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WEYL'S THEOREM FOR ALGEBRAICALLY QUASI-$*$-A OPERATORS

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ABSTRACT. In the paper, we prove the following assertions: (1) If T is an algebraically quasi-$*$-A operator, then T is polaroid. (2) If T or T^* is an algebraically quasi-$*$-A operator, then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. (3) If T^* is an algebraically quasi-$*$-A operator, then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

1. INTRODUCTION

Let H be an infinite dimensional separable Hilbert space, $B(H)$ and $K(H)$ denote, respectively, the algebra of all bounded linear operators and the ideal of compact operators on H. If $T \in B(H)$, we shall denote the set of all complex numbers by C, and henceforth shorten $T - \lambda I$ to $T - \lambda$. We write $N(T)$ and $R(T)$ for the null space and range space of T; $\alpha(T) := \dim N(T)$; $\beta(T) := \dim N(T^*)$; $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$ and $\pi(T)$ for the spectrum of T, the approximate point spectrum of T, the point spectrum of T and the set of poles of the resolvent of T. Let $p = p(T)$ be the ascent of T, i.e., the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$, if such an integer does not exist, we put $p(T) = \infty$. Analogously, let $q = q(T)$ be the descent of T, i.e., the smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$, and if such an integer does not exist, we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then

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$p(T) = q(T)$. Moreover, $0 < p(\lambda - T) = q(\lambda - T) < \infty$ precisely when λ is a pole of the resolvent of T, see Heuser [10, Proposition 50.2].

An operator $T \in B(H)$ is called Fredholm if it has closed range with finite dimension null space and its range of finite co-dimension. The index of a Fredholm operator $T \in B(H)$ is given by

$$i(T) := \alpha(T) - \beta(T).$$

An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. An operator $T \in B(H)$ is called Browder if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined in [9, 8]:

$$\sigma_e(T) := \{\lambda \in C : T - \lambda \text{ is not Fredholm}\};$$
$$w(T) := \{\lambda \in C : T - \lambda \text{ is not Weyl}\};$$
$$\sigma_b(T) := \{\lambda \in C : T - \lambda \text{ is not Browder}\}.$$

Evidently,

$$\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma_e(T) \bigcup \text{acc}\sigma(T),$$

where $\text{acc}K$ denotes the accumulation points of $K \subseteq C$.

We consider the sets

$$\Phi_+(H) := \{T \in B(H) : R(T) \text{ is closed and } \alpha(T) < \infty\};$$
$$\Phi_-(H) := \{T \in B(H) : R(T) \text{ is closed and } \alpha(T^*) < \infty\};$$
$$\Phi_+^-(H) := \{T \in B(H) : T \in \Phi_+(H) \text{ and } i(T) \leq 0\}.$$

On the other hand,

$$\sigma_{ea}(T) := \{\lambda \in C : T - \lambda \notin \Phi_+^-(H)\}$$

is the essential approximate point spectrum and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT \text{ and } K \in K(H)\}$$

is the Browder essential approximate point spectrum.

If we write $\text{iso}K = K \setminus \text{acc}K$, then we let

$$\pi_{00}(T) := \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}.$$
$$\pi_{00}^a(T) := \{\lambda \in \text{iso}\sigma_a(T) : 0 < \alpha(T - \lambda) < \infty\}.$$

We write

$$\sigma(T) \setminus \sigma_b(T) := p_{00}(T).$$

Definition 1.1. Let $T \in B(H)$.

(1) Weyl's theorem holds for T if $\sigma(T) \setminus w(T) = \pi_{00}(T)$.
(2) a-Weyl's theorem holds for T if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^a(T)$.
(3) a-Browder's theorem holds for T if $\sigma_{ea}(T) = \sigma_{ab}(T)$.

It's known from [3, 8, 15] that if $T \in B(H)$ then we have
a-Weyl's theorem $\Rightarrow$ Weyl's theorem; a-Weyl's theorem $\Rightarrow$ a-Browder's theorem.

