Banach J. Math. Anal. 6 (2012), no. 2, 168–179

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

AN INTERPOLATION THEOREM FOR SUBLINEAR OPERATORS ON NON-HOMOGENEOUS METRIC MEASURE SPACES

HAIBO LIN[1] AND DONGYONG YANG[2*]

Communicated by P. E. Jorgensen

ABSTRACT. Let $(\mathcal{X}, d, \mu)$ be a metric measure space and satisfy the so-called upper doubling condition and the geometrically doubling condition. In this paper, the authors establish an interpolation result that a sublinear operator which is bounded from the Hardy space $H^1(\mu)$ to $L^{1,\infty}(\mu)$ and from $L^\infty(\mu)$ to the BMO-type space $\mathrm{RBMO}(\mu)$ is also bounded on $L^p(\mu)$ for all $p \in (1, \infty)$. This extension is not completely straightforward and improves the existing result.

1. Introduction

The classical theory of Calderón-Zygmund operators began with the study of the convolution operators on $\mathbb{R}$. Later it has played an important role in harmonic analysis and has been developed into a large branch of analysis on metric spaces. One of the most interesting cases is the so-called space of homogeneous type introduced by Coifman and Weiss in [3]; see also [4, 5, 6]. Recall that a metric space $(\mathcal{X}, d)$ equipped with a nonnegative Borel measure μ is called a *space of homogeneous type* if $(\mathcal{X}, d, \mu)$ satisfies the following *measure doubling condition* that there exists a positive constant C_μ, depending on μ, such that for any ball $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$0 < \mu(B(x, 2r)) \le C_\mu \mu(B(x, r)). \tag{1.1}$$

Date: Received: 29 January 2012; Accepted: 2 May 2012.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 42B35; Secondary 42B25, 47B38.
Key words and phrases. RBMO(μ), upper doubling, geometrically doubling, sublinear, interpolation.

As is known to all, space of homogeneous type is a natural setting for Calderón-Zygmund operators and function spaces. Typical examples of spaces of homogeneous type include Euclidean spaces, Euclidean spaces with weighted measures satisfying the doubling condition (1.1), Heisenberg groups, connected and simply connected nilpotent Lie groups.

The measure doubling condition (1.1) above plays a key role in the classical theory of Calderón-Zygmund operators. However, recently, many classical results concerning the theory of Calderón-Zygmund operators and function spaces have been proved still valid if the doubling condition is replaced by a less demanding condition such as the polynomial growth condition; see, for example [14, 16, 17, 15, 18] and the references therein. In particular, let μ be a non-negative Radon measure on $\mathbb{R}^n$ which only satisfies the *polynomial growth condition* that there exist positive constants C and $\kappa \in (0, n]$ such that for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$\mu(\{y \in \mathbb{R}^n : |x - y| < r\}) \leq Cr^\kappa. \tag{1.2}$$

Such a measure does not need to satisfy the doubling condition (1.1). We mention that the analysis with non-doubling measures played a striking role in solving the long-standing open Painlevé's problem by Tolsa in [18].

Because measures satisfying (1.2) are only different, not more general than measures satisfying (1.1), the Calderón-Zygmund theory with non-doubling measures is not in all respects a generalization of the corresponding theory of spaces of homogeneous type. To include the spaces of homogeneous type and Euclidean spaces with a non-negative Radon measure satisfying a polynomial growth condition, Hytönen [8] introduced a new class of metric measure spaces which satisfy the so-called upper doubling condition and the geometrically doubling condition (see, respectively, Definitions 2.1 and 2.3 below), and a notion of the regularized BMO space, namely, $\mathrm{RBMO}(\mu)$ (see Definition 2.5 below). Since then, more and more papers focus on this new class of spaces; see, for example [11, 12, 10, 1, 9, 7, 13].

Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying the upper doubling condition and the geometrically doubling condition. In [10], the atomic Hardy space $H^1(\mu)$ (see Definition 2.8 below) was studied and the duality between $H^1(\mu)$ and $\mathrm{RBMO}(\mu)$ of Hytönen was established. Some of results in [10] were also independently obtained by Anh and Duong [1] via different approaches. Moreover, Anh and Duong [1, Theorem 6.4] established an interpolation result that a linear operator which is bounded from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $\mathrm{RBMO}(\mu)$ is also bounded on $L^p(\mu)$ for all $p \in (1, \infty)$.

Note that there are also many important operators in harmonic analysis which are sublinear, such as the Hardy-Littlewood maximal operator, the maximal singular integral operator, the Marcinkiewicz integral and so on. Then it is natural to ask if the interpolation theorem in [1] is true for sublinear operator in the current setting $(\mathcal{X}, d, \mu)$. The purpose of this paper is to generalize and improve the interpolation result for linear operators in [1] to sublinear operators, which is stated as follows.

