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ON LEFT AND RIGHT DECOMPOSABLY REGULAR OPERATORS

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ABSTRACT. Let $B(X,Y)$ denote the set of all bounded linear operators from Banach space X to Banach space Y. In this paper, we introduce the concepts of left and right decomposably regular operators, left and right decomposably Fredholm operators in the setting of $B(X,Y)$, and the corresponding holomorphic versions in the setting of $B(X)$. By using Harte's techniques, we obtain various characterizations of these classes of operators. As the applications of these characterizations, we can compute the topological interiors and closures of them.

1. INTRODUCTION

Throughout this paper, denote the set of all bounded (resp. compact) linear operators from Banach space X to Banach space Y by $B(X,Y)$ (resp. $K(X,Y)$), and abbreviate $B(X,X)$ and $K(X,X)$ to $B(X)$ and $K(X)$, respectively. For other classes of operators discussed below, we use similar abbreviations. For an operator $T \in B(X,Y)$, let $\ker(T)$ denote its null space, $\alpha(T)$ its nullity, $T(X)$ its range and $\beta(T)$ its defect. We also denote classes of left invertible operators, right invertible operators, invertible operators, left Fredholm operators, right Fredholm operators and Fredholm operators from X to Y by $G_l(X,Y)$, $G_r(X,Y)$, $G(X,Y)$, $\Phi_l(X,Y)$, $\Phi_r(X,Y)$ and $\Phi(X,Y)$, respectively.

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We say that $T \in B(X,Y)$ is relatively regular, in symbol $T \in R(X,Y)$, provided that there exists some $S \in B(Y,X)$ for which $T = TST$. In this case, S is called an inner generalized inverse of T. It is well known that $T \in B(X,Y)$ is relatively regular if and only if $\ker(T)$ and $T(X)$, respectively, are closed and complemented subspaces of X and Y. If S is an inner generalized inverse of T, it is well known that TS is the projection from Y onto $T(X)$, and $I - ST$ is the projection from X onto $\ker(T)$. Operators in sets $G_l(X,Y), G_r(X,Y), \Phi_l(X,Y)$ and $\Phi_r(X,Y)$ are all relatively regular operators. For a subset M of $B(X,Y)$, let $\mathrm{int}(M)$ and $\overline{M}$ denote, respectively, the topological interior and closure of M.

An operator $T \in B(X,Y)$ is decomposably regular, in symbol $T \in GR(X,Y)$, if there exists an invertible operator $S \in G(Y,X)$ for which $T = TST$. If $T = TST$ for some $S \in \Phi(Y,X)$, then T is called a decomposably Fredholm operator, in symbol $T \in \Phi R(X,Y)$. For $X = Y$, Harte (resp. Rakočevič) obtained in [5] (resp. [13]) an elegant characterization of decomposably regular operators (resp. decomposably Fredholm operators), that is

$$GR(X) = R(X) \cap \overline{G(X)}; \quad \Phi R(X) = R(X) \cap \overline{\Phi(X)}.$$

Indeed, the above two results also hold in the case that X and Y are two different Banach spaces (see Theorem 2.1(3) and Theorem 2.5(3) below), that is

$$GR(X,Y) = R(X,Y) \cap \overline{G(X,Y)};$$

$$\Phi R(X,Y) = R(X,Y) \cap \overline{\Phi(X,Y)}. \tag{1.1}$$

From (1.1), we can infer that

$$\Phi(X,Y) \subseteq \Phi R(X,Y). \tag{1.2}$$

Evidently, every idempotent operator P ($P^2 = P$) is decomposably regular, and hence, is decomposably Fredholm, but may not be Fredholm. Consequently, the inclusion (1.2) may be strict. However, we would get in Theorem 3.5(6) that

$$\mathrm{int}(\Phi R(X,Y)) = \Phi(X,Y),$$

which extends a result of Schmoeger [18, Theorem 2.2(2)] to the case that X and Y are two different Banach spaces.

Decomposably regular operators can also be characterized "spatially" (see [6, Theorem 3.8.6]):

$$T \in GR(X,Y) \Longleftrightarrow T \in R(X,Y) \text{ and } \ker(T) \approx Y/T(X). \tag{1.3}$$

For $T \in \Phi_l(X,Y) \cup \Phi_r(X,Y)$, the index of T is defined as $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. If $T \in \Phi(X,Y)$ and $\mathrm{ind}(T) = 0$, then T is said to be Weyl. If $T \in \Phi_l(X,Y)$ and $\mathrm{ind}(T) \le 0$, then T is said to be left Weyl. If $T \in \Phi_r(X,Y)$ and $\mathrm{ind}(T) \ge 0$, then T is said to be right Weyl. From (1.3), it is easy to see that

$$\{T \in \Phi(X,Y) : \mathrm{ind}(T) = 0\} \subseteq GR(X,Y). \tag{1.4}$$

As we know, every idempotent operator is decomposably regular, but may not be Weyl. Hence the inclusion (1.4) can be strict. More precisely, we can obtain that (see Theorem 3.5(3) below)

$$\mathrm{int}(GR(X,Y)) = \{T \in \Phi(X,Y) : \mathrm{ind}(T) = 0\},$$

which extends Theorem 2.1 in [18] to the case that X and Y are two different Banach spaces.

The following definition describes the classes of operators we will study.

Definition 1.1. (1) An operator $T \in B(X,Y)$ is said to be left decomposably regular, in symbol $T \in G_lR(X,Y)$, provided that there exists $S \in G_r(Y,X)$ such that $TST = T$;

(2) An operator $T \in B(X,Y)$ is said to be right decomposably regular operators, in symbol $T \in G_rR(X,Y)$, provided that there exists $S \in G_l(Y,X)$ such that $TST = T$;

(3) An operator $T \in B(X,Y)$ is said to be left decomposably Fredholm, in symbol $T \in \Phi_lR(X,Y)$, provided that there exists $S \in \Phi_r(Y,X)$ such that $TST = T$;

(4) An operator $T \in B(X,Y)$ is said to be right decomposably Fredholm, in symbol $T \in \Phi_rR(X,Y)$, provided that there exists $S \in \Phi_l(Y,X)$ such that $TST = T$.

Evidently $G_l(X,Y) \subseteq G_lR(X,Y)$ and $G_r(X,Y) \subseteq G_rR(X,Y)$. By (1.3), it is easy to observe that left (right) decomposably regular operators are genuinely more general than decomposably regular operators, and no surprise that the unilateral right (left) shifts provide the examples.

The present paper focuses mainly on the above four classes of operators and their holomorphical versions (see Definition 2.9), and is organized as follows. In Section 2, by using Harte's techniques, we characterize these classes of operators in several ways. In Section 3, as the applications of these characterizations, we compute the topological interiors and closures of them.

2. Various characterizations

Harte obtained in [5, Theorem 1.1] an elegant structure theorem of decomposably regular elements in a Banach algebra $\mathscr{A}$. Motivated by Harte's techniques therein, we will get in the following theorem the structures of left and right decomposably regular operators in the setting of $B(X,Y)$.

Theorem 2.1. (1) $G_rR(X,Y) = R(X,Y) \cap \overline{G_r(X,Y)}$;

(2) $G_lR(X,Y) = R(X,Y) \cap \overline{G_l(X,Y)}$;

(3) $GR(X,Y) = R(X,Y) \cap \overline{G(X,Y)}$.