Recently, some interesting operators were studied in [13], it was also shown in [7, 16] that Weyl's theorem holds for totally $*$-paranormal operators. In this paper, we extend this result to algebraically quasi-$*$-A operators.

2. Examples

Definition 2.1. Let $T \in B(H)$.

(1) An operator T is said to be hyponormal if $T^*T \geq TT^*$.
(2) An operator T is said to be class $*$-A if $|T^2| \geq |T^*|^2$.
(3) An operator T is said to be $*$-paranormal if $||T^2x|| \geq ||T^*x||^2$ for every unite vector $x \in H$.
(4) An operator T is said to be quasi-$*$-A if $T^*|T^2|T \geq T^*|T^*|^2T$.

We say that $T \in B(H)$ is an algebraically quasi-$*$-A operator if there exists a nonconstant complex polynomial p such that $p(T)$ is a quasi-$*$-A operator.

From [6, 17], we have the following implications:
hyponormal $\Rightarrow$ class $*$-A $\Rightarrow$ $*$-paranormal;
hyponormal $\Rightarrow$ class $*$-A $\Rightarrow$ quasi-$*$-A $\Rightarrow$ algebraically quasi-$*$-A.

By computing, we have the following Lemma 2.2.

Lemma 2.2. *Let $K = \bigoplus_{n=1}^{+\infty} H_n$, where $H_n \cong H$. For given positive operators A and B on H, define the operator $T_{A,B}$ on K as follows:*

$$T_{A,B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ A & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & B & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & B & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & B & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & B & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then the following assertions hold:

(1) *$T_{A,B}$ belongs to hyponormal if and only if $B^2 \geq A^2$.*
(2) *$T_{A,B}$ belongs to class $*$-A if and only if $B^2 \geq A^2$.*
(3) *$T_{A,B}$ belongs to quasi-$*$-A if and only if $AB^2A \geq A^4$.*
(4) *$T_{A,B}$ belongs to $*$-paranormal if and only if $B^4 - 2\lambda A^2 + \lambda^2 \geq 0$ for all $\lambda > 0$.*

Remark 2.3. It is meaningless to use this characterization for distinguishing some gaps between hyponormal operators and class $*$-A operators. However, for $*$-paranormal operators, quasi-$*$-A operators, $T_{A,B}$ has a very useful characterization. The following examples show that $*$-paranormal operators and quasi-$*$-A operators are independent.

Example 2.4. A non-class $*$-A, non-$*$-paranormal and quasi-$*$-A operator.

Take A and B as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$B^2 - A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \ngeq 0.$$

And hence $T_{A,B}$ is a non-class $*$-A operator.

On the other hand,

$$A(B^2 - A^2)A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$$

Thus $T_{A,B}$ is a quasi-$*$-A operator.

Next we show that $T_{A,B}$ is non-$*$-paranormal operator.

$$B^4 - 2\lambda A^2 + \lambda^2 = \begin{pmatrix} 8 - 2\lambda + \lambda^2 & 8 \\ 8 & 8 + \lambda^2 \end{pmatrix}.$$

If $\lambda = 1$, then

$$B^4 - 2\lambda A^2 + \lambda^2 \ngeq 0.$$

Hence $T_{A,B}$ is not a $*$-paranormal operator.

Example 2.5. A non-class $*$-A, non-quasi-$*$-A and $*$-paranormal operator.

Take A and B as

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{\frac{1}{2}} \; B = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}^{\frac{1}{4}}.$$

Then

$$B^4 - 2\lambda A^2 + \lambda^2 = \begin{pmatrix} (1-\lambda)^2 & 2(1-\lambda) \\ 2(1-\lambda) & \lambda^2 - 4\lambda + 8 \end{pmatrix} \geq 0$$

for every $\lambda > 0$. Thus $T_{A,B}$ is a $*$-paranormal operator.