Theorem 1.1. *Let T be a sublinear operator that is bounded from $L^\infty(\mu)$ to $\mathrm{RBMO}(\mu)$ and from $H^1(\mu)$ to $L^{1,\infty}(\mu)$. Then T extends boundedly to $L^p(\mu)$ for every $p \in (1, \infty)$.*

In Section 2, we collect preliminaries we need. In Section 3, for $r \in (0,1)$, we first show that the maximal function $M_r^\sharp(f)$, which is a variant of the sharp maximal function $M^\sharp(f)$ in [1], is bounded from $\mathrm{RBMO}(\mu)$ to $L^\infty(\mu)$, then we establish a weak type estimate between the doubling maximal function $N(f)$ and $M^\sharp(f)$, and we also establish a weak type estimate for $N_r(f)$ with $r \in (0,1)$, a variant of $N(f)$. Using these results we establish Theorem 1.1. We remark that the method for the proof of Theorem 1.1 is different from that of [1, Theorem 6.4]. Precisely, in the proof of [1, Theorem 6.4], the fact that the composite operator $M^\sharp \circ T$ of the sharp maximal function $M^\sharp$ and a linear operator T is a sublinear operator was used. However, as far as we know, when T is sublinear, whether the composite operator $M^\sharp \circ T$ is a sublinear operator is unclear and so the proof of [1, Theorem 6.4] is not available.

Throughout this paper, we denote by C a positive constant which is independent of the main parameters involved but may vary from line to line. The subscripts of a constant indicate the parameters it depends on. The notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \le Cg$. Also, for a μ-measurable set E, χ_E denotes its characteristic function.

2. Preliminaries

In this section, we will recall some necessary notions and notation and the Calderón-Zygmund decomposition which was established in [1]. We begin with the definition of upper doubling space in [8].

Definition 2.1. A metric measure space $(\mathcal{X}, d, \mu)$ is called *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda: \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant C_λ such that for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is non-decreasing, and for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le \lambda(x, r) \le C_\lambda \lambda(x, r/2).$$

Remark 2.2. (i) Obviously, a space of homogeneous type is a special case of the upper doubling spaces, where one can take the dominating function $\lambda(x, r) \equiv \mu(B(x, r))$. Moreover, let μ be a non-negative Radon measure on $\mathbb{R}^n$ which only satisfies the polynomial growth condition (1.2). By taking $\lambda(x, r) \equiv Cr^\kappa$, we see that $(\mathbb{R}^n, |\cdot|, \mu)$ is also an upper doubling measure space.

(ii) It was proved in [10] that there exists a dominating function $\widetilde{\lambda}$ related to λ satisfying the property that there exists a positive constant $C_{\widetilde{\lambda}}$ such that $\widetilde{\lambda} \le \lambda$, $C_{\widetilde{\lambda}} \le C_\lambda$, and for all $x, y \in \mathcal{X}$ with $d(x, y) \le r$,

$$\widetilde{\lambda}(x, r) \le C_{\widetilde{\lambda}} \widetilde{\lambda}(y, r). \tag{2.1}$$

Based on this, in this paper, we *always assume* that the dominating function λ also satisfies (2.1).

Throughout the whole paper, we also *always assume* that the underlying metric space $(\mathcal{X}, d)$ satisfies the following geometrically doubling condition introduced in [8].

Definition 2.3. A metric space $(\mathcal{X}, d)$ is called *geometrically doubling* if there exists some $N_0 \in \mathbb{N}^+ \equiv \{1, 2, \cdots\}$ such that for any ball $B(x, r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

The following coefficients $\delta(B, S)$ for all balls B and S were introduced in [8] as analogues of Tolsa's numbers $K_{Q,R}$ in [16]; see also [10].

Definition 2.4. For all balls $B \subset S$, let

$$\delta(B, S) \equiv \int_{(2S)\setminus B} \frac{d\mu(x)}{\lambda(c_B, d(x, c_B))}.$$

where and in that follows, for a ball $B \equiv B(c_B, r_B)$ and $\rho \in (0, \infty)$, $\rho B \equiv B(c_B, \rho r_B)$.

In what follows, for each $p \in (0, \infty)$, $L^p_{\rm loc}(\mu)$ denotes the *set of all functions f such that $|f|^p$ is μ-locally integrable.*

Definition 2.5. Let $\eta \in (1, \infty)$ and $p \in (0, \infty)$. A function $f \in L^p_{\rm loc}(\mu)$ is said to be in the *space* $\mathrm{RBMO}^p_\eta(\mu)$ if there exist a non-negative constant C and a complex number f_B for any ball B such that for all balls B,

$$\frac{1}{\mu(\eta B)} \int_B |f(y) - f_B|^p \, d\mu(y) \le C^p$$

and that for all balls $B \subset S$,

$$|f_B - f_S| \le C[1 + \delta(B, S)].$$

Moreover, the $\mathrm{RBMO}^p_\eta(\mu)$ *norm* of f is defined to be the minimal constant C as above and denoted by $\|f\|_{\mathrm{RBMO}^p_\eta(\mu)}$.

When $p = 1$, we write $\mathrm{RBMO}^1_\eta(\mu)$ simply by $\mathrm{RBMO}(\mu)$, which was introduced by Hytönen in [8]. Moreover, the spaces $\mathrm{RBMO}^p_\eta(\mu)$ and $\mathrm{RBMO}(\mu)$ coincide with equivalent norms, which is the special case of [7, Corollary 2.1].

Proposition 2.6. *Let $\eta \in (1, \infty)$ and $p \in (0, \infty)$. The spaces* $\mathrm{RBMO}^p_\eta(\mu)$ *and* $\mathrm{RBMO}(\mu)$ *coincide with equivalent norms.*

Remark 2.7. It was proved in [8, Lemma 4.6] that the space $\mathrm{RBMO}(\mu)$ is independent of the choice of η. By this and Proposition 2.6, it is obvious that the space $\mathrm{RBMO}^p_\eta(\mu)$ is independent of the choice of η.