Proof. (1) Let $T \in G_rR(X,Y)$. Then $T \in R(X,Y)$ and there exists $S_l \in G_l(Y,X)$ such that $TS_lT = T$. Since $S_l \in G_l(Y,X)$, there exists $S_r \in G_r(X,Y)$ such that $S_rS_l = I_Y$. It is easy to see that $T = S_rS_lT$ and $S_lTS_lT = S_lT$, and hence $T = S_rP \in B(X,Y)$, where $P = S_lT \in B(X)$ is an idempotent operator. Let $T_n = S_r(P + \frac{I_X - P}{n}) \in B(X,Y)$, for all $n \in N$. It is easy to check that $[P + n(I_X - P)][P + \frac{I_X-P}{n}] = [P + \frac{I_X-P}{n}][P + n(I_X - P)] = I_X$, that is $P + \frac{I_X-P}{n} \in G(X)$. Further, $T_n[P + n(I_X - P)]S_l = S_r[P + \frac{I_X-P}{n}][P + n(I_X - P)]S_l = S_rS_l = I_Y$, that is $T_n \in G_r(X,Y)$ for all $n \in N$. Since $T_n \to T$ $(n \to \infty)$, we have $T \in \overline{G_r(X,Y)}$, therefore $G_rR(X,Y) \subseteq R(X,Y) \cap \overline{G_r(X,Y)}$.

Conversely, suppose that $T \in R(X,Y) \cap \overline{G_r(X,Y)}$. Thus there exist $S \in B(Y,X)$ and $B_r \in G_r(X,Y)$ such that $TST = T, STS = S$ and $I_Y + (B_r - T)S \in G(Y)$. Since $B_r \in G_r(X,Y)$, there exists $B_l \in G_l(Y,X)$ such that $B_r B_l = I_Y$. Let $T' = S + (I_X - ST)B_l(I_Y - TS) \in B(Y,X)$. It is not hard to see that $T = TT'T$, and so T' is an inner generalized inverse of T.

It remains only to show that $T' \in G_l(Y,X)$. Since $I_Y + (B_r - T)S \in G(Y)$, there exists $U \in G(Y)$ such that $[I_Y + (B_r - T)S]U = U[I_Y + (B_r - T)S] = I_Y$. Hence

$$\begin{aligned}
&[I_X + S(B_r - T)][I_X - SU(B_r - T)] \\
&\qquad = I_X - SU(B_r - T) + S(B_r - T) - S(B_r - T)SU(B_r - T) \\
&\qquad = I_X - S[I_Y + (B_r - T)S]U(B_r - T) + S(B_r - T) \\
&\qquad = I_X - S(B_r - T) + S(B_r - T) \\
&\qquad = I_X
\end{aligned}$$

and

$$\begin{aligned}
&[I_X - SU(B_r - T)][I_X + S(B_r - T)] \\
&\qquad = I_X - SU(B_r - T) + S(B_r - T) - SU(B_r - T)S(B_r - T) \\
&\qquad = I_X - SU[I_Y + (B_r - T)S](B_r - T) + S(B_r - T) \\
&\qquad = I_X - S(B_r - T) + S(B_r - T) \\
&\qquad = I_X.
\end{aligned}$$

Let $V = I_X - SU(B_r - T)$. From the above two equations, we know that $V[I_X + S(B_r - T)] = [I_X + S(B_r - T)]V = I_X$, that is $V \in G(X)$. Furthermore,

$$\begin{aligned}
&[I_Y + (B_r - T)S][I_Y - (B_r - T)VS] \\
&\qquad = I_Y - (B_r - T)VS + (B_r - T)S - (B_r - T)S(B_r - T)VS \\
&\qquad = I_Y - (B_r - T)[I_X + S(B_r - T)]VS + (B_r - T)S \\
&\qquad = I_Y - (B_r - T)S + (B_r - T)S \\
&\qquad = I_Y
\end{aligned}$$

and

$$\begin{aligned}
&[I_Y - (B_r - T)VS][I_Y + (B_r - T)S] \\
&\qquad = I_Y - (B_r - T)VS + (B_r - T)S - (B_r - T)VS(B_r - T)S \\
&\qquad = I_Y - (B_r - T)V[I_X + S(B_r - T)]S + (B_r - T)S \\
&\qquad = I_Y - (B_r - T)S + (B_r - T)S \\
&\qquad = I_Y.
\end{aligned}$$

Hence, we get that $U = I_Y - (B_r - T)VS$. Let $P = ST \in B(X)$ and $Q = TS \in B(Y)$. Then P is the projection from X onto $S(Y)$ parallel to $\ker(T)$ and Q is the projection from Y onto $T(X)$ parallel to $\ker(S)$. Moreover,

$$\begin{aligned} UB_rP =& U(B_rS)T \\ =& U[I_Y + (B_r - T)S + TS - I_Y]T \\ =& U[I_Y + (B_r - T)S]T + U(TS - I_Y)T \\ =& T \end{aligned}$$

and

$$\begin{aligned} U(I_Y - Q) =& [I_Y - (B_r - T)VS](I_Y - TS) \\ =& I_Y - (B_r - T)VS - TS + (B_r - T)VSTS \\ =& I_Y - (B_r - T)VS - TS + (B_r - T)VS \\ =& I_Y - TS \\ =& I_Y - Q. \end{aligned}$$

Since $UB_rP = T$, $U(I_Y - Q) = I_Y - Q$ and $B_rB_l = I_Y$, we can get that

$$\begin{aligned} &[T + (I_Y - Q)UB_r(I_X - P)]T' \\ &=[T + (I_Y - Q)UB_r(I_X - P)][S + (I_X - ST)B_l(I_Y - TS)] \\ &=TS + [(I_Y - Q)UB_r(I_X - P)][(I_X - P)B_l(I_Y - Q)] \\ &=TS + (I_Y - Q)UB_r(I_X - P)B_l(I_Y - Q) \\ &=TS + (I_Y - Q)(UB_r - UB_rP)B_l(I_Y - Q) \\ &=TS + (I_Y - Q)(UB_r - T)B_l(I_Y - Q) \\ &=TS + (I_Y - Q)UB_rB_l(I_Y - Q) - (I_Y - Q)TB_l(I_Y - Q) \\ &=TS + (I_Y - Q)UB_rB_l(I_Y - Q) \\ &=TS + (I_Y - Q)U(I_Y - Q) \\ &=TS + (I_Y - Q)(I_Y - Q) \\ &=Q + I_Y - Q \\ &=I_Y. \end{aligned}$$

That is, $[T + (I_Y - Q)UB_r(I_X - P)]T' = I_Y$. So $T' \in G_l(Y, X)$, then $T \in G_rR(X, Y)$, and this completes the whole proof of (1).

Parts (2) and (3) can be proved similarly. $\square$

Sets of all invertible elements, left invertible elements, right invertible elements, relatively regular elements, decomposably regular elements, left decomposably regular elements and right decomposably regular elements in a complex Banach algebra $\mathscr{A}$ with identity 1 are defined as follows respectively:

$$G(\mathscr{A}) := \{a \in \mathscr{A} : \text{ there exists some } b \in \mathscr{A} \text{ such that } ab = ba = 1\};$$

$$G_l(\mathscr{A}) := \{a \in \mathscr{A} : \text{ there exists some } b \in \mathscr{A} \text{ such that } ba = 1\};$$

$$G_r(\mathscr{A}) := \{a \in \mathscr{A} : \text{ there exists some } b \in \mathscr{A} \text{ such that } ab = 1\};$$

$$R(\mathscr{A}) := \{a \in \mathscr{A} : a \in a\mathscr{A}a\};$$

$$GR(\mathscr{A}) := \{a \in \mathscr{A} : a \in aG(\mathscr{A})a\};$$

$$G_lR(\mathscr{A}) := \{a \in \mathscr{A} : a \in aG_r(\mathscr{A})a\};$$
$$G_rR(\mathscr{A}) := \{a \in \mathscr{A} : a \in aG_l(\mathscr{A})a\}.$$

Now, [5, Theorem 1.1] should be represented as follows:

$$GR(\mathscr{A}) = R(\mathscr{A}) \cap \overline{G(\mathscr{A})}.$$

Similar to the proof of Theorem 2.1, we can extend Theorem 2.1 to the Banach algebra $\mathscr{A}$.