On the other hand,

$$A(B^2 - A^2)A = \begin{pmatrix} -0.3359\cdots & -0.2265\cdots \\ -0.2265\cdots & 0.8244\cdots \end{pmatrix} \ngeq 0.$$

Hence $T_{A,B}$ is not a quasi-$*$-A operator. Therefore $T_{A,B}$ is not a class $*$-A operator.

The following example provides an operator which is algebraically quasi-$*$-A but not quasi-$*$-A operator.

Example 2.6. Let $T = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \in B(l_2 \bigoplus l_2)$. Then T is an algebraically quasi-$*$-A but not quasi-$*$-A operator.

Since $T^* = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$, we have

$$(T^{*2}T^2)^{\frac{1}{2}} - TT^* = \begin{pmatrix} 1.1213\ldots & -0.2929\ldots \\ -0.2929\ldots & -1.2929\ldots \end{pmatrix},$$

then

$$T^*((T^{*2}T^2)^{\frac{1}{2}} - TT^*)T = \begin{pmatrix} -0.7574\ldots & -1.5858\ldots \\ -1.5858\ldots & -1.2929\ldots \end{pmatrix} \ngeq 0.$$

Therefore T is not a quasi-$*$-A operator.

On the other hand, consider the complex polynomial $h(z) = (z-1)^2$. Then $h(T) = 0$, and hence T is an algebraically quasi-$*$-A operator.

3. Weyl's theorem for algebraically quasi-∗-A operators

Before we state main theorems, we need several preliminary results.

Lemma 3.1. [17] *If T is a quasi-∗-A operator and T does not have dense range, then*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \bigoplus N(T^*),$$

Where $A = T|_{\overline{R(T)}}$ is the restriction of T to $\overline{R(T)}$, and $A \in$ class ∗-A.

We say that T has the single valued extension property (abbrev. SVEP) if for every open set U of C the only analytic solution $f \colon U \to H$ of the equation

$$(T - \lambda)f(\lambda) = 0$$

for all $\lambda \in U$ is the zero function on U.

Lemma 3.2. *If T is an algebraically quasi-∗-A operator, then T has $SVEP$.*

Proof. We first suppose that T is a quasi-∗-A operator. We consider the following two cases:
Case I: If the range of T is dense, then T is class ∗-A, T has $SVEP$ by [6].
Case Π: If the range of T is not dense, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \text{ on } H = \overline{R(T)} \bigoplus N(T^*).$$

Suppose $(T - z)f(z) = 0$. Put $f(z) = f_1(z) \bigoplus f_2(z)$ on $H = \overline{R(T)} \bigoplus N(T^*)$. Then we can write

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & -z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_1(z) + T_2 f_2(z) \\ -z f_2(z) \end{pmatrix} = 0.$$

Hence $f_2(z) = 0$. Then $(T_1 - z)f_1(z) = 0$. Since T_1 is class ∗-A, T_1 has $SVEP$ by [6]. Hence $f_1(z) = 0$. Consequently, T has $SVEP$.

Now suppose that T is an algebraically quasi-∗-A operator. Then $p(T)$ is a quasi-∗-A operator for some nonconstant complex polynomial p, and hence it follows from the first part of the proof that $p(T)$ has $SVEP$. Therefore T has $SVEP$ by [11, Theorem 3.3.9]. □

Lemma 3.3. *If T is a quasi-∗-A operator with spectrum $\sigma(T) \subseteq \partial D$, where D denotes the unite disc, then T is unitary.*

Proof. If T is quasi-∗-A operator with spectrum $\sigma(T) \subseteq \partial D$, then T is invertible and hence class ∗-A. [6, Proposition 2.6] now implies it is unitary. □

Lemma 3.4. *If T is a quasi-∗-A operator, and assume that $\sigma(T) = \{\lambda\}$, then $T = \lambda I$.*

