to \infty)$ we conclude $b/x = \varphi/(1 - \varepsilon') \in c$. Thus we have proved $\mathcal{I}\left(s^0, s^{(c)}\right) \subset s_b^{*(c)}$ if $a/b \in \ell_\infty$.

Conversely let $x \in s_b^{*(c)}$. We then have $s_b^{(c)} \subset s_x^{(c)}$ and trivially $s_b^{(c)} \subset s_x^{(c)} \subset s_a^0 + s_x^{(c)}$, for all $x \in U^+$. So we have shown $s_b^{*(c)} \subset \mathcal{I}\left(s^0, s^{(c)}\right)$ and we conclude that $\mathcal{I}\left(s^0, s^{(c)}\right) = s_b^{*(c)}$ for $a/b \in \ell_\infty$.

Now determine the set $\mathcal{I}'\left(s^0, s^{(c)}\right)$. We see again that $x \in \mathcal{I}'\left(s^0, s^{(c)}\right)$ is equivalent to $s_a^0 \subset s_b^{(c)}$ and $s_x^{(c)} \subset s_b^{(c)}$, this means that $a/b \in M(c_0, c) = \ell_\infty$ and $x/b \in M(c, c) = c$. So

$$\mathcal{I}'\left(s^0, s^{(c)}\right) = \begin{cases} s_b^{(c)} & \text{if } a/b \in \ell_\infty, \\ \varnothing & \text{if } a/b \notin \ell_\infty. \end{cases}$$

We conclude $\mathcal{S}\left(s^0, s^{(c)}\right) = s_b^{*(c)} \cap s_b^{(c)} = cl^{(c)}(b)$ if $a/b \in \ell_\infty$ and $\mathcal{S}\left(s^0, s^{(c)}\right) = \varnothing$ if $a/b \notin \ell_\infty$.

ii) b) Show $\mathcal{I}\left(s, s^{(c)}\right) = s_b^{*(c)}$ if $a/b \in c_0$. Let $x \in \mathcal{I}\left(s, s^{(c)}\right)$. Since $s_b \subset s_a + s_x^{(c)}$ and $b \in s_b^{(c)}$ there are $h \in \ell_\infty$ and $\varphi \in c$ such that $b = ah + x\varphi$. We then have $1 = (a/b)h + (x/b)\varphi$ and since $a/b \in c_0$ we deduce $\varepsilon = (a/b)h \in c_0$. Since $1 - \varepsilon_n \to 1$ $(n \to \infty)$ we conclude

$$\frac{b}{x} = \frac{\varphi}{1 - \varepsilon} \in c.$$

This means that $b_n/x_n \to L$ for some $L \geq 0$ and $x_n/b_n \to 1/L \in]0, +\infty]$. Thus we have proved that if $a/b \in c_0$, then $\mathcal{I}\left(s, s^{(c)}\right) \subset s_b^{*(c)}$.

Conversely let $x \in s_b^{*(c)}$. We then have $s_b^{(c)} \subset s_x^{(c)}$ and trivially $s_b^{(c)} \subset s_x^{(c)} \subset s_a^0 + s_x^{(c)}$, for all $x \in U^+$. So we have $s_b^{*(c)} \subset \mathcal{I}\left(s, s^{(c)}\right)$ and we conclude that $\mathcal{I}\left(s, s^{(c)}\right) = s_b^{*(c)}$ if $a/b \in c_0$.

Now we need to determine $\mathcal{I}'\left(s, s^{(c)}\right)$. We see that $x \in \mathcal{I}'\left(s, s^{(c)}\right)$ if and only if $s_a \subset s_b^{(c)}$ and $s_x^{(c)} \subset s_b^{(c)}$, which is equivalent to $a/b \in M\left(\ell_\infty, c\right) = c_0$ and $x/b \in M\left(c, c\right) = c$. So

$$\mathcal{I}'\left(s, s^{(c)}\right) = \begin{cases} s_b^{(c)} & \text{if } a/b \in c_0, \\ \varnothing & \text{if } a/b \notin c_0. \end{cases}$$

We conclude $\mathcal{S}\left(s, s^{(c)}\right) = s_b^{*(c)} \cap s_b^{(c)} = cl^{(c)}\left(b\right)$ if $a/b \in c_0$. □

We immediately deduce the next corollary.