We now recall the definition of the atomic Hardy space introduced in [10]; see also [1].

Definition 2.8. Let $\rho \in (1, \infty)$ and $p \in (1, \infty]$. A function $b \in L^1(\mu)$ is called a $(p, 1)_\lambda$-*atomic block* if

(i) there exists some ball B such that $\operatorname{supp}(b) \subset B$;
(ii) $\int_{\mathcal{X}} b(x) \, d\mu(x) = 0$;

(iii) for $j = 1, 2$, there exist functions a_j supported on balls $B_j \subset B$ and $\lambda_j \in \mathbb{C}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_j\|_{L^p(\mu)} \le [\mu(\rho B_j)]^{1/p-1}[1+\delta(B_j,\, B)]^{-1}.$$

Moreover, let

$$|b|_{H^{1,\,p}_{atb}(\mu)} \equiv |\lambda_1| + |\lambda_2|.$$

A function $f \in L^1(\mu)$ is said to belong to the *atomic Hardy space* $H^{1,\,p}_{atb}(\mu)$ if there exist $(p,\, 1)_\lambda$-atomic blocks $\{b_j\}_{j\in\mathbb{N}}$ such that $f = \sum_{j=1}^\infty b_j$ and

$$\sum_{j=1}^{\infty} |b_j|_{H^{1,\,p}_{atb}(\mu)} < \infty.$$

The *norm* of f in $H^{1,\,p}_{atb}(\mu)$ is defined by

$$\|f\|_{H^{1,\,p}_{atb}(\mu)} \equiv \inf\left\{\sum_j |b_j|_{H^{1,\,p}_{atb}(\mu)}\right\},$$

where the infimum is taken over all the possible decompositions of f as above.

Remark 2.9. It was proved in [10] that for each $p \in (1,\, \infty]$, the atomic Hardy space $H^{1,\,p}_{atb}(\mu)$ is independent of the choice of ρ, and that for all $p \in (1,\, \infty)$, the spaces $H^{1,\,p}_{atb}(\mu)$ and $H^{1,\,\infty}_{atb}(\mu)$ coincide with equivalent norms. Thus, in the following, we denote $H^{1,\,p}_{atb}(\mu)$ simply by $H^1(\mu)$.

At the end of this section, we recall the $(\alpha,\, \beta)$-doubling property of some balls and the Calderón-Zygmund decomposition established by Anh and Duong [1, Theorem 6.3].

Given $\alpha,\, \beta \in (1,\, \infty)$, a ball $B \subset \mathcal{X}$ is called $(\alpha,\, \beta)$-*doubling* if $\mu(\alpha B) \le \beta\mu(B)$. It was proved in [8] that if a metric measure space $(\mathcal{X},\, d,\, \mu)$ is upper doubling and $\beta > C_\lambda^{\log_2 \alpha} \equiv \alpha^\nu$, then for every ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$ such that $\alpha^j B$ is $(\alpha,\, \beta)$-doubling. Moreover, let $(\mathcal{X},\, d)$ be geometrically doubling, $\beta > \alpha^n$ with $n \equiv \log_2 N_0$ and μ a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [8] also showed that for μ-almost every $x \in \mathcal{X}$, there exist arbitrarily small $(\alpha,\, \beta)$-doubling balls centered at x. Furthermore, the radius of these balls may be chosen to be of the form $\alpha^{-j} r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0,\, \infty)$. Throughout this paper, for any $\alpha \in (1,\, \infty)$ and ball B, $\widetilde{B}^\alpha$ denotes the *smallest* $(\alpha,\, \beta_\alpha)$-*doubling ball of the form* $\alpha^j B$ *with* $j \in \mathbb{Z}_+$, where

$$\beta_\alpha \equiv \max\left\{\alpha^{3n},\, \alpha^{3\nu}\right\} + 30^n + 30^\nu = \alpha^{3\max\{n,\,\nu\}} + 30^n + 30^\nu. \tag{2.2}$$

Lemma 2.10. *Let* $p \in [1,\, \infty)$, $f \in L^p(\mu)$ *and* $\ell \in (0, \infty)$ *(*$\ell > \ell_0$ *if* $\mu(\mathcal{X}) < \infty$, *where* $\ell_0 \equiv \gamma_0\|f\|_{L^1(\mu)}/\mu(\mathcal{X})$ *and* γ_0 *is any fixed positive constant satisfying that* $\gamma_0 > max\{C_\lambda^{3\log_2 6},\, 6^{3n}\}$, C_λ *is as in* (2.1) *and* $n = \log_2 N_0$*). Then*

(i) *there exists an almost disjoint family* $\{6B_j\}_j$ *of balls such that* $\{B_j\}_j$ *is pairwise disjoint,*

$$\frac{1}{\mu(6^2 B_j)} \int_{B_j} |f(x)|^p \, d\mu(x) > \frac{\ell^p}{\gamma_0} \quad \textit{for all } j,$$

$$\frac{1}{\mu(6^2\eta B_j)}\int_{\eta B_j}|f(x)|^p\,d\mu(x)\le\frac{\ell^p}{\gamma_0}\quad \textit{for all } j \textit{ and all } \eta>2,$$

and

$$|f(x)|\le\ell\quad \textit{for } \mu-\textit{almost every } x\in\mathcal{X}\setminus(\cup_j 6B_j);$$

