SHARP MAXIMAL FUNCTION INEQUALITIES AND BOUNDEDNESS OF TOEPLITZ TYPE OPERATOR RELATED TO GENERAL FRACTIONAL INTEGRAL OPERATORS

LANZHE LIU

Communicated by L. P. Castro

ABSTRACT. In this paper, we establish sharp maximal function inequalities for the Toeplitz type operator related to some general fractional integral operators. As an application, we obtain the boundedness of the operator on Lebesgue, Morrey and Triebel-Lizorkin spaces. The operators include Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

1. Introduction and definitions

As the development of singular integral operators (see [6],[22]), their commutators have been well studied. In [3],[20],[21], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [7],[17], the boundedness of the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(\mathbb{R}^n)(1 < p < \infty)$ spaces is obtained. In [1], some singular integral operators with general kernel are introduced, and the boundedness of the operators and their commutators generated by BMO and Lipschitz functions is obtained (see [1],[10]). In [8],[9], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness of the operators

generated by BMO and Lipschitz functions is obtained. In this paper, we will study the Toeplitz type operator generated by some general fractional integral operators with the Lipschitz and BMO functions.

First, let us introduce some notations. Throughout this paper, C denotes a positive constant, which is not necessarily the same at each occurrence, Q denotes a cube of $\mathbb{R}^n$ with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [6],[22])

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $BMO(\mathbb{R}^n)$ if $M^{\#}(f)$ belongs to $L^\infty(\mathbb{R}^n)$ and define $||f||_{BMO} = ||M^{\#}(f)||_{L^\infty}$. It has been known that (see [22])

$$||f - f_{2^k Q}||_{BMO} \le Ck||f||_{BMO}.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$ and $1 \le r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

The A_p weight is defined by (see [6]), for $1 < p < \infty$,

$$A_p = \left\{ w \in L^1_{loc}(\mathbb{R}^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x) dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}$$

and

$$A_1 = \{w \in L^p_{loc}(\mathbb{R}^n) : M(w)(x) \le Cw(x), a.e.\}.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(\mathbb{R}^n)$ be the homogeneous Triebel-Lizorkin space (see [17]).

For $\beta > 0$, the Lipschitz space $\mathrm{Lip}_\beta(\mathbb{R}^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y \in \mathbb{R}^n \\ x \ne y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Definition 1.1. Let φ be a positive, increasing function on R^+ for which there exists a constant $D > 0$ such that

$$\varphi(2t) \le D\varphi(t) \text{ for } t \ge 0.$$

Let f be a locally integrable function on $\mathbb{R}^n$. Set, for $1 \le p < \infty$,

$$||f||_{L^{p,\varphi}} = \sup_{x \in \mathbb{R}^n, \ d>0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x,d)$ denotes a cube of $\mathbb{R}^n$ with sides parallel to the axes, whose center is x and side length is d. The generalized Morrey space is defined by

$$L^{p,\varphi}(\mathbb{R}^n) = \{f \in L^1_{loc}(\mathbb{R}^n) : ||f||_{L^{p,\varphi}} < \infty\}.$$

If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(\mathbb{R}^n) = L^{p,\delta}(\mathbb{R}^n)$, which is the classical Morrey spaces (see [18],[19]). If $\varphi(d) = 1$, then $L^{p,\varphi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, which is the Lebesgue spaces (see [6]).

Since the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [4],[5],[11],[16]).

In this paper, we will study some singular integral operators as following (see [1]).

Definition 1.2. Let $F_t(x,y)$ be defined on $\mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty)$ and b be a locally integrable function on $\mathbb{R}^n$. Set

$$F_t(f)(x) = \int_{\mathbb{R}^n} F_t(x,y)f(y)dy$$

for every bounded and compactly supported function f. And F_t satisfies: there is a sequence of positive constant numbers $\{C_j\}$ such that for any $j \ge 1$,

$$\int_{2|y-z|<|x-y|} (||F_t(x,y) - F_t(x,z)|| + ||F_t(y,x) - F_t(z,x)||)dx \le C,$$

and

$$\left(\int_{2^j|z-y| \le |x-y| < 2^{j+1}|z-y|} (||F_t(x,y) - F_t(x,z)|| + ||F_t(y,x) - F_t(z,x)||)^q dy \right)^{1/q}$$
$$\le \ C_j(2^j|z-y|)^{-n/q'},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$.

Let H be the Banach space $H = \{h : ||h|| < \infty\}$. For each fixed $x \in \mathbb{R}^n$, it is view $F_t(f)(x)$ as the mapping from $[0, +\infty)$ to H. Set

$$T(f)(x) = ||F_t(f)(x)||,$$

which T is bounded on $L^2(\mathbb{R}^n)$. Let b be a locally integrable function on $\mathbb{R}^n$. The Toeplitz type operator related to T is defined by

$$T^b(f) = ||F_t^b(f)||,$$

where

$$F_t^b(f) = \sum_{k=1}^m (F_t^{k,1} M_b I_\alpha F_t^{k,2} + F_t^{k,3} I_\alpha M_b F_t^{k,4}),$$

moreover, $F_t^{k,1}(f)$ are $F_t(f)$ or $\pm I$(the identity operator), $F_t^{k,2}(f)$, $F_t^{k,3}(f)$ and $F_t^{k,4}(f)$ are the linear operators with $T^{k,2}(f) = ||F_t^{k,2}(f)||$, $T^{k,4}(f) = ||F_t^{k,4}(f)||$

and $T^{k,3} = \pm I$, $k = 1, \cdots, m$, $M_b(f) = bf$ and I_α is the fractional integral operator($0 < \alpha < n$) (see [2]).