Corollary 4.5. *Let a, $b \in U^+$. Then*

i) a) $\mathcal{S}\left(s^0, s\right) = \mathcal{S}\left(s^{(c)}, s^0\right) = \mathcal{S}\left(s, s^0\right) = cl\left(b\right)$ for $a/b \in c_0$;

b) $\mathcal{S}\left(s^0, s\right) = \mathcal{S}\left(s^{(c)}, s^0\right) = \mathcal{S}\left(s, s^0\right) = \varnothing$ if $a/b \notin \ell_\infty$;

ii) a) $\mathcal{S}\left(s, s^{(c)}\right) = \mathcal{S}\left(s^0, s^{(c)}\right) = cl^{(c)}\left(b\right)$ for $a/b \in c_0$;

b) $\mathcal{S}\left(s, s^{(c)}\right) = \mathcal{S}\left(s^0, s^{(c)}\right) = \varnothing$ if $a/b \notin \ell_\infty$.

Remark 4.6. It can easily be seen that each of the equations $s_a^0 + s_x = s_b$, or $s_a^0 + s_x^{(c)} = s_b^{(c)}$ has a solution if and only if $a/b \in \ell_\infty$, and each of the equations $s_a^{(c)} + s_x^0 = s_b^0$, $s_a + s_x^0 = s_b^0$, or $s_a + s_x^{(c)} = s_b^{(c)}$ has a solution if and only if $a/b \in c_0$.

Remark 4.7. If anyone of the five equations has a solution, then $a/b \in \ell_\infty$.

Illustrate the previous results with the next example, where we put $cl\left(r_2\right) = cl\left(\left(r_2^n\right)_n\right)$ and $cl^{(c)}\left(r_2\right) = cl^{(c)}\left(\left(r_2^n\right)_n\right)$.

Example 4.8. Let r_1, $r_2 > 0$ and consider the next two statements

P_1: For every y, $z \in s$ the conditions $y_n/r_1^n \to l_1$ and $z_n/x_n \to l_2$ imply together $\left(y_n + z_n\right)/r_2^n \to l_3$ $(n \to \infty)$ for some scalars l_1, l_2 and l_3;

P_2: For every $t \in s$ we have $t_n/r_2^n \to 0$ $(n \to \infty)$ if and only if there are u and $v \in s$ such that $t = u + v$ and $u_n/r_1^n \to L$ and $v_n/x_n \to 0$ $(n \to \infty)$ for some scalar L.

The set S_1 of all $x \in U^+$ such that P_1 and P_2 hold is determined by the next system

$$s_{r_1}^{(c)} + s_x^{(c)} \subset s_{r_2}^{(c)} \tag{4.1}$$

and

$$s_{r_1}^{(c)} + s_x^0 = s_{r_2}^0. \tag{4.2}$$

It can easily be seen that (4.1) is equivalent to $r_1 \leq r_2$ and $x \in s_{r_2}^{(c)}$ and by Theorem 4.1 i) b) (SSE) (4.2) is equivalent to $x \in cl\left(r_2\right)$ if $\left(r_1/r_2\right)^n \to 0$ $(n \to \infty)$, that is for $r_1 < r_2$. We conclude that if $r_1 < r_2$, then the set S_1 is equal to $s_{r_2}^{(c)} \cap cl\left(r_2\right) = cl^{(c)}\left(r_2\right)$, so $x \in S_1$ if and only if $x_n \sim k r_2^n$ with $k > 0$. If $r_1 \geq r_2$, then $S_1 = \varnothing$.

5. Application to special (SSE) with operators

In this section we consider two systems of (SSE) with operators defined by $s_a^0 + s_x(\Delta) = s_b$, $s_x \supset s_b$ and $s_a + s_x^{(c)}(\Delta) = s_b^{(c)}$, $s_x^{(c)} \supset s_b^{(c)}$. Then we solve the (SSE) defined by $\chi_a(C(\lambda)D_\tau) + s_x^{(c)}(C(\mu)D_\tau) = s_b^{(c)}$ where χ is either s^0, or s.

5.1. **The sets $\widehat{C}$ and $\widehat{C_1}$.** In the following we need the next definitions and results. First recall that for any nonzero sequence $\eta = (\eta_n)_{n\geq 1}$, the triangle $C(\eta)$ is defined by

$$[C(\eta)]_{nm} = \begin{cases} \dfrac{1}{\eta_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

It can be shown that the matrix $\Delta(\eta)$ defined by

$$[\Delta(\eta)]_{nm} = \begin{cases} \eta_n & \text{if } m = n, \\ -\eta_{n-1} & \text{if } m = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise;} \end{cases}$$

is the inverse of $C(\eta)$, that is $C(\eta)(\Delta(\eta)\xi) = \Delta(\eta)(C(\eta)\xi)$ for all $\xi \in s$. It is well known that $\Delta = \Delta(e) \in (s,s)$ is the operator of first-difference and we have $\Delta\xi_n = \xi_n - \xi_{n-1}$ for all $n \geq 1$ with $\xi_0 = 0$. The inverse $\Delta^{-1} = \Sigma$ is defined by $\Sigma_{nm} = 1$ for $m \leq n$. We also use the sets

$$\widehat{C_1} = \left\{a \in U^+ : C(a)a \in \ell_\infty\right\} \text{ and } \widehat{C} = \left\{a \in U^+ : C(a)a \in c\right\}.$$