(ii) *for each* j, *let* S_j *be a* $(3\times 6^2,\ C_\lambda^{\log_2(3\times 6^2)+1})$*-doubling ball concentric with* B_j *satisfying that* $r_{S_j}>6^2 r_{B_j}$, *and* $\omega_j\equiv\chi_{6B_j}/(\sum_k\chi_{6B_k})$. *Then there exists a family* $\{\varphi_j\}_j$ *of functions such that for each* j, $\operatorname{supp}(\varphi_j)\subset S_j$, φ_j *has a constant sign on* S_j *and*

$$\int_{\mathcal{X}}\varphi_j(x)\,d\mu(x)=\int_{6B_j}f(x)\omega_j(x)\,d\mu(x),$$

$$\sum_j|\varphi_j(x)|\le\gamma\ell\quad \textit{for } \mu-\textit{almost every } x\in\mathcal{X},$$

where γ *is some positive constant depending only on* $(\mathcal{X},\ \mu)$, *and there exists a positive constant* C, *independent of* f, ℓ *and* j, *such that*

$$\|\varphi_j\|_{L^\infty(\mu)}\mu(S_j)\le C\int_{\mathcal{X}}|f(x)\omega_j(x)|\,d\mu(x),$$

and if $p\in(1,\ \infty)$,

$$\left\{\int_{S_j}|\varphi_j(x)|^p\,d\mu(x)\right\}^{1/p}[\mu(S_j)]^{1/p'}\le\frac{C}{\ell^{p-1}}\int_{\mathcal{X}}|f(x)\omega_j(x)|^p\,d\mu(x);$$

(iii) *if for any* j, *choosing* S_j *in* (ii) *to be the smallest* $(3\times 6^2,\ C_\lambda^{\log_2(3\times 6^2)+1})$*-doubling ball of* $(3\times 6^2)B_j$, *then* $h\equiv\sum_j(f\omega_j-\varphi_j)\in H^{1,\,p}_{atb}(\mu)$ *and there exists a positive constant* C, *independent of* f *and* ℓ, *such that*

$$\|h\|_{H^{1,\,p}_{atb}(\mu)}\le\frac{C}{\ell^{p-1}}\|f\|^p_{L^p(\mu)}.$$

3. Proof Theorem 1.1

To prove Theorem 1.1, we also need some maximal functions in [8, 1] as follows. Let $f\in L^1_{\rm loc}(\mu)$ and $x\in\mathcal{X}$. The *doubling Hardy-Littlewood maximal function* $N(f)(x)$ and the *sharp maximal function* $M^\sharp(f)(x)$ are respectively defined by setting,

$$N(f)(x)\equiv\sup_{\substack{B\ni x\\ B(6,\,\beta_6)-\text{doubling}}}\frac{1}{\mu(B)}\int_B|f(y)|\,d\mu(y),$$

and

$$M^\sharp(f)(x)\equiv\sup_{B\ni x}\frac{1}{\mu(5B)}\int_B|f(y)-m_{\widetilde{B}^6}(f)|\,d\mu(y)$$
$$+\sup_{\substack{x\in B\subset S\\ B,\,S(6,\,\beta_6)-\text{doubling}}}\frac{|m_B(f)-m_S(f)|}{1+\delta(B,\,S)},$$

where for any $f \in L^1_{\rm loc}(\mu)$ and ball B, $m_B(f)$ means its average over B, namely, $m_B(f) \equiv \frac{1}{\mu(B)} \int_B f(x)\,d\mu(x)$. It was showed in [9, Lemma 2.3] that for any $p \in [1, \infty)$, Nf is bounded from $L^p(\mu)$ to $L^{p,\infty}(\mu)$.

Lemma 3.1. *Let $f \in \mathrm{RBMO}(\mu)$, $r \in (0,1)$ and $M^\sharp_r(f) \equiv [M^\sharp(|f|^r)]^{1/r}$. Then we have $M^\sharp_r f \in L^\infty(\mu)$, and moreover,*

$$\|M^\sharp_r f\|_{L^\infty(\mu)} \lesssim \|f\|_{\mathrm{RBMO}(\mu)}.$$

Proof. From Remark 2.7, we deduce that for any ball B,

$$\left|f_{\widetilde{B}^6} - m_{\widetilde{B}^6}(f)\right| \le \frac{1}{\mu(\widetilde{B}^6)} \int_{\widetilde{B}^6} \left|f(x) - f_{\widetilde{B}^6}\right| \,d\mu(x) \lesssim \|f\|_{\mathrm{RBMO}(\mu)}.$$