Note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operators T^b. The Toeplitz type operators T^b are the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [20],[21]). The main purpose of this paper is to prove sharp maximal inequalities for the Toeplitz type operators T^b. As the application, we obtain the the L^p-norm inequality and the boundedness of T^b on Triebel-Lizorkin spaces. The operators include Littlewood-Paley operators, Marcinkiewicz operators and Bochner-Riesz operator.

2. Some Preliminary Lemmas

We begin with some preliminary lemmas.

Lemma 2.1. *(see [1]) Let T be the integral operator as Definition 1.2. Then T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

Lemma 2.2. *(see [17]). For $0 < \beta < 1$ and $1 < p < \infty$, we have*

$$||f||_{\dot{F}_p^{\beta,\infty}} \approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p}$$
$$\approx \left\| \sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right\|_{L^p}.$$

Lemma 2.3. *(see [6]). Let $0 < p < \infty$ and $w \in \cup_{1 \le r < \infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{\mathbb{R}^n} M(f)(x)^p w(x) dx \le C \int_{\mathbb{R}^n} M^\#(f)(x)^p w(x) dx.$$

Lemma 2.4. *(see [2],[6]). Suppose that $0 < \alpha < n$, $1 \le s < p < n/\alpha$ and $1/r = 1/p - \alpha/n$. Then*

$$||I_\alpha(f)||_{L^r} \le C||f||_{L^p}$$

and

$$||M_{\alpha,s}(f)||_{L^r} \le C||f||_{L^p}.$$

Lemma 2.5. *Let $1 < p < \infty$, $0 < D < 2^n$. Then, for any smooth function f for which the left-hand side is finite,*

$$||M(f)||_{L^{p,\varphi}} \le C||M^\#(f)||_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in $\mathbb{R}^n$, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [6]. Noticing that $M(\chi_Q) \le 1$ and $M(\chi_Q)(x) \le d^n/(|x - x_0| - d)^n$

if $x \in Q^c$. By Lemma 2.3, we have, for $f \in L^{p,\varphi}(\mathbb{R}^n)$,

$$\int_Q M(f)(x)^p dx = \int_{R^n} M(f)(x)^p \chi_Q(x) dx$$

$$\leq \int_{\mathbb{R}^n} M(f)(x)^p M(\chi_Q)(x) dx \leq C \int_{\mathbb{R}^n} M^{\#}(f)(x)|^p M(\chi_Q)(x) dx$$

$$= C \left(\int_Q M^{\#}(f)(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^{\#}(f)(x)^p M(\chi_Q)(x) dx \right)$$

$$\leq C \left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} M^{\#}(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx \right)$$

$$\leq C \left(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^{\#}(f)(x)^p 2^{-kn} dy \right)$$

$$\leq C ||M^{\#}(f)||_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} 2^{-kn} \varphi(2^{k+1}d)$$

$$\leq C ||M^{\#}(f)||_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d)$$

$$\leq C ||M^{\#}(f)||_{L^{p,\varphi}}^p \varphi(d),$$

thus

$$\left(\frac{1}{\varphi(d)} \int_Q M(f)(x)^p dx \right)^{1/p} \leq C \left(\frac{1}{\varphi(d)} \int_Q M^{\#}(f)(x)^p dx \right)^{1/p}$$

and

$$||M(f)||_{L^{p,\varphi}} \leq C ||M^{\#}(f)||_{L^{p,\varphi}}.$$

This finishes the proof.   □

Lemma 2.6. *Let* $0 < \alpha < n$, $0 < D < 2^n$, $1 \leq s < p < n/\alpha$ *and* $1/r = 1/p - \alpha/n$. *Then*

$$||I_\alpha(f)||_{L^{r,\varphi}} \leq C ||f||_{L^{p,\varphi}}$$

and

$$||M_{\alpha,s}(f)||_{L^{r,\varphi}} \leq C ||f||_{L^{p,\varphi}}.$$

The proof of the Lemma is similar to that of Lemma 2.5 by Lemma 2.4, we omit the details.

3. Theorems And Proofs

Now we are in the position to prove the following theorems. Suppose that T is the integral operator as Definition 1.2 in the following theorems.

Theorem 3.1. *Let* $0 < \beta < 1$, $q' \le s < \infty$, $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ *and the sequence* $\{C_j\} \in l^1$. *If* $F_t^1(g) = 0$ *for any* $g \in L^u(\mathbb{R}^n)(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that for any* $f \in C_0^\infty(\mathbb{R}^n)$ *and* $\tilde{x} \in \mathbb{R}^n$,

$$M^\#(T^b(f))(\tilde{x}) \le C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^m (M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Proof. It suffices to prove that for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T^b(f)(x) - C_0|\, dx \le C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^m (M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k=1,\cdots,m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$
\begin{aligned}
F_t^b(f)(x) &= \sum_{k=1}^m F_t^{k,1} M_b I_\alpha F_t^{k,2}(f)(x) + \sum_{k=1}^m F_t^{k,3} I_\alpha M_b F_t^{k,4}(f)(x) = A_b(x) + B_b(x) \\
&= A_{b-b_Q}(x) + B_{b-b_Q}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{b-b_Q}(x) &= \sum_{k=1}^m F_t^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha F_t^{k,2}(f)(x) + \sum_{k=1}^m F_t^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} I_\alpha F_t^{k,2}(f)(x) \\
&= A_1(x) + A_2(x)
\end{aligned}
$$