Note that if a, $b \in \widehat{C_1}$ then the sum $a+b$ and the product ab are in $\widehat{C_1}$. It can easily be seen that any sequence of the form $(R^n)_n$ with $R > 1$ belongs to $\widehat{C_1}$. It is known that $\widehat{C}$ which is equal to the set $\widehat{\Gamma}$ of all $x \in U^+$ such that $\lim_{n\to\infty}(x_{n-1}/x_n) < 1$. Here we use the next lemmas which are consequences of [5, Proposition 2.1, p. 1786], and [6] and of the fact that $s_a(\Delta) \subset s_a$ is equivalent to $\Sigma \in (s_a, s_a)$ and $D_{1/a}\Sigma D_a \in S_1$, which in turn is $a \in \widehat{C_1}$.

Lemma 5.1. *Let $a \in U^+$. Then*
i) The following statements are equivalent
a) $a \in \widehat{C_1}$,
b) $s_a(\Delta) \subset s_a$,
c) $s_a(\Delta) = s_a$.
ii) $s_a^{(c)}(\Delta) = s_a^{(c)}$ if and only if $a \in \widehat{\Gamma}$.

We also have the next elementary result.

Lemma 5.2. *Let a, $b \in U^+$ and assume $s_a = s_b$. Then $a \in \widehat{C_1}$ if and only if $b \in \widehat{C_1}$.*

5.2. **Solvability of two systems of (SSE) with operators defined by $s_a^0 +$ $s_x(\Delta) = s_b$, $s_x \supset s_b$ and $s_a + s_x^{(c)}(\Delta) = s_b^{(c)}$, $s_x^{(c)} \supset s_b^{(c)}$.** Now consider the next statement: what are the sequences $x \in U^{+}$ such that for every $y \in s$ we have $y_n/b_n = O(1)$ $(n \to \infty)$ if and only if there are u, $v \in s$ such that $y = u + v$ and

$$\frac{u_n}{a_n} \to 0,\ \frac{v_n - v_{n-1}}{x_n} = O(1) \quad (n \to \infty) \text{ and } x_n \geq K b_n \text{ for all } n\ ?$$

This statement is equivalent to the system

$$s_a^0 + s_x(\Delta) = s_b \text{ and } s_x \supset s_b. \tag{5.1}$$

We then have the following.

Proposition 5.3. *Let $\mathcal{S}_1$ be the set of all $x \in U^{+}$ such that (5.1) holds. Then*
i) if $b \notin \widehat{C}_1$ then $\mathcal{S}_1 = \varnothing$;
ii) if $b \in \widehat{C}_1$ then

$$\mathcal{S}_1 = \begin{cases} cl(b) & \text{if } a/b \in \ell_\infty, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. i) First show $\mathcal{S}_1 \neq \varnothing$ implies $b \in \widehat{C}_1$. Let $x \in \mathcal{S}_1$. Then by (5.1) we have $s_x(\Delta) \subset s_x$ and $x \in \widehat{C}_1$ by Lemma 5.1. Then

$$s_x = s_x(\Delta) \subset s_a^0 + s_x(\Delta) = s_b \subset s_x$$

and $s_b = s_x$. By Lemma 5.2 we conclude $b \in \widehat{C}_1$. So we have shown i).

ii) Let $b \in \widehat{C}_1$. Then

$$s_x(\Delta) \subset s_a^0 + s_x(\Delta) = s_b \subset s_x$$

and as we have just seen this implies $x \in \widehat{C}_1$ and $s_x = s_b$. Then by Lemma 5.1 we have $x \in \mathcal{S}_1$ if and only if $s_a^0 + s_x = s_b$ and we conclude by Theorem 4.4 i) a). □

Now consider the next question. What are the sequences $x \in U^{+}$ such that for every $y \in s$ we have $y_n/b_n \to L$ $(n \to \infty)$ if and only if there are u, $v \in s$ such that $y = u + v$, and

$$u_n/a_n = O(1),\ (v_n - v_{n-1})/x_n \to L'\,(n \to \infty) \text{ and } x_n/b_n \to L'' \in\,]0, \infty],$$

for some scalars L, L', and L'' ?