Proof. We consider the following two cases:
Case I: if $\lambda = 0$, then $A = 0$ in Lemma 3.1, and so $T = 0$.
Case Π: if $\lambda \neq 0$, then T is invertible and class ∗-A. [6, Theorem 2.9] implies $T = \lambda I$. □

An operator $T \in B(H)$ is called polaroid if iso$\sigma(T) \subset \pi(T)$. In general, if T is polaroid, then it is isoloid, however, the converse isn't true. Consider the following example, let $T \in B(l_2)$ is defined by

$$T(x_1, x_2, x_3 \cdots) = (\frac{x_2}{2}, \frac{x_3}{3} \cdots).$$

Then T is a compact quasinilpotent operator with $\alpha(T) = 1$, thus T is isoloid, however, since T doesn't have finite ascent, T is not polaroid. In [6] it is showed that every $*$-paranormal operator is polaroid, we can prove more.

Lemma 3.5. *If T is an algebraically quasi-$*$-A operator, then T is polaroid.*

Proof. We first show that quasi-$*$-A operator is polaroid. Suppose T is a quasi-$*$-A operator. Let $\lambda \in$ iso$\sigma(T)$. Using the spectral projection $P := \frac{1}{2\pi i}\int_{\partial D}(\mu - T)^{-1}d\mu$, where D is a closed disk of center λ which contains no other point of $\sigma(T)$. We can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix},$$

where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Since T_1 is a quasi-$*$-A operator, it follows from Lemma 3.4 that $T_1 - \lambda = 0$, therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda$ has finite ascent and descent, and hence λ is a pole of the resolvent of T, thus $\lambda \in$ iso$\sigma(T)$ implies $\lambda \in \pi(T)$. Hence T is polaroid.

Next we show that algebraically quasi-$*$-A operator is polaroid. If T is an algebraically quasi-$*$-A operator, then $p(T)$ is quasi-$*$-A operator for some non-constant polynomial p. Hence it follows from the first part of the proof that $p(T)$ is polaroid. Now apply [5, Lemma 3.3] to conclude that $p(T)$ polaroid implies T polaroid. □

Corollary 3.6. *If T is an algebraically quasi-$*$-A operator, then T is isoloid.*

In the following theorem, recall that $H(\sigma(T))$ is the space of functions analytic in an open neighborhood of $\sigma(T)$.

Theorem 3.7. *Suppose T or T^* is an algebraically quasi-$*$-A operator. Then Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose that T is an algebraically quasi-$*$-A operator. We first show that Weyl's theorem holds for T. We use the fact [4, Theorem 2.2] that if T is polaroid then Weyl's theorem holds for T if and only if T has $SVEP$ at points of $\lambda \in \sigma(T) \setminus w(T)$. We have that T is polaroid by Lemma 3.5 and T has $SVEP$ by Lemma 3.2. Hence T satisfies Weyl's theorem.

Next we show that Weyl's theorem holds for $f(T)$. Since T is isoloid, by [12] we have $\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(w(T))$, where the last equality holds since T satisfies Weyl's theorem. Since T has $SVEP$, by [1, Corollary 2.6], we have $f(w(T)) = w(f(T))$. Therefore we have $\sigma(f(T)) \setminus \pi_{00}(f(T)) = w(f(T))$. Hence Weyl's theorem holds for $f(T)$.

Suppose that T^* is an algebraically quasi-$*$-A operator. We first show that Weyl's theorem holds for T. Since T^* has $SVEP$ and is polaroid, T is polaroid.

And T^* has $SVEP$ implies T^* satisfies Browder's theorem, then T satisfies Browder's theorem. Therefore Weyl's theorem holds for T. Since T^* has $SVEP$, by [1, Corollary 2.6], we have $f(w(T)) = w(f(T))$. Noting that T is isoloid, as in the proof of the first part, we have that Weyl's theorem holds for $f(T)$. This completes the proof. □

From the proof of Theorem 3.7, we have that the Weyl spectrum obeys the spectral mapping theorem for algebraically quasi-$*$-A operator.