and

$$
\begin{aligned}
B_{b-b_Q}(x) &= \sum_{k=1}^m F_t^{k,3} I_\alpha M_{(b-b_Q)\chi_{2Q}} F_t^{k,4}(f)(x) + \sum_{k=1}^m F_t^{k,3} I_\alpha M_{(b-b_Q)\chi_{(2Q)^c}} F_t^{k,4}(f)(x) \\
&= B_1(x) + B_2(x).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{|Q|} \int_Q |A_{b-b_Q}(f)(x) - A_2(x_0)|\, dx \\
&\le \frac{1}{|Q|} \int_Q |A_1(x)|\, dx + \frac{1}{|Q|} \int_Q |A_2(x) - A_2(x_0)|\, dx = I_1 + I_2
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{|Q|} \int_Q |B_{b-b_Q}(f)(x) - B_2(x_0)|\, dx \\
&\le \frac{1}{|Q|} \int_Q |B_1(x)|\, dx + \frac{1}{|Q|} \int_Q |B_2(x) - B_2(x_0)|\, dx = I_3 + I_4.
\end{aligned}
$$

For I_1, by Hölder's inequality and Lemma 2.1, we obtain, for $1/r = 1/s - \alpha/n$,

$$
\frac{1}{|Q|} \int_Q ||F_t^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha F_t^{k,2}(f)(x)|| dx
$$

$$
= \frac{1}{|Q|} \int_Q |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)| dx
$$

$$
\leq \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^s dx \right)^{1/s}
$$

$$
\leq C|Q|^{-1/s} \left(\int_{\mathbb{R}^n} |M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^s dx \right)^{1/s}
$$

$$
\leq C|Q|^{-1/s} \left(\int_{2Q} (|b(x) - b_Q| |I_\alpha T^{k,2}(f)(x)|)^s dx \right)^{1/s}
$$

$$
\leq C|Q|^{-1/s} ||b||_{\mathrm{Lip}_\beta} |2Q|^{\beta/n} |2Q|^{1/s - \beta/n} \left(\frac{1}{|2Q|^{1-s\beta/n}} \int_{2Q} |I_\alpha T^{k,2}(f)(x)|^s dx \right)^{1/s}
$$

$$
\leq C||b||_{\mathrm{Lip}_\beta} M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}),
$$

thus

$$
I_1 \leq \sum_{k=1}^m \frac{1}{|Q|} \int_{\mathbb{R}^n} ||F_t^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha F_t^{k,2}(f)(x)|| dx \leq C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^m M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}).
$$

For I_2, by the boundedness of T and recalling that $s > q'$, we get, for $x \in Q$,

$$
||F_t^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} I_\alpha F_t^{k,2}(f)(x) - F_t^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} I_\alpha F_t^{k,2}(f)(x_0)||
$$

$$
\leq \int_{(2Q)^c} |b(y) - b_{2Q}| ||F_t(x,y) - F_t(x_0,y)|| |I_\alpha T^{k,2}(f)(y)| dy
$$

$$
= \sum_{j=1}^\infty \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |b(y) - b_{2Q}| ||F_t(x,y) - F_t(x_0,y)|| |I_\alpha T^{k,2}(f)(y)| dy
$$

$$
\leq C||b||_{\mathrm{Lip}_\beta} \sum_{j=1}^\infty |2^{j+1}Q|^{\beta/n} \left(\int_{2^j d \leq |y-x_0| < 2^{j+1} d} ||F_t(x,y) - F_t(x_0,y)||^q dy \right)^{1/q}
$$

$$
\times \left(\int_{2^{j+1}Q} |I_\alpha T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'}
$$

$$
\leq C||b||_{\mathrm{Lip}_\beta} \sum_{j=1}^\infty |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'} |2^{j+1}Q|^{1/q' - \beta/n}
$$

$$
\times \left(\frac{1}{|2^{j+1}Q|^{1-s\beta/n}} \int_{2^{j+1}Q} |I_\alpha T^{k,2}(f)(y)|^s dy \right)^{1/s}
$$

$$
\leq C||b||_{\mathrm{Lip}_\beta} M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) \sum_{j=1}^\infty C_j
$$

$$
\leq C||b||_{\mathrm{Lip}_\beta} M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}),
$$