The answer to this question is given by the following proposition, which can be shown as in Proposition 5.3.

Proposition 5.4. *Let $\mathcal{S}$ be the set of all $x \in U^{+}$ such that*

$$s_a + s_x^{(c)}(\Delta) = s_b^{(c)} \text{ and } s_x^{(c)} \supset s_b^{(c)}.$$

We have

$$\mathcal{S} = \begin{cases} cl^{(c)}(b) & \text{if } a/b \in c_0 \text{ and } b \in \widehat{\Gamma}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Example 5.5. Let $r_1, r_2 > 0$. The system $s_{r_1} + s_x^{(c)}(\Delta) = s_{r_2}^{(c)}$ and $r_2^n/x_n \to l$ for some scalar l, has solutions if and only if $r_1 < r_2$ and $r_2 > 1$, and they are given by $x_n \sim k r_2^n$ $(n \to \infty)$ for some $k > 0$.

5.3. **On the (SSE) with operators $\chi_a(C(\lambda)D_\tau)+s_x^{(c)}(C(\mu)D_\tau)=s_b^{(c)}$ where χ is either s^0, or s.** Here let $\Phi\left(\chi,s^{(c)}\right)$ be the set of all $x\in U^+$ such that

$$\chi_a(C(\lambda)D_\tau)+s_x^{(c)}(C(\mu)D_\tau)=s_b^{(c)} \text{ where } \chi \text{ is either } s^0, \text{ or } s. \tag{5.2}$$

For $\chi=s^0$ the solvability of (SSE) (5.2) consists in determining all the sequences $x\in U^+$ such that the condition $y_n/b_n\to l$ $(n\to\infty)$ holds if and only if there are u, $v\in s$ such that $y=u+v$ and

$$\frac{\tau_1u_1+\cdots+\tau_nu_n}{\lambda_na_n}\to 0 \text{ and } \frac{\tau_1v_1+\cdots+\tau_nv_n}{\mu_nx_n}\to l' \ (n\to\infty) \text{ for all } y\in s,$$

and for some scalars l, l'.

To state the next theorem we need a lemma where we use the set

$$\mathcal{S}'(\chi)=\left\{x\in U^+:\chi_{a\lambda}+s_{\mu x}^{(c)}=s_{b\tau}^{(c)}\right\},$$

where χ is either s^0 or s. We have the following.

Lemma 5.6. *We have*

$$\Phi\left(\chi,s^{(c)}\right)=\begin{cases}\mathcal{S}'(\chi) & \text{if } b\tau\in\widehat{C},\\ \varnothing & \text{if } b\tau\notin\widehat{C_1}.\end{cases}$$

where χ is either s^0, or s.

Proof. Since $C^{-1}(\nu)=\Delta(\nu)$ for any nonzero sequence ν, we have (5.2) equivalent to

$$D_{1/\tau}\Delta(\lambda)\chi_a+D_{1/\tau}\Delta(\mu)s_x^{(c)}=D_{1/\tau}\Delta\chi_{a\lambda}+D_{1/\tau}\Delta s_{\mu x}^{(c)}=s_b^{(c)}$$

and to

$$\chi_{a\lambda}+s_{\mu x}^{(c)}=s_b^{(c)}\left(D_{1/\tau}\Delta\right)=s_{b\tau}^{(c)}(\Delta). \tag{5.3}$$

So if $b\tau\in\widehat{C}$, then we have $s_{b\tau}^{(c)}(\Delta)=s_{b\tau}^{(c)}$ and since (5.3) is equivalent to (5.2), we conclude $\Phi\left(\chi,s^{(c)}\right)=\mathcal{S}'(\chi)$.

It remains to show that $\Phi\left(\chi,s^{(c)}\right)\neq\varnothing$ implies $b\tau\in\widehat{C_1}$. For this let $\xi\in\Phi\left(\chi,s^{(c)}\right)$, that is $\chi_{a\lambda}+s_{\mu\xi}^{(c)}=s_{b\tau}^{(c)}(\Delta)$. First we have $s_{a\lambda}^0\subset\chi_{a\lambda}\subset s_{a\lambda}$ and $s_{\mu\xi}^{(c)}\subset s_{\mu\xi}$ which imply together

$$s_{a\lambda+\mu\xi}^0=s_{a\lambda}^0+s_{\mu\xi}^0\subset\chi_{a\lambda}+s_{\mu\xi}^{(c)}\subset s_{a\lambda}+s_{\mu\xi}=s_{a\lambda+\mu\xi}.$$

Then

$$s_{a\lambda+\mu\xi}^0\subset s_{b\tau}^{(c)}(\Delta)\subset s_{a\lambda+\mu\xi}. \tag{5.4}$$