On the other hand, by Proposition 2.6 and Remark 2.7, we see that

$$\|f\|_{\mathrm{RBMO}(\mu)} \sim \|f\|_{\mathrm{RBMO}^r_5(\mu)}.$$

From these facts, it follows that

$$\begin{aligned}
&\frac{1}{\mu(5B)} \int_B \left||f(x)|^r - m_{\widetilde{B}^6}(|f|^r)\right| \,d\mu(x)\\
&\quad\le \frac{1}{\mu(5B)} \int_B \left[\left||f(x)|^r - \left|m_{\widetilde{B}^6}(f)\right|^r\right| + \left|\left|m_{\widetilde{B}^6}(f)\right|^r - m_{\widetilde{B}^6}(|f|^r)\right|\right] \,d\mu(x)\\
&\quad\lesssim \frac{1}{\mu(5B)} \int_B |f(x) - f_B|^r \,d\mu(x) + \left|f_B - f_{\widetilde{B}^6}\right|^r + \left|f_{\widetilde{B}^6} - m_{\widetilde{B}^6}(f)\right|^r\\
&\qquad + \frac{1}{\mu(\widetilde{B}^6)} \int_{\widetilde{B}^6} \left|f(x) - f_{\widetilde{B}^6}\right|^r \,d\mu(x)\\
&\quad\lesssim \left[1 + \delta\left(B, \widetilde{B}^6\right)\right]^r \|f\|^r_{\mathrm{RBMO}(\mu)} \lesssim \|f\|^r_{\mathrm{RBMO}(\mu)},
\end{aligned}$$

where the last inequality follows from the fact that $\delta(B, \widetilde{B}^6) \lesssim 1$, which holds by [10, Lemma 2.1].

On the other hand, for any $(6, \beta_6)$-doubling balls $B \subset S$,

$$\begin{aligned}
|m_B(|f|^r) - m_S(|f|^r)| &\le |m_B(|f|^r) - |f_B|^r| + ||f_B|^r - |f_S|^r| + ||f_S|^r - m_S(|f|^r)|\\
&\le m_B(|f - f_B|^r) + |f_B - f_S|^r + m_S(|f - f_S|^r)\\
&\lesssim [1 + \delta(B, S)]^r \|f\|^r_{\mathrm{RBMO}(\mu)}.
\end{aligned}$$

Combining these two inequalities finishes the proof of Lemma 3.1. $\square$

Lemma 3.2. *Let $p \in [1, \infty)$ and $f \in L^1_{\rm loc}(\mu)$ such that $\int_{\mathcal{X}} f(x)\,d\mu(x) = 0$ if $\mu(\mathcal{X}) < \infty$. If for any $R > 0$,*

$$\sup_{0<\ell<R} \ell^p \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}) < \infty,$$

we then have

$$\sup_{\ell>0} \ell^p \mu(\{x \in \mathcal{X} : N(f)(x) > \ell\}) \lesssim \sup_{\ell>0} \ell^p \mu\left(\left\{x \in \mathcal{X} : M^\sharp(f)(x) > \ell\right\}\right).$$

Proof. Recall the λ-good inequality in [1] that for some fixed constant $\nu \in (0,1)$ and all $\epsilon \in (0,\infty)$, there exists some $\delta > 0$ such that for any $\ell > 0$,

$$\mu\left(\left\{x \in \mathcal{X} :\, N(f)(x) > (1+\epsilon)\ell,\, M^\sharp(f)(x) \le \delta\ell\right\}\right) \\ \le \nu\mu(\{x \in \mathcal{X} :\, N(f)(x) > \ell\}).$$

From this, it then follows that for R large enough and any $\epsilon > 0$,

$$\begin{aligned} &\sup_{0<\ell<R} \ell^p \mu(\{x \in \mathcal{X} :\, N(f)(x) > \ell\}) \\ &\quad\le \sup_{0<\ell<R} [(1+\epsilon)\ell]^p \mu(\{x \in \mathcal{X} :\, N(f)(x) > (1+\epsilon)\ell\}) \\ &\quad\le \nu(1+\epsilon)^p \sup_{0<\ell<R} \ell^p \mu(\{x \in \mathcal{X} :\, N(f)(x) > \ell\}) \\ &\qquad + (1+\epsilon)^p \sup_{\ell>0} \ell^p \mu\left(\left\{x \in \mathcal{X} :\, M^\sharp(f)(x) > \delta\ell\right\}\right). \end{aligned}$$

Choosing ϵ small enough such that $\nu(1+\epsilon)^p < 1$, our assumption then implies that

$$\sup_{0<\ell<R} \ell^p \mu(\{x \in \mathcal{X} :\, N(f)(x) > \ell\}) \lesssim \sup_{\ell>0} \ell^p \mu\left(\left\{x \in \mathcal{X} :\, M^\sharp(f)(x) > \ell\right\}\right).$$

Letting $R \to \infty$ then leads to the conclusion, which completes the proof of Lemma 3.2. □

Lemma 3.3. *Let $r \in (0,\,1)$ and $N_r(f) \equiv [N(|f|^r)]^{1/r}$. Then for any $p \in [1,\,\infty)$, there exists a positive constant C, depending on r, such that for suitable function f and any $\ell > 0$,*

$$\mu(\{x \in \mathcal{X} :\, N_r(f)(x) > \ell\}) \le C\ell^{-p} \sup_{\tau \ge \ell} \tau^p \mu(\{x \in \mathcal{X} :\, |f(x)| > \tau\}).$$

Proof. For each fixed $\ell > 0$ and function f, decompose f as

$$f(x) = f(x)\chi_{\{x \in \mathcal{X}:\, |f(x)| \le \ell\}}(x) + f(x)\chi_{\{x \in \mathcal{X}:\, |f(x)| > \ell\}} \equiv f_1(x) + f_2(x).$$