Theorem 2.2. (1) $G_rR(\mathscr{A}) = R(\mathscr{A}) \cap \overline{G_r(\mathscr{A})}$;
(2) $G_lR(\mathscr{A}) = R(\mathscr{A}) \cap \overline{G_l(\mathscr{A})}$.

Before providing the characterizations of left and right decomposably Fredholm operators in $B(X, Y)$, we give an example to illustrate that there exists some operator which is left (right) decomposably Fredholm but not decomposably Fredholm.

For any Hilbert space X, let $\dim_H X$ denote the Hilbert dimension of X, that is the cardinality of an orthonormal basis of X. We set $\mathrm{nul}_H(T) = \dim_H \ker(T)$ and $\mathrm{def}_H(T) = \dim_H T(X)^{\perp}$. For a separable Hilbert space X, Rakočevič has proved the following:

Proposition 2.3. ([13, *Theorem* 5]) *Let X be a separable Hilbert space. Then*

$$R(X) \cap \overline{\Phi(X)} = \Phi(X) \cup \{T \in B(X) : \mathrm{nul}_H(T) = \mathrm{def}_H(T) \text{ and } T(X) \text{ is closed}\}.$$

Example 2.4. Let H be the direct sum of countably many copies of $l_2(\mathbb{N})$, that is,

$$H = \{(x_j)_{j=1}^{\infty} : x_j \in l_2(\mathbb{N}) \text{ and } \sum_{i=1}^{\infty} \| x_j \| < \infty\}.$$

(1) Let $S : l_2(\mathbb{N}) \longrightarrow l_2(\mathbb{N})$ be the unilateral right shift operator defined by

$$S(z_1, z_2, z_3, \cdots) = (0, z_1, z_2, \cdots) \quad \text{for all} \quad (z_n) \in l_2(\mathbb{N}).$$

The operator $\widehat{S}$ on H is defined by

$$\widehat{S}(x_1, x_2, x_3, \cdots) = (Sx_1, Sx_2, Sx_3, \cdots) \quad \text{for all} \quad (x_n) \in H.$$

Note that $\widehat{S}(H)$ is closed, $\alpha(\widehat{S}) = 0$ and $\beta(\widehat{S}) = \infty$. Then $\widehat{S} \in G_l(H) \subseteq G_lR(H) \subseteq \Phi_lR(H)$. But since $\widehat{S} \notin \Phi R(H)$ by Proposition 2.3 and (1.1), we have $\Phi R(H) \subsetneqq \Phi_lR(H)$.

(2) Let $T : l_2(\mathbb{N}) \longrightarrow l_2(\mathbb{N})$ be the unilateral left shift operator defined by

$$T(z_1, z_2, z_3, \cdots) = (z_2, z_3, z_4, \cdots) \quad \text{for all} \quad (z_n) \in l_2(\mathbb{N}).$$

The operator $\widehat{T}$ on H is defined by

$$\widehat{T}(x_1, x_2, x_3, \cdots) = (Tx_1, Tx_2, Tx_3, \cdots) \quad \text{for all} \quad (x_n) \in H.$$

Noting that $\widehat{T}(H)$ is closed, $\alpha(\widehat{T}) = \infty$ and $\beta(\widehat{T}) = 0$, we have $\widehat{T} \in G_r(H) \subseteq G_rR(H) \subseteq \Phi_rR(H)$. But since $\widehat{T} \notin \Phi R(H)$ by Proposition 2.3 and (1.1), we have $\Phi R(H) \subsetneqq \Phi_rR(H)$.

Rakočevič [13, Theorem 3] adopted Harte's technique used in [5, Theorem 1.1] to prove that

$$\Phi R(X) = R(X) \cap \overline{\Phi(X)} = \mathcal{P}(X)\Phi(X),$$

where $\mathcal{P}(X) = \{P \in B(X) : P^2 = P\}$. Next, we will prove that similar results also hold for left and right decomposably Fredholm operators. It is remarked that here we need to deal with compact perturbations.

Theorem 2.5. (1) $\Phi_r R(X,Y) = R(X,Y) \cap \overline{\Phi_r(X,Y)}$;

(2) $\Phi_l R(X,Y) = R(X,Y) \cap \overline{\Phi_l(X,Y)}$;

(3) $\Phi R(X,Y) = R(X,Y) \cap \overline{\Phi(X,Y)}$.

Proof. We will prove (2), omitting the similar proofs of (1) and (3).

(2) Let $T \in \Phi_l R(X,Y)$. Then $T \in R(X,Y)$ and there exists $S_r \in \Phi_r(Y,X)$ for which $TS_rT = T$. Since $S_r \in \Phi_r(Y,X)$, there exist $S_l \in \Phi_l(X,Y)$ and $K_1 \in K(X)$ such that $S_rS_l = I_X + K_1$. It is easy to see that $TS_rS_l = T + TK_1$ and $TS_rTS_r = TS_r$, and hence $TS_r(S_l - TK_1) = T + TK_1 - TS_rTK_1 = T$, thus $T = PC \in B(X,Y)$, where $C = S_l - TK_1 \in \Phi_l(X,Y)$ and $P = TS_r \in B(Y)$ is an idempotent operator. Let $T_n = (P + \frac{I_Y - P}{n})C \in B(X,Y)$ for all $n \in N$. It is easy to check that$[P + n(I_Y - P)][P + \frac{I_Y-P}{n}] = [P + \frac{I_Y-P}{n}][P + n(I_Y - P)] = I_Y$, that is, $P + \frac{I_Y-P}{n} \in G(Y)$ for all $n \in N$. Further, $S_r[P + n(I_Y - P)]T_n = S_r[P + n(I_Y - P)][P + \frac{I_Y-P}{n}]C = S_rC = S_r(S_l - TK_1) = I_X + K_1 - S_rTK_1$, that is, $T_n \in \Phi_l(X,Y)$ for all $n \in N$. Since $T_n \to T$ $(n \to \infty)$, $T \in \overline{\Phi_l(X,Y)}$. Thus $\Phi_l R(X,Y) \subseteq R(X,Y) \cap \overline{\Phi_l(X,Y)}$.

Conversely, suppose that $T \in R(X,Y) \cap \overline{\Phi_l(X,Y)}$. Then there exist $S \in B(Y,X)$ and $B_l \in \Phi_l(X,Y)$ for which $TST = T, STS = S$ and $I_X + S(B_l - T) \in G(X)$. Since $B_l \in \Phi_l(X,Y)$, there exist $B_r \in \Phi_r(Y,X)$ and $K_1 \in K(X)$ such that $B_rB_l = I_X + K_1$. Let $T' = S + (I_X - ST)B_r(I_Y - TS) \in B(Y,X)$. It is not hard to see that $T = TT'T$, and so T' is an inner generalized inverse of T.