thus

$$
\begin{aligned}
I_2 \ \leq \ & \frac{1}{|Q|}\int_Q \sum_{k=1}^{m} ||F_t^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha F_t^{k,2}(f)(x) \\
& -F_t^{k,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha F_t^{k,2}(f)(x_0)||dx \\
\leq \ & C||b||_{\mathrm{Lip}_\beta}\sum_{k=1}^{m} M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
I_3 \leq & \sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{\mathbb{R}^n}||I_\alpha M_{(b-b_Q)\chi_{2Q}}F_t^{k,4}(f)(x)||^r dx\right)^{1/r} \\
\leq & C\sum_{k=1}^{m}|Q|^{-1/r}\left(\int_{2Q}(|b(x)-b_Q||T^{k,4}(f)(x)|)^s dx\right)^{1/s} \\
\leq & C||b||_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}|Q|^{-1/r}|2Q|^{\beta/n}|2Q|^{1/s-(\beta+\alpha)/n} \\
& \times\left(\frac{1}{|2Q|^{1-s(\beta+\alpha)/n}}\int_{2Q}|T^{k,4}(f)(x)|^s dx\right)^{1/s} \\
\leq & C||b||_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}M_{\beta+\alpha,s}(T^{k,6}(f))(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
I_4 \leq & \sum_{k=1}^{m}\frac{1}{|Q|}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right|||F_t^{k,4}(f)(y)||dydx \\
\leq & C\sum_{k=1}^{m}\sum_{j=1}^{\infty}||b||_{\mathrm{Lip}_\beta}|2^{j+1}Q|^{\beta/n}\int_{2^j d\leq|y-x_0|<2^{j+1}d}\frac{d}{|x_0-y|^{n-\alpha+1}}|T^{k,4}(f)(y)|dy \\
\leq & C||b||_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}\sum_{j=1}^{\infty}(2^j d)^\beta d(2^j d)^{-n+\alpha-1}(2^j d)^{n(1-1/s)}(2^j d)^{n/s-\beta-\alpha} \\
& \times\left(\frac{1}{|2^{j+1}Q|^{1-s(\beta+\alpha)/n}}\int_{2^{j+1}Q}|T^{k,4}(f)(y)|^s dy\right)^{1/s} \\
\leq & C||b||_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})\sum_{j=1}^{\infty}2^{-j} \\
\leq & C||b||_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 3.1. $\qquad\qquad\square$

Theorem 3.2. *Let* $0 < \beta < 1$, $q' \leq s < \infty$, $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ *and the sequence* $\{2^{j\beta}C_j\} \in l^1$. *If* $F_t^1(g) = 0$ *for any* $g \in L^u(\mathbb{R}^n)(1 < u < \infty)$, *then there exists a*

constant $C > 0$ such that for any $f \in C_0^\infty(\mathbb{R}^n)$ and $\tilde{x} \in \mathbb{R}^n$,

$$\sup_{Q \ni \tilde{x}} \inf_{c \in \mathbb{C}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T^b(f)(x) - c|\, dx \leq C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^m (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Proof. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0, the following inequality holds:

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T^b(f)(x) - C_0|\, dx \leq C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^m (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, \cdots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 3.1, we have, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q |T^b(f)(x) - A_2(x_0) - B_2(x_0)|\, dx$$

$$\leq \quad \frac{1}{|Q|^{1+\beta/n}} \int_Q |A_1(x)|dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |A_2(x) - A_2(x_0)|dx$$

$$+ \quad \frac{1}{|Q|^{1+\beta/n}} \int_Q |B_1(x)|dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |B_2(x) - B_2(x_0)|dx$$

$$= \quad I_5 + I_6 + I_7 + I_8.$$

By using the same argument as in the proof of Theorem 3.1, we get, for $1/r = 1/s - \alpha/n$,

$$I_5 \leq |Q|^{-\beta/n} \sum_{k=1}^m \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^s dx \right)^{1/s}$$

$$\leq C|Q|^{-\beta/n} \sum_{k=1}^m |Q|^{-1/s} \left(\int_{2Q} (|b(x) - b_Q||I_\alpha T^{k,2}(f)(x)|)^s dx \right)^{1/s}$$

$$\leq C|Q|^{-\beta/n} \sum_{k=1}^m |Q|^{-1/s} ||b||_{\mathrm{Lip}_\beta} |2Q|^{\beta/n} |Q|^{1/s} \left(\frac{1}{|Q|} \int_{2Q} |I_\alpha T^{k,2}(f)(x)|^s dx \right)^{1/s}$$

$$\leq C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^m M_s(I_\alpha T^{k,2}(f))(\tilde{x}),$$

$$I_6 \leq |Q|^{-\beta/n} \sum_{k=1}^{m} \frac{1}{|Q|} \int_Q \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1}d} |b(y) - b_{2Q}|$$

$$\times ||F_t(x,y) - F_t(x_0,y)|| |I_\alpha T^{k,2}(f)(y)| dy dx$$

$$\leq |Q|^{-\beta/n} \sum_{k=1}^{m} \frac{C}{|Q|} \int_Q \sum_{j=1}^{\infty} ||b||_{\mathrm{Lip}_\beta} |2^{j+1}Q|^{\beta/n} \left(\int_{2^{j+1}Q} |I_\alpha T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'}$$

$$\times \left(\int_{2^j d \leq |y-x_0| < 2^{j+1}d} ||F_t(x,y) - F_t(x_0,y)||^q dy \right)^{1/q} dx$$

$$\leq C ||b||_{\mathrm{Lip}_\beta} |Q|^{-\beta/n} \sum_{k=1}^{m} \sum_{j=1}^{\infty} |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'} |2^{j+1}Q|^{1/q'}$$

$$\times \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |I_\alpha T^{k,2}(f)(y)|^s dy \right)^{1/s}$$

$$\leq C ||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} M_s(I_\alpha T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} 2^{j\beta} C_j$$

$$\leq C ||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} M_s(I_\alpha T^{k,2}(f))(\tilde{x}),$$