By the boundedness of N form $L^p(\mu)$ to $L^{p,\infty}(\mu)$, we obtain that

$$\begin{aligned} \mu(\{x \in \mathcal{X} :\, N_r(f)(x) > 2^{1/r}\ell\}) &\le \mu(\{x \in \mathcal{X} :\, N(|f_2|^r)(x) > \ell^r\}) \\ &\lesssim \ell^{-rp} \int_{\mathcal{X}} |f_2(x)|^{rp}\, d\mu(x) \\ &\lesssim \mu(\{x \in \mathcal{X} :\, |f(x)| > \ell\}) \\ &\quad + \ell^{-rp} \int_\ell^\infty \tau^{rp-1} \mu(\{x \in \mathcal{X} :\, |f(x)| > \tau\})\, d\tau \\ &\lesssim \mu(\{x \in \mathcal{X} :\, |f(x)| > \ell\}) \\ &\quad + \ell^{-p} \sup_{\tau > \ell} \tau^p \mu(\{x \in \mathcal{X} :\, |f(x)| > \tau\}), \end{aligned}$$

which implies our desired result. □

Proof of Theorem 1.1. By the Marcinkiewicz interpolation theorem, we only need to prove that for all $f \in L^p(\mu)$ with $p \in (1,\,\infty)$ and $\ell > 0$,

$$\mu(\{x \in \mathcal{X} :\, |Tf(x)| > \ell\}) \lesssim \ell^{-p} \|f\|_{L^p(\mu)}^p. \tag{3.1}$$

We further consider the following two cases.

Case (i) $\mu(\mathcal{X})=\infty$. Let $L_b^\infty(\mu)$ be the *space of bounded functions with bounded supports* and

$$L_{b,0}^\infty(\mu)\equiv\left\{f\in L_b^\infty(\mu):\ \int_{\mathcal{X}} f(x)\,d\mu(x)=0\right\}.$$

Then in this case, $L_{b,0}^\infty(\mu)$ is dense in $L^p(\mu)$ for all $p\in(1,\infty)$. Let $r\in(0,1)$ and $N_r(g)\equiv[N(|g|^r)]^{1/r}$ for any $g\in L^r_{\rm loc}(\mu)$. Notice that $|Tf|\le N_r(Tf)$ μ-almost everywhere on $\mathcal{X}$. Then by a standard density argument, to prove (3.1), it suffices to prove that for all $f\in L_{b,0}^\infty(\mu)$ and $p\in(1,\infty)$,

$$\sup_{\ell>0}\ell^p\mu\left(\{x\in\mathcal{X}:\ N_r(Tf)(x)>\ell\}\right)\lesssim\|f\|^p_{L^p(\mu)}. \tag{3.2}$$

For each fixed $\ell>0$, applying Lemma 2.10, we obtain that $f=g+h$, where h is as Lemma 2.10 and $g\equiv f-h$, such that

$$\|g\|_{L^\infty(\mu)}\lesssim\ell,\quad h\in H^1(\mu) \tag{3.3}$$

and

$$\|h\|_{H^1(\mu)}\lesssim\ell^{1-p}\|f\|^p_{L^p(\mu)}. \tag{3.4}$$

For each $r\in(0,1)$, define $M_r^\sharp(f)\equiv\{M^\sharp(|f|^r)\}^{1/r}$. Then, (3.3) together with the boundedness of T from $L^\infty(\mu)$ to $\mathrm{RBMO}(\mu)$ and the proof of Lemma 3.1 shows that the function $M_r^\sharp(Tg)$ is bounded by a multiple of ℓ. Hence, if c_0 is a sufficiently large constant, we have

$$\left\{x\in\mathcal{X}:\ M_r^\sharp(Tg)(x)>c_0\ell\right\}=\emptyset. \tag{3.5}$$

On the other hand, since both f and h belong to $H^1(\mu)$, we see that $g\in H^1(\mu)$ and

$$\|g\|_{H^1(\mu)}\le\|f\|_{H^1(\mu)}+\|h\|_{H^1(\mu)}\lesssim\|f\|_{H^1(\mu)}+\ell^{1-p}\|f\|^p_{L^p(\mu)}.$$

By this together with the boundedness of T from $H^1(\mu)$ to $L^{1,\infty}(\mu)$ and Lemma 3.3, we have that for any $p\in(1,\infty)$ and $R>0$,

$$\begin{aligned}\sup_{0<\ell<R}\ell^p\mu\left(\{x\in\mathcal{X}:\ N_r(Tg)(x)>\ell\}\right)&\lesssim\sup_{0<\ell<R}\ell^{p-1}\sup_{\tau\ge\ell}\tau\mu(\{x\in\mathcal{X}:\ |Tg(x)|>\tau\})\\&<\infty.\end{aligned}$$

From this, (3.5), Lemma 3.2 and the fact that $N_r\circ T$ is quasi-linear, we deduce that there exists a positive constant $\widetilde{C}$ such that

$$\begin{aligned}\sup_{\ell>0}&\ell^p\mu\left(\left\{x\in\mathcal{X}:\ N_r(Tf)(x)>\widetilde{C}c_0\ell\right\}\right)\\&\le\sup_{\ell>0}\ell^p\mu\left(\{x\in\mathcal{X}:\ N_r(Tg)(x)>c_0\ell\}\right)\\&\quad+\sup_{\ell>0}\ell^p\mu\left(\{x\in\mathcal{X}:\ N_r(Th)(x)>c_0\ell\}\right)\\&\lesssim\sup_{\ell>0}\ell^p\mu\left(\left\{x\in\mathcal{X}:\ M_r^\sharp(Tg)(x)>c_0\ell\right\}\right)\\&\quad+\sup_{\ell>0}\ell^p\mu\left(\{x\in\mathcal{X}:\ N_r(Th)(x)>c_0\ell\}\right)\\&\lesssim\sup_{\ell>0}\ell^p\mu\left(\{x\in\mathcal{X}:\ N_r(Th)(x)>\ell\}\right).\end{aligned}\tag{3.6}$$