Corollary 3.8. *Suppose T or T^* is an algebraically quasi-$*$-A operator. Then for every $f \in H(\sigma(T))$, we have $f(w(T)) = w(f(T))$.*

Corollary 3.9. *Suppose T or T^* is an algebraically quasi-$*$-A operator. If F is an operator commuting with T and for which there exists a positive integer n such that F^n has a finite rank, then Weyl's theorem holds for $f(T) + F$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T or T^* is an algebraically quasi-$*$-A operator. By Lemma 3.5 and Theorem 3.7, we have that T is isoloid and Weyl's theorem holds for $f(T)$. Observe that if T is isoloid then $f(T)$ is isoloid. The result follows from [13, Theorem 2.4]. □

Theorem 3.10. *Suppose T or T^* is an algebraically quasi-$*$-A operator. Then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$.*

Proof. Let $f \in H(\sigma(T))$. It suffices to show that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda \in \Phi_+^-(H)$ and

$$f(T) - \lambda = (T - \lambda_1) \cdots (T - \lambda_k) g(T), \tag{3.1}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k \in C$ and $g(T)$ is invertible. Since the operators on the right side of (3.1) commute, $T - \lambda_i \in \Phi_+(H)$. Suppose T is an algebraically quasi-$*$-A operator. Then $i(T - \lambda_j) \leq 0$ for each $j = 1, 2, \cdots, k$. Therefore $\lambda \notin f(\sigma_{ea}(T))$.

Suppose T^* is an algebraically quasi-$*$-A operator. It follows by [2, Theorem 2.8] that $i(T - \lambda_j) \geq 0$ for each $j = 1, 2, \cdots, k$. Since

$$0 \leq \sum_{j=1}^{k} i(T - \lambda_j) = i(f(T) - \lambda) \leq 0,$$

$T - \lambda_j$ is Weyl for each $j = 1, 2, \cdots, k$. Therefore $\lambda \notin f(\sigma_{ea}(T))$, and hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. This completes the proof. □

Theorem 3.11. *Suppose T^* is an algebraically quasi-$*$-A operator. Then a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose T^* is an algebraically quasi-$*$-A operator. We first prove that a-Weyl's theorem holds for T. Since T^* has $SVEP$ and T is polaroid, $\overline{\sigma_a(T)} = \overline{\sigma(T)} = \sigma(T^*)$, $\pi_{00}(T^*) = \overline{\pi_{00}^a(T)}$ and $\overline{\sigma_{ea}(T)} = \overline{w(T)} = w(T^*)$. Since T^* satisfies Weyl's theorem, T satisfies a-Weyl's theorem.

Next we prove that a-Weyl's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Let $f \in H(\sigma(T))$. Since T satisfies a-Weyl's theorem, we have that a-Browder's theorem holds for T. Hence $\sigma_{ea}(T) = \sigma_{ab}(T)$. Since T^* is an algebraically quasi-$*$-A operator, it follows from Theorem 3.10 that $\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{ea}(T)) = \sigma_{ea}(f(T))$, and hence a-Browder's theorem holds for $f(T)$. We use the fact [3, Theorem 3.8] that if T satisfies a-Browder's theorem then a-Weyl's theorem holds for T if $R(T-\lambda)$ is closed for each $\lambda \in \pi^a_{00}(T)$. Hence it suffices to show that if $\lambda \in \pi^a_{00}(f(T))$, then $R(f(T)-\lambda)$ is closed. Since $f(T^*)$ has SVEP, $\pi_{00}(f(T)) = \pi^a_{00}(f(T))$, and hence $R(f(T)-\lambda)$ is closed for each $\lambda \in \pi^a_{00}(f(T))$. □

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