It remains only to show that $T' \in \Phi_r(Y,X)$. Since $I_X + S(B_l - T) \in G(X)$, there exists $U_0 \in G(X)$ such that $[I_X + S(B_l - T)]U_0 = U_0[I_X + S(B_l - T)] = I_X$. Hence

$$\begin{aligned}
&[I_Y + (B_l - T)S][I_Y - (B_l - T)U_0S] \\
&\quad = I_Y - (B_l - T)U_0S + (B_l - T)S - (B_l - T)S(B_l - T)U_0S \\
&\quad = I_Y - (B_l - T)[I_X + S(B_l - T)]U_0S + (B_l - T)S \\
&\quad = I_Y - (B_l - T)S + (B_l - T)S \\
&\quad = I_Y
\end{aligned}$$

and

$$\begin{aligned}
&[I_Y - (B_l - T)U_0S][I_Y + (B_l - T)S] \\
&\quad = I_Y + (B_l - T)S - (B_l - T)U_0S - (B_l - T)U_0S(B_l - T)S \\
&\quad = I_Y + (B_l - T)S - (B_l - T)U_0[I_X + S(B_l - T)]S \\
&\quad = I_Y + (B_l - T)S - (B_l - T)S \\
&\quad = I_Y.
\end{aligned}$$

Let $V_0 = I_Y - (B_r - T)U_0S$. From the above equations, we know that $V_0[I_Y + (B_l - T)S] = [I_Y + (B_l - T)S]V_0 = I_Y$, that is $V_0 \in G(Y)$. Further,

$$\begin{aligned}
&[I_X + S(B_l - T)][I_X - SV_0(B_l - T)]\\
&\qquad = I_X - SV_0(B_l - T) + S(B_l - T) - S(B_l - T)SV_0(B_l - T)\\
&\qquad = I_X + S(B_l - T) - S[I_Y + (B_l - T)S]V_0(B_l - T)\\
&\qquad = I_X + S(B_l - T) - S(B_l - T)\\
&\qquad = I_Y
\end{aligned}$$

and

$$\begin{aligned}
&[I_X - SV_0(B_l - T)][I_X + S(B_l - T)]\\
&\qquad = I_X + S(B_l - T) - SV_0(B_l - T) - SV_0(B_l - T)S(B_l - T)\\
&\qquad = I_X + S(B_l - T) - SV_0[I_Y + (B_l - T)S](B_l - T)\\
&\qquad = I_X + S(B_l - T) - S(B_l - T)\\
&\qquad = I_X.
\end{aligned}$$

Hence, we get that $U_0 = I_X - SV_0(B_l - T)$. Let $P = ST \in B(X)$ and $Q = TS \in B(Y)$. Then P is the projection from X onto $S(Y)$ parallel to $\ker(T)$ and Q is the projection from Y onto $T(X)$ parallel to $\ker(S)$. Moreover,

$$\begin{aligned}
QB_lU_0 &= T(SB_l)U_0\\
&= T[I_X + S(B_l - T) + ST - I_X]U_0\\
&= T[I_X + S(B_l - T)]U_0 + T(ST - I_X)U_0\\
&= T
\end{aligned}$$

and

$$\begin{aligned}
(I_X - P)U_0 &= (I_X - ST)[I_X - SV_0(B_l - T)]\\
&= I_X - ST - SV_0(B_l - T) + STSV_0(B_l - T)\\
&= I_X - ST - SV_0(B_l - T) + SV_0(B_l - T)\\
&= I_X - ST\\
&= I_X - P.
\end{aligned}$$

Since $QB_lU_0 = T$, $(I_X - P)U_0 = I_X - P$ and $B_rB_l = I_X + K_1$, we can get that

$$\begin{aligned}
&T'[T + (I_Y - Q)B_lU_0(I_X - P)]\\
&\quad =[S + (I_X - ST)B_r(I_Y - TS)][T + (I_Y - Q)B_lU_0(I_X - P)]\\
&\quad =ST + [(I_X - ST)B_r(I_Y - TS)][(I_Y - Q)B_lU_0(I_X - P)]\\
&\quad =ST + (I_X - ST)B_r(I_Y - Q)B_lU_0(I_X - P)\\
&\quad =ST + (I_X - P)B_r(B_lU_0 - QB_lU_0)(I_X - P)\\
&\quad =ST + (I_X - P)B_r(B_lU_0 - T)(I_X - P)\\
&\quad =ST + (I_X - P)B_rB_lU_0(I_X - P) + (I_X - P)B_rT(I_X - P)\\
&\quad =ST + (I_X - P)B_rB_lU_0(I_X - P)\\
&\quad =ST + (I_X - P)(I_X + K_1)U_0(I_X - P)\\
&\quad =ST + (I_X - P)U_0(I_X - P) + (I_X - P)K_1U_0(I_X - P)\\
&\quad =ST + (I_X - P)(I_X - P) + (I_X - P)K_1U_0(I_X - P)\\
&\quad =P + I_X - P + (I_X - P)K_1U_0(I_X - P)\\
&\quad =I_X + (I_X - P)K_1U_0(I_X - P).
\end{aligned}$$

That is, $T'[T + (I_Y - Q)B_lU_0(I_X - P)] = I_X + (I_X - P)K_1U_0(I_X - P)$. So $T' \in \Phi_r(Y, X)$, then $T \in \Phi_lR(X, Y)$, and this complete the whole proof of (2). $\square$

Decomposably regular operators are characterized "spatially" as in (1.3). Our next result extends it to left (right) decomposably regular operators. Before this, the notions of embedded spaces, essentially embedded spaces and essentially isomorphic spaces are needed.

Definition 2.6. Let X and Y be Banach spaces. We say that X can be embedded in Y and write $X \preceq Y$ if there exists a left invertible operator $J : X \to Y$. We say that X can be essentially embedded in Y and write $X \preceq_e Y$ if there exists a left Fredholm operator $F : X \to Y$. We say that X and Y are essentially isomorphic and write $X \approx_e Y$ if $\Phi(X, Y) \neq \varnothing$.

The notion of embedded spaces and the notion similar to essentially embedded spaces were introduced by Djordjević [4] to investigate the perturbations of spectra of operator matrices. The notion of essentially isomorphic spaces was introduced by González and Herrera [9]. Obviously, $X \preceq Y$ if and only if there exists a right invertible operator $J_1 : Y \to X$; $X \preceq_e Y$ if and only if there exists a right Fredholm operator $F_1 : Y \to X$; $X \approx_e Y$ if and only if $Y \approx_e X$.

Theorem 2.7. (1) $T \in G_lR(X, Y) \Longleftrightarrow T \in R(X, Y)$ *and* $\ker(T) \preceq Y/T(X)$;
(2) $T \in G_rR(X, Y) \Longleftrightarrow T \in R(X, Y)$ *and* $Y/T(X) \preceq \ker(T)$.
And these equivalences yield the following inclusions directly:

$$\{T \in \Phi_l(X, Y) : \mathrm{ind}(T) \leq 0\} \subseteq G_lR(X, Y); \tag{2.1}$$

$$\{T \in \Phi_r(X, Y) : \mathrm{ind}(T) \geq 0\} \subseteq G_rR(X, Y). \tag{2.2}$$

Proof. (1) Suppose that $T \in G_lR(X, Y)$. Then $T \in R(X, Y)$ and there exists $S \in G_r(Y, X)$ such that $TST = T$. Since $S \in G_r(Y, X)$, there exists $L \in G_l(X, Y)$ such that $SL = I_X$.

The operators $U : \ker(T) \longrightarrow Y/T(X)$ and $V : Y/T(X) \longrightarrow \ker(T)$ are defined as follows respectively:

$$U(x) = L(x) + T(X), \text{ for all } x \in \ker(T);$$

$$V(y + T(X)) = (I - ST)S(I - TS)(y), \text{ for all } y + T(X) \in Y/T(X).$$

For all $x \in \ker(T)$, we can check that

$$\begin{aligned} VU(x) =& V(L(x) + T(X)) \\ =& (I - ST)S(I - TS)(L(x)) \\ =& (S - STS)(L(x)) \\ =& (SL - STSL)(x) \\ =& x - ST(x) \\ =& x. \end{aligned}$$

That is, $VU = I_{\ker(T)}$, hence $U \in G_l(\ker(T), Y/T(X))$. This proves $\ker(T) \preceq Y/T(X)$.

Conversely, suppose that $T \in R(X, Y)$ and $\ker(T) \preceq Y/T(X)$. Then there exists $S \in B(Y, X)$ such that $TST = T$. Let $P = TS \in B(Y)$ and $Q = ST \in B(X)$. Therefore P is the projection from Y onto $T(X)$ and $I-Q$ is the projection from X onto $\ker(T)$.