$$I_7 \leq |Q|^{-\beta/n} \sum_{k=1}^{m} \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |I_\alpha M_{(b-b_Q)\chi_{2Q}} T^{k,4}(f)(x)|^r dx \right)^{1/r}$$

$$\leq C |Q|^{-\beta/n-1/r} \sum_{k=1}^{m} \left(\int_{2Q} (|b(x) - b_Q| |T^{k,4}(f)(x)|)^s dx \right)^{1/s}$$

$$\leq C ||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} |Q|^{-\beta/n-1/r} |2Q|^{\beta/n} |Q|^{1/s-\alpha/n} \left(\frac{1}{|Q|^{1-s\alpha/n}} \int_{2Q} |T^{k,4}(f)(x)|^s dx \right)^{\frac{1}{s}}$$

$$\leq C ||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}),$$

$$I_8 \leq |Q|^{-\beta/n-1} \sum_{k=1}^{m} \int_Q \int_{(2Q)^c} |b(y) - b_{2Q}| \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}} \right| |T^{k,4}(f)(y)| dy dx$$

$$\leq C |Q|^{-\beta/n} \sum_{k=1}^{m} \sum_{j=1}^{\infty} ||b||_{\mathrm{Lip}_\beta} |2^{j+1}Q|^{\beta/n}$$

$$\times \int_{2^j d \leq |y-x_0| < 2^{j+1}d} \frac{d}{|x_0-y|^{n-\alpha+1}} |T^{k,4}(f)(y)| dy$$

$$\leq C ||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} \sum_{j=1}^{\infty} d^{-\beta} (2^j d)^\beta d (2^j d)^{-n+\alpha-1} (2^j d)^{n(1-1/s)} (2^j d)^{n/s-\alpha}$$

$$\times \left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}} \int_{2^{j+1}Q} |T^{k,4}(f)(y)|^s dy \right)^{1/s}$$

$$\leq C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}) \sum_{j=1}^{\infty} 2^{j(\beta-1)} \leq C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}).$$

This completes the proof of Theorem 3.2. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.3. *Let $q' \leq s < \infty$, $b \in BMO(\mathbb{R}^n)$ and the sequence $\{jC_j\} \in l^1$. If $F_t^1(g) = 0$ for any $g \in L^u(\mathbb{R}^n)(1 < u < \infty)$, then there exists a constant $C > 0$ such that for any $f \in C_0^\infty(\mathbb{R}^n)$ and $\tilde{x} \in \mathbb{R}^n$,*

$$M^\#(T^b(f))(\tilde{x}) \leq C||b||_{BMO} \sum_{k=1}^{m} (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Proof. It suffices to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|} \int_Q |T^b(f)(x) - C_0| \, dx \leq C||b||_{BMO} \sum_{k=1}^{m} (M_s(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{k,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{k,1}$ are $T(k = 1, \cdots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 3.1, we have, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$\frac{1}{|Q|} \int_Q |T^b(f)(x) - A_2(x_0) - B_2(x_0)| \, dx$$

$$\leq \quad \frac{1}{|Q|} \int_Q |A_1(x)| dx + \frac{1}{|Q|} \int_Q |A_2(x) - A_2(x_0)| dx$$

$$+ \quad \frac{1}{|Q|} \int_Q |B_1(x)| dx + \frac{1}{|Q|} \int_Q |B_2(x) - B_2(x_0)| dx = I_9 + I_{10} + I_{11} + I_{12}.$$

By using the same argument as in the proof of Theorem 3.1, we get, for $1 < r_1 < s$, $1 < p < \infty, 1 < r_2 < s$ with $1/p + 1/q + 1/r_2 = 1$, $1/r_3 = 1/p - \alpha/n$ with $1 < p < s$,

$$I_9 \leq \sum_{k=1}^{m} \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |T^{k,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{k,2}(f)(x)|^{r_1} dx \right)^{1/r_1}$$

$$\leq C \sum_{k=1}^{m} |Q|^{-1/r} \left(\int_{2Q} (|b(x) - b_Q||I_\alpha T^{k,2}(f)(x)|)^{r_1} dx \right)^{1/r_1}$$

$$\leq C \sum_{k=1}^{m} \left(\frac{1}{|Q|} \int_{2Q} |I_\alpha T^{k,2}(f)(x)|^s dx \right)^{1/s} \left(\frac{1}{|Q|} \int_{2Q} |b(x) - b_Q|^{sr_1/(s-r_1)} dx \right)^{\frac{s-r_1}{sr_1}}$$