From the boundedness of N from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ and the boundedness of T from $H^1(\mu)$ to $L^{1,\infty}(\mu)$, it follows that

$$\begin{aligned}
&\mu(\{x \in \mathcal{X} : N_r(Th)(x) > \ell\}) \\
&\quad \le \mu\left(\left\{x \in \mathcal{X} : N\left(|Th|^r \chi_{\{x \in \mathcal{X}: |(Th)(x)| > \ell/2^{\frac{1}{r}}\}}\right) > \frac{\ell^r}{2}\right\}\right) \\
&\quad \lesssim \ell^{-r} \int_{\mathcal{X}} \left|(Th)(x)\chi_{\{x\in\mathcal{X}: |(Th)(x)|>\ell/2^{\frac{1}{r}}\}}(x)\right|^r d\mu(x) \\
&\quad \lesssim \ell^{-r}\mu\left(\left\{x \in \mathcal{X} : |(Th)(x)| > \ell/2^{\frac{1}{r}}\right\}\right) \int_0^{\ell/2^{\frac{1}{r}}} s^{r-1}\, ds \\
&\qquad + \ell^{-r} \int_{\ell/2^{\frac{1}{r}}}^{\infty} s^{r-1} \mu(\{x \in \mathcal{X} : |(Th)(x)| > s\})\, ds \\
&\quad \lesssim \mu\left(\left\{x \in \mathcal{X} : |(Th)(x)| > \ell/2^{\frac{1}{r}}\right\}\right) + \frac{1}{\ell} \sup_{s \ge \ell/2^{\frac{1}{r}}} s\mu(\{x \in \mathcal{X} : |(Th)(x)| > s\}) \\
&\quad \lesssim \frac{\|h\|_{H^1(\mu)}}{\ell} \lesssim \ell^{-p}\|f\|^p_{L^p(\mu)},
\end{aligned}$$

which together with (3.6) yields (3.2).

Case (ii) $\mu(\mathcal{X}) < \infty$. In this case, assume that $f \in L^\infty_b(\mu)$. Notice that if $\ell \in (0, \ell_0]$, where ℓ_0 is as in Lemma 2.10, then (3.1) holds trivially. Thus, we only have to consider the case when $\ell > \ell_0$. Let $r \in (0, 1)$, $N_r(f)$ be as in Lemma 3.3 and $M^\sharp_r$ as in Case (i). For each fixed $\ell > \ell_0$, applying Lemma 2.10, we obtain that $f = g + h$ with g and h satisfying (3.3) and (3.4), which together with the boundedness of T from $L^\infty(\mu)$ to $\mathrm{RBMO}(\mu)$ and the proof of Lemma 3.1 yields (3.5) for $M^\sharp_r(Tg)$. We now claim that

$$\mathrm{F} \equiv \frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} |Tg(x)|^r\, d\mu(x) \lesssim \ell^r, \tag{3.7}$$

where the constant depends on $\mu(\mathcal{X})$ and r. In fact, since $\mu(\mathcal{X}) < \infty$, we regard $\mathcal{X}$ as a ball. Then $g_0 \equiv g - \frac{1}{\mu(\mathcal{X})}\int_{\mathcal{X}} g(x)\, d\mu(x) \in H^1(\mu)$. On the other hand, $|T1|^r \in \mathrm{RBMO}(\mu)$ because of the fact that $T1 \in \mathrm{RBMO}(\mu)$ and Lemma 3.1. This together with $\mu(\mathcal{X}) < \infty$ implies that

$$\int_{\mathcal{X}} |T1(x)|^r\, d\mu(x) < \infty.$$

Then by the boundedness of T from $H^1(\mu)$ to $L^{1,\infty}(\mu)$ and (3.3), we have

$$\begin{aligned}\int_{\mathcal{X}} |Tg(x)|^r \, d\mu(x) &\le \int_{\mathcal{X}} \left\{ |Tg_0(x)|^r + \left| T\left[\frac{1}{\mu(\mathcal{X})} \int_{\mathcal{X}} g(y)\, d\mu(y) \right](x) \right|^r \right\} d\mu(x) \\ &\lesssim r \int_0^{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})} t^{r-1} \mu\left(\{x \in \mathcal{X} : |Tg_0(x)| > t\}\right) dt \\ &\quad + r \int_{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})}^{\infty} t^{r-1} \mu\left(\{x \in \mathcal{X} : |Tg_0(x)| > t\}\right) dt + \ell^r \\ &\lesssim \mu(\mathcal{X}) \int_0^{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})} t^{r-1}\, dt \\ &\quad + \|g_0\|_{H^1(\mu)} \int_{\|g_0\|_{H^1(\mu)}/\mu(\mathcal{X})}^{\infty} t^{r-2}\, dt + \ell^r \\ &\lesssim [\mu(\mathcal{X})]^{1-r} \|g_0\|_{H^1(\mu)}^r + \ell^r \lesssim \ell^r,\end{aligned}$$

which implies (3.7).