The operators $\widehat{T} : Q(X) \to P(Y)$ and $\widehat{S} : P(Y) \to Q(X)$ are defined as follows respectively:

$$\widehat{T}(Q(x)) = T(Q(x)), \text{ for all } x \in X;$$

$$\widehat{S}(P(y)) = S(P(y)), \text{ for all } y \in Y.$$

Since $Y/T(X) = Y/P(Y) \approx \ker(P)$, there exists $U_0 \in G_l(\ker(Q), \ker(P))$. Let $V_0 \in G_r(\ker(P), \ker(Q))$ be a left inverse of U_0.

The operator $V : Y \to X$ is defined by

$$V(y) = \widehat{S}P(y) + V_0(I - P)(y), \text{ for all } y \in Y.$$

Hence for all $x \in X$, we have

$$\begin{aligned} TVT(x) =& T[\widehat{S}P(Tx) + V_0(I - P)(Tx)] \\ =& T\widehat{S}P(Tx) \\ =& T\widehat{S}T(x) \\ =& TST(x) \\ =& T(x), \end{aligned}$$

that is, $TVT = T$.

The operator $W : X \to Y$ is defined by

$$W(x) = \widehat{T}Q(x) + U_0(I - Q)(x), \text{ for all } x \in X.$$

Then for all $x \in X$, we can obtain that

$$\begin{aligned}VW(x) =&[\widehat{S}P + V_0(I-P)][\widehat{T}Q + U_0(I-Q)](x)\\ =&[\widehat{S}P\widehat{T}Q + V_0(I-P)U_0(I-Q)](x)\\ =&[\widehat{S}\widehat{T}Q + V_0U_0(I-Q)](x)\\ =&Q(x) + (I-Q)(x)\\ =&x.\end{aligned}$$

That is $VW = I_X$, so $V \in G_r(Y,X)$, and this induces that $T \in G_lR(X,Y)$.

(2) Suppose that $T \in G_rR(X,Y)$. Then $T \in R(X,Y)$ and there exists $S \in G_l(Y,X)$ such that $TST = T$. Since $S \in G_l(Y,X)$, there exists $L \in G_r(X,Y)$ such that $LS = I_Y$.

The operators $U : \ker(T) \longrightarrow Y/T(X)$ and $V : Y/T(X) \longrightarrow \ker(T)$ are defined as follows respectively:

$$U(x) = L(x) + T(X), \text{ for all } x \in \ker(T);$$

$$V(y + T(X)) = (I - ST)S(I - TS)(y), \text{ for all } y + T(X) \in Y/T(X).$$

For all $y + T(X) \in Y/T(X)$, we can check that

$$\begin{aligned}UV(y + T(X)) =&U(I - ST)S(I - TS)(y)\\ =&U(S - STS)(y)\\ =&L(S - STS)(y) + T(X)\\ =&(LS - LSTS)(y) + T(X)\\ =&y - TS(y) + T(X)\\ =&y + T(X).\end{aligned}$$

That is, $UV = I_{Y/T(X)}$, hence $U \in G_r(\ker(T), Y/T(X))$. This proves $Y/T(X) \preceq \ker(T)$.

Conversely, suppose that $T \in R(X,Y)$ and $Y/T(X) \preceq \ker(T)$. Then there exists $S \in B(Y,X)$ such that $TST = T$. Let $P = TS \in B(Y)$ and $Q = ST \in B(X)$. Therefore P is the projection from Y onto $T(X)$ and $I-Q$ is the projection from X onto $\ker(T)$.

The operators $\widehat{T} : Q(X) \to P(Y)$ and $\widehat{S} : P(Y) \to Q(X)$ are defined as follows respectively:

$$\widehat{T}(Q(x)) = T(Q(x)), \text{ for all } x \in X;$$

$$\widehat{S}(P(y)) = S(P(y)), \text{ for all } y \in Y.$$

Since $Y/T(X) = Y/P(Y) \approx \ker(P)$, there exists $U_0 \in G_r(\ker(Q), \ker(P))$. Let $V_0 \in G_l(\ker(P), \ker(Q))$ be a right inverse of U_0.

The operator $V : Y \to X$ is defined by

$$V(y) = \widehat{S}P(y) + V_0(I - P)(y), \text{ for all } y \in Y.$$

Hence for all $x \in X$, we have

$$\begin{aligned} TVT(x) =& T[\widehat{S}P(Tx) + V_0(I-P)(Tx)] \\ =& T\widehat{S}P(Tx) \\ =& T\widehat{S}T(x) \\ =& TST(x) \\ =& T(x), \end{aligned}$$

that is $TVT = T$.

The operator $W : X \to Y$ is defined by

$$W(x) = \widehat{T}Q(x) + U_0(I-Q)(x), \text{ for all } x \in X.$$

Then for all $x \in X$, we can obtain that

$$\begin{aligned} WV(y) =& [\widehat{T}Q + U_0(I-Q)][\widehat{S}P + V_0(I-P)](y) \\ =& [\widehat{T}Q\widehat{S}P + U_0(I-Q)V_0(I-P)](y) \\ =& [\widehat{T}\widehat{S}P + U_0V_0(I-P)](y) \\ =& P(y) + (I-P)(y) \\ =& y. \end{aligned}$$

That is $WV = I_Y$, so $V \in G_l(Y, X)$, and this induces that $T \in G_rR(X, Y)$.

For $T \in \Phi_l(X, Y)$ with $\operatorname{ind}(T) \leq 0$, it is easy to know that $T \in R(X, Y)$ and there exists a finite-dimensional subspace W of $Y/T(X)$ such that $\ker(T) \approx W$. Let $J_1 : \ker(T) \longrightarrow W$ be the above isomorphic and $J_2 : W \longrightarrow Y/T(X)$ be the naturally embedded operator. Hence $U = J_2J_1 \in G_l(\ker(T), Y/T(X))$, that is, $\ker(T) \preceq Y/T(X)$, and this proves (2.1). Similarly, we can get (2.2). □

Similar to the proof of Theorem 2.7, we can show the following:

Theorem 2.8. (1) $T \in \Phi_lR(X, Y) \Longleftrightarrow T \in R(X, Y)$ *and* $\ker(T) \preceq_e Y/T(X)$.
(2) $T \in \Phi_rR(X, Y) \Longleftrightarrow T \in R(X, Y)$ *and* $Y/T(X) \preceq_e \ker(T)$.
(3) $T \in \Phi R(X, Y) \Longleftrightarrow T \in R(X, Y)$ *and* $\ker(T) \approx_e Y/T(X)$.

Next, we turn to the discussion of holomorphical versions.

For $T \in B(X)$, we say that T is semi-regular, in symbol $T \in S(X)$, if $T(X)$ is closed and $\ker(T) \subseteq T^n(X)$ for all $n \in \mathbb{N}$. If $T \in R(X) \cap S(X)$, then T is called an Saphar operator. This class of operators has been studied by Saphar [14] (see also [3]). Operators in this class have an important property: $T \in B(X)$ is a Saphar operator if and only if there exist a neighborhood $U \subseteq \mathbb{C}$ of 0 and a holomorphic function $F : U \longrightarrow B(X)$ such that

$$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I, \tag{2.3}$$

for all $\lambda \in U$. For its proof see [11, Théorème 2.6] or [15, Theorem 1.4]. If $F(\lambda) \in G(X)$ for all $\lambda \in U$ in (2.3), we say that T is holomorphically decomposably regular, in symbol $T \in HG(X)$. If $F(\lambda) \in \Phi(X)$ for all $\lambda \in U$ in (2.3), we say that T is holomorphically decomposably Fredholm, in symbol $T \in H\Phi(X)$.