$$\leq C||b||_{BMO} \sum_{k=1}^{m} M_s(I_\alpha T^{k,2}(f))(\tilde{x}),$$

$$
\begin{aligned}
I_{10} \quad \leq \quad & \sum_{k=1}^{m} \frac{1}{|Q|} \int_{Q} \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1}d} |b(y) - b_{2Q}| \\
\times \quad & ||F_t(x,y) - F_t(x_0,y)|||I_\alpha T^{k,2}(f)(y)| dy dx \\
\leq \quad & \sum_{k=1}^{m} \frac{C}{|Q|} \int_{Q} \sum_{j=1}^{\infty} \left(\int_{2^j d \leq |y-x_0| < 2^{j+1}d} ||F_t(x,y) - F_t(x_0,y)||^q dy \right)^{1/q} \\
\times \quad & \left(\int_{2^{j+1}Q} |b(y) - b_Q|^p dy \right)^{1/p} \left(\int_{2^{j+1}Q} |I_\alpha T^{k,2}(f)(y)|^{r_2} dy \right)^{1/r_2} dx \\
\leq \quad & C||b||_{BMO} \sum_{k=1}^{m} \sum_{j=1}^{\infty} C_j (2^j d)^{-n/q'} j (2^j d)^{n/p} (2^j d)^{n/s} \\
\times \quad & \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |I_\alpha T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
\leq \quad & C||b||_{BMO} \sum_{k=1}^{m} M_s(I_\alpha T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} j C_j \\
\leq \quad & C||b||_{BMO} \sum_{k=1}^{m} M_s(I_\alpha T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
I_{11} \quad \leq \quad & \sum_{k=1}^{m} \left(\frac{1}{|Q|} \int_{\mathbb{R}^n} |I_\alpha M_{(b-b_Q)\chi_{2Q}} T^{k,4}(f)(x)|^{r_3} dx \right)^{1/r_3} \\
\leq \quad & C|Q|^{-1/r_3} \sum_{k=1}^{m} \left(\int_{2Q} (|b(x) - b_Q||T^{k,4}(f)(x)|)^p dx \right)^{1/p} \\
\leq \quad & C \sum_{k=1}^{m} \left(\frac{1}{|Q|} \int_{2Q} |b(x) - b_Q|^{ps/(s-p)} dx \right)^{(s-p)/ps} \\
\times \quad & \left(\frac{1}{|Q|^{1-s\alpha/n}} \int_{2Q} |T^{k,4}(f)(x)|^s dx \right)^{1/s} \\
\leq \quad & C||b||_{BMO} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}),
\end{aligned}
$$

$$
\begin{aligned}
I_{12} \quad \leq \quad & |Q|^{-1} \sum_{k=1}^{m} \int_{Q} \int_{(2Q)^c} |b(y) - b_{2Q}| \\
\times \quad & \left| \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}} \right| |T^{k,4}(f)(y)| dy dx \\
\leq \quad & C \sum_{k=1}^{m} \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1}d} |b(y) - b_{2Q}| \frac{d}{|x_0-y|^{n-\alpha+1}} |T^{k,4}(f)(y)| dy
\end{aligned}
$$

$$\leq \quad C \sum_{k=1}^{m} \sum_{j=1}^{\infty} \frac{d}{(2^j d)^{n-\alpha+1}} (2^j d)^{n(1-1/s)} \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |b(y) - b_Q|^{s'} dy \right)^{1/s'}$$

$$\times \quad (2^j d)^{n/s-\alpha} \left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}} \int_{2^{j+1}Q} |T^{k,4}(f)(y)|^s dy \right)^{1/s}$$

$$\leq \quad C||b||_{BMO} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}) \sum_{j=1}^{\infty} j2^{-j}$$

$$\leq \quad C||b||_{BMO} \sum_{k=1}^{m} M_{\alpha,s}(T^{k,4}(f))(\tilde{x}).$$

This completes the proof of Theorem 3.3 □

Theorem 3.4. *Let* $0 < \beta < \min(1, n/q')$, $q' < p < n/(\alpha + \beta)$, $1/r = 1/p - (\alpha + \beta)/n$, $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ *and the sequence* $\{C_j\} \in l^1$. *If* $F_t^1(g) = 0$ *for any* $g \in L^u(\mathbb{R}^n)(1 < u < \infty)$ *and* $T^{k,4}(f) = ||F_t^{k,4}(f)||$ *are the bounded operators on* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty (1 \leq k \leq m)$, *then* T^b *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^r(\mathbb{R}^n)$.

Proof. Choose $q' < s < p$ in Theorem 3.1 and set $1/v = 1/p - \beta/n$. We have, by Lemmas 2.3 and 2.4,

$$||T^b(f)||_{L^r} \leq ||M(T^b(f))||_{L^r} \leq C||M^{\#}(T^b(f))||_{L^r}$$

$$\leq \quad C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} (||M_{\beta,s}(I_\alpha T^{k,2}(f))||_{L^r} + ||M_{\beta+\alpha,s}(T^{k,4}(f))||_{L^r})$$

$$\leq \quad C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} (||I_\alpha T^{k,2}(f)||_{L^v} + ||T^{k,4}(f)||_{L^p})$$

$$\leq \quad C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} (||T^{k,2}(f)||_{L^p} + ||f||_{L^p})$$

$$\leq \quad C||b||_{\mathrm{Lip}_\beta}||f||_{L^p}.$$

This completes the proof of the theorem. □

Theorem 3.5. *Let* $0 < \beta < \min(1, n/q')$, $q' < p < n/(\alpha + \beta)$, $1/r = 1/p - (\alpha + \beta)/n$, $0 < D < 2^n$, $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ *and the sequence* $\{C_j\} \in l^1$. *If* $F_t^1(g) = 0$ *for any* $g \in L^u(\mathbb{R}^n)(1 < u < \infty)$ *and* $T^{k,4}(f) = ||F_t^{k,4}(f)||$ *are the bounded operators on* $L^{p,\varphi}(\mathbb{R}^n)$ *for* $1 < p < \infty (1 \leq k \leq m)$, *then* T^b *is bounded from* $L^{p,\varphi}(\mathbb{R}^n)$ *to* $L^{r,\varphi}(\mathbb{R}^n)$.