Observe that $\int_{\mathcal{X}} (|Tg|^r - \mathrm{F})\, d\mu(x) = 0$ and for any $R > 0$,

$$\sup_{0<\ell<R} \ell^p \mu\left(\{x \in \mathcal{X} : N(|Tg|^r - \mathrm{F})(x) > \ell\}\right) \le R^p \mu(\mathcal{X}) < \infty.$$

From this together with Lemma 3.2, $M_r^\sharp(\mathrm{F}) = 0$, (3.7) and an argument similar to that used in Case (i), we conclude that there exists a positive constant $\widetilde{c}$ such that

$$\begin{aligned}&\sup_{\ell>\ell_0} \ell^p \mu\left(\{x \in \mathcal{X} : N_r(Tf)(x) > \widetilde{c} c_0 \ell\}\right) \\ &\quad \le \sup_{\ell>\ell_0} \ell^p \mu\left(\{x \in \mathcal{X} : N(|Tg|^r - \mathrm{F})(x) > (c_0\ell)^r\}\right) \\ &\qquad + \sup_{\ell>\ell_0} \ell^p \mu\left(\{x \in \mathcal{X} : N_r(Th)(x) > c_0\ell\}\right) \\ &\quad \lesssim \sup_{\ell>0} \ell^p \mu\left(\left\{x \in \mathcal{X} : M_r^\sharp(Tg)(x) > c_0\ell\right\}\right) \\ &\qquad + \sup_{\ell>0} \ell^p \mu\left(\{x \in \mathcal{X} : N_r(Th)(x) > c_0\ell\}\right) \\ &\quad \lesssim \sup_{\ell>0} \ell^p \mu\left(\{x \in \mathcal{X} : N_r(Th)(x) > \ell\}\right) \lesssim \|f\|_{L^p(\mu)}^p,\end{aligned}$$

where in the first inequality we choose c_0 large enough such that $\mathrm{F} \le (c_0\ell)^r$. This completes the proof of Theorem 1.1.

□

Acknowledgement. The authors want to express their deep thanks to the referee for his/her very valuable remarks which made this article more readable. The authors are very grateful to their advisor professor Dachun Yang for advice and encouragements. The first author is supported by the Mathematical Tianyuan Youth Fund (Grant No. 11026120) of National Natural Science Foundation of China and Chinese Universities Scientific Fund (Grant No. 2011JS043). The second (corresponding) author is supported by the National Natural Science Foundation (Grant No. 11101339) of China.

References

1. B.T. Anh and X.T. Duong, *Hardy spaces, regularized BMO spaces and the boundeness of Calderón-Zygmund operators on non-homogeneous spaces*, J. Geom. Anal., doi: 10.1007/s12220-011-9268-y.
2. R.R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), 249–254.
3. R.R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur Certains Espaces Homogenes*, Lecture Notes in Math. **242**, Springer, Berlin, 1971.
4. R.R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.
5. P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688.
6. J. Heinenon, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, New York, 2001.
7. G. Hu, Y. Meng and D. Yang, *A new characterization of regularized BMO spaces on non-homogeneous spaces and its applications*, preprint.
8. T. Hytönen, *A framework for non-homogeneous analysis on metric spaces, and the* RBMO *space of Tolsa*, Publ. Mat. **54** (2010), 485–504.
9. T. Hytönen, S. Liu, Da. Yang and Do. Yang, *Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces*, Canad. J. Math. doi:10.4153/CJM-2011-065-2.
10. T. Hytönen, Da. Yang and Do. Yang, *The Hardy space H^1 on non-homogeneous metric spaces*, Math. Proc. Cambridge Philos. Soc. **153** (2012), no. 1, 9–31.
11. T. Hytönen and H. Martikainen, *Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces*, J. Geom. Anal., doi: 10.1007/s1220-011-9230-z.
12. H. Lin and D. Yang, *Spaces of type BLO on non-homogeneous metric measure spaces*, Front. Math. China **6** (2011), 271–292.
13. S. Liu, Da. Yang and Do. Yang, *Boundedness of Calderón-Zygmund operators on non-homogeneous metric measure spaces: equivalent characterizations*, J. Math. Anal. Appl. **386** (2012), 258–272.
14. J. Mateu, P. Mattila, A. Nicolau and J. Orobitg, *BMO for nondoubling measures*, Duke Math. J. **102** (2000), 533–565.
15. F. Nazarov, S. Treil and A. Volberg, *The Tb-theorem on non-homogeneous spaces*, Acta Math. **190** (2003), 151–239.
16. X. Tolsa, *BMO, H^1 and Calderón-Zygmund operators for non doubling measures*, Math. Ann. **319** (2001), 89–149.
17. X. Tolsa, *Littlewood-Paley theory and the $T(1)$ theorem with non-doubling measures*, Adv. Math. **164** (2001), 57–116.
18. X. Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. **190** (2003), 105–149.

[1] College of Science, China Agricultural University, Beijing 100083, People's Republic of China.

E-mail address: haibolincau@126.com

[2] School of Mathematical Sciences, Xiamen University, Xiamen 361005, People's Republic of China.

E-mail address: dyyang@xmu.edu.cn