The first inclusion gives $I\in\left(s_{a\lambda+\mu\xi}^0,s_{b\tau}^{(c)}(\Delta)\right)$ and $D_{1/b\tau}\Delta D_{a\lambda+\mu\xi}\in(c_0,c)$. Since $(c_0,c)\subset(c_0,s_1)=S_1$ we deduce

$$\frac{a_n\lambda_n+\mu_n\xi_n}{b_n\tau_n}\leq K \text{ for all } n \text{ and for some } K>0.$$

The second inclusion of (5.4) yields $\Delta^{-1}=\Sigma\in\left(s_{b\tau}^{(c)},s_{a\lambda+\mu\xi}\right)$, that is

$$D_{1/(a\lambda+\mu\xi)}\Sigma D_{b\tau}\in(c,\ell_\infty)=S_1$$

and

$$\frac{b_1\tau_1 + \cdots + b_n\tau_n}{a_n\lambda_n + \mu_n\xi_n} \leq K' \text{ for all } n \text{ and for some } K' > 0.$$

We deduce

$$\frac{b_1\tau_1 + \cdots + b_n\tau_n}{b_n\tau_n} = \frac{b_1\tau_1 + \cdots + b_n\tau_n}{a_n\lambda_n + \mu_n\xi_n}\frac{a_n\lambda_n + \mu_n\xi_n}{b_n\tau_n} \leq KK' \text{ for all } n.$$

We conclude $b\tau \in \widehat{C_1}$. This concludes the proof. □

As a direct consequence of Lemma 5.6 and Theorem 4.4 we obtain the next result.

Theorem 5.7. *Let a, b, λ, μ, $\tau \in U^+$. Then*

i) a) if $b\tau \notin \widehat{C}_1$ then $\Phi\left(s^0, s^{(c)}\right) = \varnothing$;

b) if $b\tau \in \widehat{C}$ then

$$\Phi\left(s^0, s^{(c)}\right) = \begin{cases} cl^{(c)}\left(b\tau/\mu\right) & \text{if } a\lambda/b\tau \in c_0, \\ \varnothing & \text{otherwise.} \end{cases}$$

ii) a) if $b\tau \notin \widehat{C}_1$ then $\Phi\left(s, s^{(c)}\right) = \varnothing$;

b) if $b\tau \in \widehat{C}$ then

$$\Phi\left(s, s^{(c)}\right) = \begin{cases} cl^{(c)}\left(b\tau/\mu\right) & \text{if } a\lambda/b\tau \in \ell_\infty, \\ \varnothing & \text{otherwise.} \end{cases}$$

We are led to state the next corollary where the (SSE) is totally solved.

Corollary 5.8. *Let a, λ, $\mu \in U^+$ and $R > 0$. Let $\Phi_R\left(\chi, s^{(c)}\right)$ be the set of the solutions of the equation*

$$\chi_a\left(C\left(\lambda\right)\right) + s_x^{(c)}\left(C\left(\mu\right)\right) = s_R^{(c)},$$

where χ is either s^0, or s. We have

i) a) if $R \leq 1$ then $\Phi_R\left(s^0, s^{(c)}\right) = \varnothing$;

b) if $R > 1$, then

$$\Phi_R\left(s^0, s^{(c)}\right) = \begin{cases} cl^{(c)}\left(\left(R^n/\mu_n\right)_n\right) & \text{if } a_n\lambda_n/R^n \to 0 \ \ (n \to \infty), \\ \varnothing & \text{otherwise.} \end{cases}$$

ii) a) if $R \leq 1$ then $\Phi_R\left(s, s^{(c)}\right) = \varnothing$;

b) if $R > 1$, then

$$\Phi_R\left(s, s^{(c)}\right) = \begin{cases} cl^{(c)}\left(\left(R^n/\mu_n\right)_n\right) & \text{if } \left(a_n\lambda_n/R^n\right)_{n\geq 1} \in s_1, \\ \varnothing & \text{otherwise.} \end{cases}$$

Proof. The proof is a direct consequence of Theorem 5.7. Indeed, if $R \leq 1$, then $\left(R^n\right)_{n\geq 1} \notin \widehat{C_1}$. Since $\widehat{C} = \widehat{\Gamma}$ and $\lim_{n\to\infty}\left(R^{n-1}/R^n\right) = 1/R < 1$ we deduce that if $R > 1$, then $\left(R^n\right)_{n\geq 1} \in \widehat{C}$. □

Acknowledgement. The work of the second author is supported by Grant No. 174025 of the Ministry of Science, Technology and Development, Republic of Serbia.

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