Proof. Choose $q' < s < p$ in Theorem 3.1 and set $1/v = 1/p - \beta/n$. We have, by Lemmas 2.5 and 2.6,

$$||T^b(f)||_{L^{r,\varphi}} \leq ||M(T^b(f))||_{L^{r,\varphi}} \leq C||M^{\#}(T^b(f))||_{L^{r,\varphi}}$$

$$\leq \; C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} (||M_{\beta,s}(I_\alpha T^{k,2}(f))||_{L^{r,\varphi}} + ||M_{\beta+\alpha,s}(T^{k,4}(f))||_{L^{r,\varphi}})$$

$$\leq \; C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} (||I_\alpha T^{k,2}(f)||_{L^{v,\varphi}} + ||T^{k,4}(f)||_{L^{p,\varphi}})$$

$$\leq \; C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} (||T^{k,2}(f)||_{L^{p,\varphi}} + ||f||_{L^{p,\varphi}})$$

$$\leq \; C||b||_{\mathrm{Lip}_\beta} ||f||_{L^{p,\varphi}}.$$

This completes the proof of the theorem. $\qquad\square$

Theorem 3.6. *Let* $0 < \beta < \min(1, n/q')$, $q' < p < n/\alpha$, $1/r = 1/p - \alpha/n$, $b \in \mathrm{Lip}_\beta(\mathbb{R}^n)$ *and the sequence* $\{2^{j\beta} C_j\} \in l^1$. *If* $F_t^1(g) = 0$ *for any* $g \in L^u(\mathbb{R}^n)(1 < u < \infty)$ *and* $T^{k,4}(f) = ||F_t^{k,4}(f)||$ *are the bounded operators on* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty(1 \leq k \leq m)$, *then* T^b *is bounded from* $L^p(\mathbb{R}^n)$ *to* $\dot{F}_r^{\beta,\infty}(\mathbb{R}^n)$.

Proof. Choose $q' < s < p$ in Theorem 3.2. We have, by Lemmas 2.2, 2.3 and 2.4,

$$||T^b(f)||_{\dot{F}_r^{\beta,\infty}} \leq C \left|\left| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |T^b(f)(x) - C_0| \, dx \right|\right|_{L^r}$$

$$\leq \; C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} (||M_s(I_\alpha T^{k,2}(f))||_{L^r} + ||M_{\alpha,s}(T^{k,4}(f))||_{L^r})$$

$$\leq \; C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} (||I_\alpha T^{k,2}(f)||_{L^r} + ||T^{k,4}(f)||_{L^p})$$

$$\leq \; C||b||_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} (||T^{k,2}(f)||_{L^p} + ||f||_{L^p})$$

$$\leq \; C||b||_{\mathrm{Lip}_\beta} ||f||_{L^p}.$$

This completes the proof of the theorem. $\qquad\square$

Theorem 3.7. *Let* $q' < p < n/\alpha$, $1/r = 1/p - \alpha/n$, $b \in BMO(\mathbb{R}^n)$ *and the sequence* $\{jC_j\} \in l^1$. *If* $F_t^1(g) = 0$ *for any* $g \in L^u(\mathbb{R}^n)(1 < u < \infty)$ *and* $T^{k,4}(f) = ||F_t^{k,4}(f)||$ *are the bounded operators on* $L^p(\mathbb{R}^n)$ *for* $1 < p < \infty(1 \leq k \leq m)$, *then* T^b *is bounded from* $L^p(\mathbb{R}^n)$ *to* $L^r(\mathbb{R}^n)$.

Proof. Choose $q' < s < p$ in Theorem 3.3, we have, by Lemmas 2.3 and 2.4,

$$||T^b(f)||_{L^r} \leq ||M(T^b(f))||_{L^r} \leq C||M^{\#}(T^b(f))||_{L^r}$$

$$\leq \; C||b||_{BMO} \sum_{k=1}^{m} (||M_s(I_\alpha T^{k,2}(f))||_{L^r} + ||M_{\alpha,s}(T^{k,4}(f))||_{L^r})$$

$$\leq \; C||b||_{BMO} \sum_{k=1}^{m} (||I_\alpha T^{k,2}(f)||_{L^r} + ||T^{k,4}(f)||_{L^p})$$

$$\leq \; C||b||_{BMO} ||f||_{L^p}.$$

This completes the proof of the theorem. $\qquad\square$

Theorem 3.8. *Let $0 < D < 2^n$, $q' < p < n/\alpha$, $1/r = 1/p - \alpha/n$, $b \in BMO(\mathbb{R}^n)$ and the sequence $\{jC_j\} \in l^1$. If $F_t^1(g) = 0$ for any $g \in L^u(\mathbb{R}^n)(1 < u < \infty)$ and $T^{k,4}(f) = \|F_t^{k,4}(f)\|$ are the bounded operators on $L^{p,\varphi}(\mathbb{R}^n)$ for $1 < p < \infty(1 \leq k \leq m)$, then T^b is bounded from $L^{p,\varphi}(\mathbb{R}^n)$ to $L^{r,\varphi}(\mathbb{R}^n)$.*