These two classes of operators are characterized in [18] and [2] (or [1]) as follows respectively:

$$HG(X) = S(X) \cap GR(X) = S(X) \cap R(X) \cap \overline{G(X)}; \tag{2.4}$$

$$H\Phi(X) = S(X) \cap \Phi R(X) = S(X) \cap R(X) \cap \overline{\Phi(X)}. \tag{2.5}$$

Another characterization can also be found in [18] and [1] respectively:

$$\begin{aligned} T \in HG(X) \iff & \text{there exist some } R \in G(X) \text{ and sequences} \\ & \{S_n\}_{n=1}^{\infty} \subseteq G(X), \{T_n\}_{n=1}^{\infty} \subseteq G(X) \text{ such that} \\ & (T - S_n)T_n(T - S_n) = (T - S_n), TS_n = S_nT \\ & \text{for all } n \in \mathbb{N} \text{ and } \lim_{n\to\infty}(||S_n|| + ||T_n - R||) = 0; \end{aligned} \tag{2.6}$$

$$\begin{aligned} T \in H\Phi(X) \iff & \text{there exist some } R \in \Phi(X) \text{ and sequences} \\ & \{S_n\}_{n=1}^{\infty} \subseteq G(X), \{T_n\}_{n=1}^{\infty} \subseteq \Phi(X) \text{ such that} \\ & (T - S_n)T_n(T - S_n) = (T - S_n), TS_n = S_nT \\ & \text{for all } n \in \mathbb{N} \text{ and } \lim_{n\to\infty}(||S_n|| + ||T_n - R||) = 0. \end{aligned} \tag{2.7}$$

Definition 2.9. (1) An operator $T \in B(X)$ is said to be holomorphically left decomposably regular, in symbol $T \in HG_lR(X)$, if $F(\lambda) \in G_r(X)$ for all $\lambda \in U$ in (2.3);

(2) An operator $T \in B(X)$ is said to be holomorphically right decomposably regular, in symbol $T \in HG_rR(X)$, if $F(\lambda) \in G_l(X)$ for all $\lambda \in U$ in (2.3);

(3) An operator $T \in B(X)$ is said to be holomorphically left decomposably Fredholm, in symbol $T \in H\Phi_lR(X)$, if $F(\lambda) \in \Phi_r(X)$ for all $\lambda \in U$ in (2.3);

(4) An operator $T \in B(X)$ is said to be holomorphically right decomposably Fredholm, in symbol $T \in H\Phi_rR(X)$, if $F(\lambda) \in \Phi_l(X)$ for all $\lambda \in U$ in (2.3).

The following theorem generalizes (2.4) to holomorphically left (right) decomposably regular operators.

Theorem 2.10. (1) $HG_l(X) = S(X) \cap G_lR(X) = S(X) \cap R(X) \cap \overline{G_l(X)}$.

(2) $HG_r(X) = S(X) \cap G_rR(X) = S(X) \cap R(X) \cap \overline{G_r(X)}$.

Proof. (1) If $T \in HG_l(X)$, then there exist a neighborhood $U \subseteq \mathbb{C}$ of 0 and a holomorphic function $F : U \longrightarrow B(X)$ such that $F(\lambda) \in G_r(X)$ and $(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I$, for all $\lambda \in U$. Thus, we have that $T \in G_lR(X)$. By (2.3), we get that $T \in S(X)$.

Conversely, if $T \in S(X) \cap G_lR(X)$, then there exists $S \in G_r(X)$ such that $TST = T$. An function F is defined by $F(\lambda) = (I - \lambda S)^{-1}S$ for all $|\lambda| < ||S||^{-1}$. Therefore, $F(\lambda) \in G_r(X)$ for all $|\lambda| < ||S||^{-1}$. From Lemma 6 of [12, Chapter II, Section 13], we get that $(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I$ for all $|\lambda| < ||S||^{-1}$, and this induces that $T \in HG_l(X)$.

Consequently, $HG_l(X) = S(X) \cap G_lR(X)$. By Theorem 2.1(2), we have that $S(X) \cap G_lR(X) = S(X) \cap R(X) \cap \overline{G_l(X)}$, and this completes the proof of (1).

(2) It can be proved similarly to (1). ∎

The next theorem, whose proof is similar to Theorem 2.10, generalizes (2.5) to holomorphically left (right) decomposably Fredholm operators.

Theorem 2.11. (1) $H\Phi_l(X) = S(X) \cap \Phi_l R(X) = S(X) \cap R(X) \cap \overline{\Phi_l(X)}$.

(2) $H\Phi_r(X) = S(X) \cap \Phi_r R(X) = S(X) \cap R(X) \cap \overline{\Phi_r(X)}$.

The following theorem generalizes (2.6) to holomorphically left (right) decomposably regular operators.

Theorem 2.12. (1) $T \in HG_l(X) \iff$ *there exist some* $R \in G_r(X)$ *and sequences* $\{S_n\}_{n=1}^{\infty} \subseteq G(X), \{T_n\}_{n=1}^{\infty} \subseteq G_r(X)$ *such that* $TS_n = S_nT, (T - S_n)T_n(T - S_n) = T - S_n$ *for all* $n \in \mathbb{N}$ *and* $\lim\limits_{n\to\infty}(||S_n|| + ||T_n - R||) = 0$.

(2) $T \in HG_r(X) \iff$ *there exist some* $R \in G_l(X)$ *and sequences* $\{S_n\}_{n=1}^{\infty} \subseteq G(X), \{T_n\}_{n=1}^{\infty} \subseteq G_l(X)$ *such that* $TS_n = S_nT, (T - S_n)T_n(T - S_n) = T - S_n$ *for all* $n \in \mathbb{N}$ *and* $\lim\limits_{n\to\infty}(||S_n|| + ||T_n - R||) = 0$.

Proof. (1) If $T \in HG_l(X)$, then there exist an open disc $\mathbb{D}(0, r) \subseteq \mathbb{C}$ and a holomorphic function $F : \mathbb{D}(0, r) \to B(X)$ such that $F(\lambda) \in G_r(X)$ and $(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I$, for all $\lambda \in \mathbb{D}(0, r)$. Let $\{k_n\}_{n=1}^{\infty}$ be a sequence such that $\frac{1}{k_n} < r$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \frac{1}{k_n} = 0$. Let $R = F(0)$, $S_n = \frac{1}{k_n} I$ and $T_n = F(\frac{1}{k_n})$ for all $n \in \mathbb{N}$. It is easy to see that $R \in G_r(X)$, $\{S_n\}_{n=1}^{\infty} \subseteq G(X)$, $\{T_n\}_{n=1}^{\infty} \subseteq G_r(X)$, $TS_n = S_nT$ and $(T - S_n)T_n(T - S_n) = T - S_n$. Since F is continuous, we have that $\lim\limits_{n\longrightarrow\infty}(||S_n|| + ||T_n - R||) = 0$.

Conversely, if there exist some $R \in G_r(X)$ and sequences $\{S_n\}_{n=1}^{\infty} \subseteq G(X)$, $\{T_n\}_{n=1}^{\infty} \subseteq G_r(X)$ such that $TS_n = S_nT, (T - S_n)T_n(T - S_n) = T - S_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty}(||S_n|| + ||T_n - R||) = 0$. Then it follows from Theorem 9 in [7] that $T \in R(X) \cap S(X) \subseteq S(X)$. Since $(T - S_n)T_n(T - S_n) = T - S_n$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty}(||S_n|| + ||T_n - R||) = 0$, $TRT = T$. Further, we have that $T \in G_l R(X)$ since $R \in G_r(X)$. Hence by Theorem 2.10(1), we know that $T \in HG_l(X)$.