Proof. Choose $q' < s < p$ in Theorem 3.3, we have, by Lemmas 2.5 and 2.6,

$$\|T^b(f)\|_{L^{r,\varphi}} \leq \|M(T^b(f))\|_{L^{r,\varphi}} \leq C\|M^\#(T^b(f))\|_{L^{r,\varphi}}$$

$$\leq C\|b\|_{BMO} \sum_{k=1}^m (\|M_s(I_\alpha T^{k,2}(f))\|_{L^{r,\varphi}} + \|M_{\alpha,s}(T^{k,4}(f))\|_{L^{r,\varphi}})$$

$$\leq C\|b\|_{BMO} \sum_{k=1}^m (\|I_\alpha T^{k,2}(f)\|_{L^{r,\varphi}} + \|T^{k,4}(f)\|_{L^{p,\varphi}})$$

$$\leq C\|b\|_{BMO}\|f\|_{L^{p,\varphi}}.$$

This completes the proof of the theorem. $\qquad\square$

4. Applications

In this section we shall apply Theorems 3.1-3.8 to some particular operators such as the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

Application 1. Littlewood-Paley operator.

Fixed $\varepsilon > 0$. Let ψ be a fixed function which satisfies:

(1) $\int_{\mathbb{R}^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1)}$,

(3) $|\psi(x + y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$;

Let $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$ and $F_t(f)(x) = \int_{\mathbb{R}^n} f(y)\psi_t(x - y)dy$. The Littlewood-Paley operator is defined(see [23])

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

Set H be the space

$$H = \left\{ h : \|h\| = \left(\int_0^\infty |h(t)|^2 dt/t \right)^{1/2} < \infty \right\}.$$

Let b be a locally integrable function on $\mathbb{R}^n$. The Toeplitz type operator related to the Littlewood-Paley operator is defined by

$$g_\psi^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^b = \sum_{k=1}^m (F_t^{k,1} M_b I_\alpha F_t^{k,2} + F_t^{k,3} I_\alpha M_b F_t^{k,4}),$$

moreover, $F_t^{k,1}(f)$ are $F_t(f)$ or $\pm I$(the identity operator), $T^{k,2}(f) = \|F_t^{k,2}(f)\|$ and $T^{k,4}(f) = \|F_t^{k,4}(f)\|$ are the bounded linear operators on $L^p(R^n)$ for $1 <$

$p < \infty$, $T^{k,3} = \pm I$, $k = 1, \cdots, m$, $M_b(f) = bf$ and I_α is the fractional integral operator($0 < \alpha < n$). It is easily to see that g_ψ^b satisfies the conditions of Theorems 3.1-3.8 (see [12],[13],[14]), thus these theorems hold for g_ψ^b.

Application 2. Marcinkiewicz operator.

Fixed $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on $\mathbb{R}^n$ with $\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$. Set $F_t(f)(x) = \int_{|x-y| \le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy$. The Marcinkiewicz operator is defined by (see [24])

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}.$$

Set H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3 \right)^{1/2} < \infty \right\}.$$

Let b be a locally integrable function on R^n. The Toeplitz type operator related to the Marcinkiewicz operator is defined by

$$\mu_\Omega^b(f)(x) = \left(\int_0^\infty |F_t^b(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^b = \sum_{k=1}^m (F_t^{k,1} M_b I_\alpha F_t^{k,2} + F_t^{k,3} I_\alpha M_b F_t^{k,4}),$$

moreover, $F_t^{k,1}(f)$ are $F_t(f)$ or $\pm I$(the identity operator), $T^{k,2}(f) = ||F_t^{k,2}(f)||$ and $T^{k,4}(f) = ||F_t^{k,4}(f)||$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$, $T^{k,3} = \pm I$, $k = 1, \cdots, m$, $M_b(f) = bf$ and I_α is the fractional integral operator($0 < \alpha < n$). It is easily to see that μ_Ω^b satisfies the conditions of Theorems 3.1-3.8 (see [12],[13],[14],[24]), thus these theorems hold for μ_Ω^b.

Application 3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $\widehat{F_t^\delta(f)}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n} B^\delta(z/t)$ for $t > 0$. The maximal Bochner-Riesz operator is defined by(see [15])

$$B_{\delta,*}(f)(x) = \sup_{t>0} |F_t^\delta(f)(x)|.$$

Set H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$. Let b be a locally integrable function on $\mathbb{R}^n$. The Toeplitz type operator related to the maximal Bochner-Riesz operator is defined by

$$B_{\delta,*}^b(f)(x) = \sup_{t>0} |B_{\delta,t}^b(f)(x)|,$$

where

$$B_{\delta,t}^b = \sum_{k=1}^m (F_t^{k,1} M_b I_\alpha F_t^{k,2} + F_t^{k,3} I_\alpha M_b F_t^{k,4}),$$

moreover, $F_t^{k,1}(f)$ are $F_t(f)$ or $\pm I$(the identity operator), $T^{k,2}(f) = ||F_t^{k,2}(f)||$ and $T^{k,4}(f) = ||F_t^{k,4}(f)||$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$, $T^{k,3} = \pm I$, $k = 1, \cdots, m$, $M_b(f) = bf$ and I_α is the fractional integral

operator($0 < \alpha < n$). It is easily to see that $B_{\delta,*}^b$ satisfies the conditions of Theorems 3.1-3.8 (see [12],[13]), thus these theorems hold for $B_{\delta,*}^b$.

Acknowledgement. The author would like to express his gratitude to the referee for his comments and suggestions.

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College of Mathematics and Econometrics, Hunan University, Changsha 410082, P. R. of China.
E-mail address: lanzheliu@163.com