(2) It can be proved similarly to (1). □

The next theorem, whose proof is similar to Theorem 2.12, generalizes (2.7) to holomorphically left (right) decomposably Fredholm operators.

Theorem 2.13. (1) $T \in H\Phi_l(X) \iff$ *there exist some* $R \in \Phi_r(X)$ *and sequences* $\{S_n\}_{n=1}^{\infty} \subseteq G(X), \{T_n\}_{n=1}^{\infty} \subseteq \Phi_r(X)$ *such that* $TS_n = S_nT, (T - S_n)T_n(T - S_n) = T - S_n$ *for all* $n \in \mathbb{N}$ *and* $\lim\limits_{n\to\infty}(||S_n|| + ||T_n - R||) = 0$.

(2) $T \in H\Phi_r(X) \iff$ *there exist some* $R \in \Phi_l(X)$ *and sequences* $\{S_n\}_{n=1}^{\infty} \subseteq G(X)$, $\{T_n\}_{n=1}^{\infty} \subseteq \Phi_l(X)$ *such that* $TS_n = S_nT, (T - S_n)T_n(T - S_n) = T - S_n$ *for all* $n \in \mathbb{N}$ *and* $\lim\limits_{n\to\infty}(||S_n|| + ||T_n - R||) = 0$.

3. Topological interiors and closures

As the direct applications of the characterizations obtained in the above section, we can get the following:

Theorem 3.1. (1) $\overline{G_l R(X, Y)} = \overline{G_l(X, Y)}$.

(2) $\overline{G_rR(X,Y)} = \overline{G_r(X,Y)}$.
(3) $\overline{GR(X,Y)} = \overline{G(X,Y)}$.
(4) $\overline{\Phi_lR(X,Y)} = \overline{\Phi_l(X,Y)}$.
(5) $\overline{\Phi_rR(X,Y)} = \overline{\Phi_r(X,Y)}$.
(6) $\overline{\Phi R(X,Y)} = \overline{\Phi(X,Y)}$.

Proof. Here we only prove (1), and the rest is similar.

(1) By Theorem 2.1(2), we have that

$$G_l(X,Y) \subseteq \overline{G_lR(X,Y)} = R(X,Y) \cap \overline{G_l(X,Y)} \subseteq \overline{G_l(X,Y)},$$

which implies that $\overline{G_lR(X,Y)} = \overline{G_l(X,Y)}$. □

Theorem 3.2. (1) $\overline{HG_l(X)} = \overline{G_l(X)}$.
(2) $\overline{HG_r(X)} = \overline{G_r(X)}$.
(3) $\overline{HG(X)} = \overline{G(X)}$.
(4) $\overline{H\Phi_l(X)} = \overline{\Phi_l(X)}$.
(5) $\overline{H\Phi_r(X)} = \overline{\Phi_r(X)}$.
(6) $\overline{H\Phi(X)} = \overline{\Phi(X)}$.

Proof. Part (6) has been proved in [1, Theorem 2.2(2)], and parts (4) and (5) can be proved similarly to it, so we omit the proofs of them here.

(1) From Theorem 2.10(1), we can get that

$$G_l(X) \subseteq HG_l(X) = R(X) \cap S(X) \cap \overline{G_l(X)} \subseteq \overline{G_l(X)},$$

which implies that $\overline{HG_l(X)} = \overline{G_l(X)}$.

Parts (2) and (3) can be proved similarly. □

Next, we compute the topological interiors of the classes of operators studied in this paper. We begin by stating two lemmas which will be used repeatedly throughout the sequel.

Lemma 3.3. *Sets $G_l(X,Y)\backslash G_r(X,Y)$ and $G_r(X,Y)\backslash G_l(X,Y)$ are open.*

Proof. Suppose that $T \in G_l(X,Y)\backslash G_r(X,Y)$. Then there exists $S \in G_r(Y,X)$ such that $ST = I_X$. For $R \in B(X,Y)$ such that $||R-T|| < ||S||^{-1}$, we have that $||I_X - SR|| \leq ||S||||R-T|| < 1$, therefore $U = SR \in G(X)$, so, $U^{-1}SR = I_X$. If $R \in G(X,Y)$, then $R(U^{-1}S) = I_Y$. Since $U^{-1}SR = I_X$, $S(RU^{-1}) = I_X$. Hence $RU^{-1} \in G(X,Y)$. Since $RU^{-1} = RU^{-1}I_X = RU^{-1}ST = T$, $T \in G(X,Y)$, a contradiction. Hence $G_l(X,Y)\backslash G_r(X,Y)$ is open.

We can prove similarly that $G_r(X,Y)\backslash G_l(X,Y)$ is open. □

Lemma 3.4. *Sets $\Phi_l(X,Y)\backslash \Phi_r(X,Y)$ and $\Phi_r(X,Y)\backslash \Phi_l(X,Y)$ are open.*

Proof. Suppose that $T \in \Phi_l(X,Y)\backslash \Phi_r(X,Y)$. Then there exist $S \in \Phi_r(Y,X)$ and $K_1 \in K(X)$ such that $ST = I_X - K_1$. For $R \in B(X,Y)$ such that $||R-T|| < ||S||^{-1}$, we have that $||I_X - (SR+K_1)|| \leq ||S||||R-T|| < 1$, therefore $U = SR + K_1 \in G(X)$, that is, $U^{-1}(SR+K_1) = I_X$. Hence, $U^{-1}SR = I_X - U^{-1}K_1$, thus $R \in \Phi_l(X,Y)$. If $R \in \Phi(X,Y)$, then there exists $K_2 \in K(Y)$ such that $(RU^{-1})S = R(U^{-1}S) = I_Y + K_2$. Since $U^{-1}SR = I_X - U^{-1}K_1$, $S(RU^{-1}) =$

$I_X - K_1U^{-1}$, hence $RU^{-1} \in \Phi(X,Y)$. Since $RU^{-1} = RU^{-1}I_X = RU^{-1}(I_X - K_1 + K_1) = RU^{-1}ST + RU^{-1}K_1 = (I_Y + K_2)T + RU^{-1}K_1 = T + K_2T + RU^{-1}K_1$, $T \in \Phi(X,Y)$, a contradiction. Hence $\Phi_l(X,Y)\backslash\Phi_r(X,Y)$ is open.

We can prove similarly that $\Phi_r(X,Y)\backslash\Phi_l(X,Y)$ is open. □

By using a result of Goldberg [8, Theorem V.2.6], Schmoeger [16, Theorem 6] proved that

$$\mathrm{int}(R(X)) = \Phi_l(X) \cup \Phi_r(X).$$

Moreover, noting that [8, Theorem V.2.6] also holds in the case that X and Y are two different Banach spaces, we can get that

$$\mathrm{int}(R(X,Y)) = \Phi_l(X,Y) \cup \Phi_r(X,Y).$$

Parts (3) and (6) of the following theorem extend [18, Theorem 2.1] and [18, Theorem 2.2(2)], respectively, to the case that X and Y are two different Banach spaces.

Theorem 3.5. (1) $\mathrm{int}(G_lR(X,Y)) = \{T \in \Phi_l(X,Y) : \mathrm{ind}(T) \le 0\}$.
(2) $\mathrm{int}(G_rR(X,Y)) = \{T \in \Phi_r(X,Y) : \mathrm{ind}(T) \ge 0\}$.
(3) $\mathrm{int}(GR(X,Y)) = \{T \in \Phi(X,Y) : \mathrm{ind}(T) = 0\}$.
(4) $\mathrm{int}(\Phi_lR(X,Y)) = \Phi_l(X,Y)$.
(5) $\mathrm{int}(\Phi_rR(X,Y)) = \Phi_r(X,Y)$.
(6) $\mathrm{int}(\Phi R(X,Y)) = \Phi(X,Y)$.

Proof. We will prove (1) and (4), omitting the similar proofs of the others.

(1) Since $\{T \in \Phi_l(X,Y) : \mathrm{ind}(T) \le 0\}$ is open and $\{T \in \Phi_l(X,Y) : \mathrm{ind}(T) \le 0\} \subseteq G_lR(X,Y)$ (see (2.1)), we get that

$$\{T \in \Phi_l(X,Y) : \mathrm{ind}(T) \le 0\} \subseteq \mathrm{int}(G_lR(X,Y)).$$

For the converse inclusion, suppose that $T \in \mathrm{int}(G_lR(X,Y)) = \mathrm{int}(R(X,Y) \cap \overline{G_l(X,Y)})$, then $T \in \mathrm{int}(R(X,Y)) = \Phi_l(X,Y) \cup \Phi_r(X,Y)$. Since $T \in \overline{G_l(X,Y)}$, there exists a sequence $\{T_n\}_{n=1}^{\infty} \subseteq G_l(X,Y)$ for which $||T - T_n|| \to 0$ $(n \to \infty)$. We can claim that $T \in \Phi_l(X,Y)$. If not, $T \notin \Phi_l(X,Y)$. Noting that $T \in \Phi_l(X,Y) \cup \Phi_r(X,Y)$, we have $T \in \Phi_r(X,Y)\backslash\Phi_l(X,Y)$. By Lemma 3.4, we know that $T_n \in \Phi_r(X,Y)\backslash\Phi_l(X,Y)$ for enough large $n \in \mathbb{N}$, and this contradicts with the fact that $\{T_n\}_{n=1}^{\infty} \subseteq G_l(X,Y) \subseteq \Phi_l(X,Y)$. From the continuation of the index (cf. [10, Proposition 2.c.9]), we get that $\mathrm{ind}(T) = \mathrm{ind}(T_n) \le 0$ for enough large $n \in \mathbb{N}$, therefore $\mathrm{int}(G_lR(X,Y)) \subseteq \{T \in \Phi_l(X,Y) : \mathrm{ind}(T) \le 0\}$.

Consequently, $\mathrm{int}(G_lR(X,Y)) = \{T \in \Phi_l(X,Y) : \mathrm{ind}(T) \le 0\}$.

(4) Since $\Phi_l(X,Y)$ is open and $\Phi_l(X,Y) \subseteq \Phi_lR(X,Y)$, we get that $\Phi_l(X,Y) \subseteq \mathrm{int}(\Phi_lR(X,Y))$.

For the converse inclusion, suppose that $T \in \mathrm{int}(\Phi_lR(X,Y)) = \mathrm{int}(R(X,Y) \cap \overline{\Phi_l(X,Y)})$. Then $T \in \mathrm{int}(R(X,Y)) = \Phi_l(X,Y) \cup \Phi_r(X,Y)$. Since $T \in \overline{\Phi_l(X,Y)}$, there exists a sequence $\{T_n\}_{n=1}^{\infty} \subseteq \Phi_l(X,Y)$ such that $||T - T_n|| \to 0$ $(n \to \infty)$. We can claim that $T \in \Phi_l(X,Y)$. If not, $T \notin \Phi_l(X,Y)$. Noting that $T \in \Phi_l(X,Y) \cup \Phi_r(X,Y)$, we have $T \in \Phi_r(X,Y)\backslash\Phi_l(X,Y)$. By Lemma 3.4, we know that $T_n \in \Phi_r(X,Y)\backslash\Phi_l(X,Y)$ for enough large $n \in \mathbb{N}$, and this contradicts with the fact that $\{T_n\}_{n=1}^{\infty} \subseteq \Phi_l(X,Y)$, therefore $\mathrm{int}(G_lR(X,Y)) \subseteq \Phi_l(X,Y)$.

Consequently, $\mathrm{int}(\Phi_lR(X,Y)) = \Phi_l(X,Y)$. □

Part (6) of the following theorem generalizes [1, Theorem 2.2(1)] by computing precisely the topological interior of the class of the holomorphically decomposably Fredholm operators.

Theorem 3.6. (1) $\mathrm{int}(HG_l(X)) = G_l(X)$.
(2) $\mathrm{int}(HG_r(X)) = G_r(X)$.
(3) $\mathrm{int}(HG(X)) = G(X)$.
(4) $\mathrm{int}(H\Phi_l(X)) = (G_l(X) \cup G_r(X)) \cap \Phi_l(X)$.
(5) $\mathrm{int}(H\Phi_r(X)) = (G_l(X) \cup G_r(X)) \cap \Phi_r(X)$.
(6) $\mathrm{int}(H\Phi(X)) = (G_l(X) \cup G_r(X)) \cap \Phi(X)$.

Proof. We will prove (1) and (4), omitting the similar proofs of the others.

(1) Since $G_l(X)$ is open and $G_l(X) \subseteq HG_l(X)$, we get that $G_l(X) \subseteq \mathrm{int}(HG_l(X))$.

For the converse inclusion, suppose that $T \in \mathrm{int}(HG_l(X)) = \mathrm{int}(R(X) \cap S(X) \cap \overline{G_l(X)})$. Then $T \in \mathrm{int}(R(X) \cap S(X)) = G_l(X) \cup G_r(X)$ by [17, Theorem 2(b)]. Since $T \in \overline{G_l(X)}$, there exists a sequence $\{T_n\}_{n=1}^{\infty} \subseteq G_l(X)$ for which $||T - T_n|| \to 0$ $(n \to \infty)$. We can claim that $T \in G_l(X)$. If not, $T \notin G_l(X)$. Noting that $T \in G_l(X) \cup G_r(X)$, we have $T \in G_r(X) \backslash G_l(X)$. By Lemma 3.3, we know that $T_n \in G_r(X) \backslash G_l(X)$ for enough large $n \in \mathbb{N}$, and this contradicts with the fact that $\{T_n\}_{n=1}^{\infty} \subseteq G_l(X)$, thus $\mathrm{int}(HG_l(X)) \subseteq G_l(X)$.

Consequently, $\mathrm{int}(HG_l(X)) = G_l(X)$.

(4) Since $(G_l(X) \cup G_r(X)) \cap \Phi_l(X)$ is open and $(G_l(X) \cup G_r(X)) \cap \Phi_l(X) \subseteq H\Phi_l(X)$, we get that $(G_l(X) \cup G_r(X)) \cap \Phi_l(X) \subseteq \mathrm{int}(H\Phi_l(X))$.

For the converse inclusion, suppose that $T \in \mathrm{int}(H\Phi_l(X)) = \mathrm{int}(R(X) \cap S(X) \cap \overline{\Phi_l(X)})$. Then $T \in \mathrm{int}(R(X) \cap S(X)) = G_l(X) \cup G_r(X)$. Since $T \in \overline{\Phi_l(X)}$, there exists a sequence $\{T_n\}_{n=1}^{\infty} \subseteq \Phi_l(X)$ for which $||T_n - T|| \to 0$ $(n \to \infty)$. We can claim that $T \in \Phi_l(X)$. If not, $T \notin \Phi_l(X)$. Noting that $T \in G_l(X) \cup G_r(X) \subseteq \Phi_l(X) \cup \Phi_r(X)$, we have $T \in \Phi_r(X) \backslash \Phi_l(X)$. By Lemma 3.4, we know that $T_n \in \Phi_r(X) \backslash \Phi_l(X)$ for enough large $n \in \mathbb{N}$, and this contradicts with the fact that $\{T_n\}_{n=1}^{\infty} \subseteq \Phi_l(X)$, thus $\mathrm{int}(H\Phi_l(X)) \subseteq (G_l(X) \cup G_r(X)) \cap \Phi_l(X)$.

Consequently, $\mathrm{int}(H\Phi_l(X)) = (G_l(X) \cup G_r(X)) \cap \Phi_l(X)$. □

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