APPROXIMATION BY POLYNOMIALS IN REARRANGEMENT INVARIANT QUASI BANACH FUNCTION SPACES

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ABSTRACT. In the present work we deal with the approximation properties of certain linear polynomial operators in rearrangement invariant quasi Banach function spaces. We obtain some Jackson type direct theorem and sharp converse theorem of trigonometric approximation with respect to fractional positive order moduli of smoothness in these spaces.

1. INTRODUCTION AND THE MAIN RESULTS

Approximation by trigonometric polynomials in rearrangement invariant Banach function spaces and some rearrangement invariant quasi Banach function spaces was investigated by many mathematicians. Denseness problems of the trigonometric system in the Orlicz class $\varphi(L)$ was obtained in [26]. Direct and converse theorems of trigonometric approximation the Lebesgue spaces L_p, $0 < p < 1$, was obtained by the intermediate approximation in [25]. Also there is a different [22] method to obtain direct theorem of trigonometric approximation in L_p, $0 < p < 1$, by means of linear polynomial operators. For a complete discussions of the problems of approximation theory for L_p, $0 < p < \infty$, and rearrangement invariant Banach function spaces we can refer to the book [6]. For the variable exponent Lebesgue spaces similar problems of approximation theory was considered in [2, 1, 3].

In this work we investigate the approximation properties of certain linear polynomial operators in rearrangement invariant quasi Banach function spaces. We

obtain some Jackson type direct theorem and sharp converse theorem of trigono-metric approximation with respect to fractional order moduli of smoothness in these spaces. We give some definitions.

Let $\mathcal{M}$ be the set of all measurable functions defined on $\mathsf{T} := [0, 2\pi]$ and let $\mathcal{M}^+$ be the subset of functions from $\mathcal{M}$ whose values lie in $[0, \infty]$. By χ_E we denote the characteristic function of a measurable set $E \subset \mathsf{T}$. A mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a *function norm* if for all constants $a \geq 0$, for all functions f, g, f_n $(n = 1, 2, 3, \ldots)$, and for all measurable subsets E of T, the following properties hold:

(i) $\rho(f) = 0$ iff $f = 0$ a.e.; $\rho(af) = a\rho(f)$; $\rho(f + g) \leq \rho(f) + \rho(g)$,

(ii) if $0 \leq g \leq f$ a.e., then $\rho(g) \leq \rho(f)$, (iii) if $0 \leq f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$,

(iv) $\rho(\chi_E) < \infty$ holds for every set E having a finite Lebesgue measure $|E| < \infty$,

(v) $\int_E f(x) \, dx \leq C_E \, \rho(f)$ holds for every set E having a finite Lebesgue measure $|E| < \infty$, with a constant $C_E \in (0, \infty)$, depending on E and ρ but independent of f.

If ρ is a function norm, its *associate norm* ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup \left\{ \int_{\mathsf{T}} f(x) \, g(x) \, dx : f \in \mathcal{M}^+, \, \rho(f) \leq 1 \right\}, \quad g \in \mathcal{M}^+.$$

If ρ is a function norm, then ρ' is itself a function norm [5, p. 8, Th. 2.2]. Let ρ be a function norm. The collection of functions

$$X := X(\rho) := \{ f \in \mathcal{M} : \rho(|f|) < \infty \}$$

is called *Banach function space* (shortly BFS). For each $f \in X$ we define

$$\|f\|_X := \rho(|f|).$$

A Banach function space X equipped with the norm $\|\cdot\|_X$ is a Banach space [5, p. 3,5, Ths. 1.4 and 1.6]. Let ρ' be the associate norm of a function norm ρ. The Banach function space $X(\rho')$ determined by the function norm ρ' is called the *associate space* of $X = X(\rho)$ and is denoted by X'. It is well-known [5, p. 9] that

$$\|f\|_X = \sup \left\{ \int_{\mathsf{T}} |f(x) \, g(x)| \, dx : g \in X', \, \|g\|_{X'} \leq 1 \right\} \tag{1.1}$$

hold. The distribution function μ_f of a measurable function f is defined as

$$\mu_f(\lambda) = meas \{ x \in [0, 2\pi] : |f(x)| > \lambda \}, \, \lambda \geq 0.$$

A Banach function norm is *rearrangement invariant* if $\rho(f) = \rho(g)$ for every pair of functions f, g which are equimeasurable, that is $\mu_f(\lambda) = \mu_g(\lambda)$.

Given a Banach function space X, for each $r \in (0, \infty)$, we define $X_r := \{ f \in \mathcal{M} : f^r \in X \}$ and r-norm as

$$\|f\|_{X_r} := \||f|^r\|_X^{1/r}.$$

The X_p spaces and generalized Orlicz spaces have been studied and used in [17, 20, 21, 18, 13, 11, 12]. Hardy type inequalities in X_p are investigated in [10].

Throughout this work by C, c, c_i we denote the constants which are absolute or depend only on the parameters given in their brakets.

A *quasi Banach function norm* is a mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ such that it satisfies (ii)-(iv) of above definition of function norm but (i) satisfies as a quasinorm, namely, $\rho(f) = 0$ iff $f = 0$ a.e.; $\rho(af) = a\rho(f)$; $\rho(f + g) \le c(\rho(f) + \rho(g))$. If a quasi Banach function norm ρ is rearrangement invariant then the collection of functions $X(\rho) = \{f \in \mathcal{M} : \rho(|f|) < \infty\}$ will be called *rearrangement invariant quasi Banach function space* (shortly RIQBFS). A quasi BFS X is said to be *p-convex* for some $p \in (0, 1]$ if there is a c such that for all $f_1, \ldots, f_N \in X$ we have

$$\left\| \left(\sum_{i=1}^{N} |f_i|^p \right)^{1/p} \right\|_X \le c \left(\sum_{i=1}^{N} \|f_i\|_X^p \right)^{1/p}. \tag{1.2}$$

In this case the condition (1.2) is equivalent to the fact that $X_{1/p}$ is a rearrangement invariant BFS. From (1.1) one can be see that $\|\cdot\|_X$ be equivalently represented [8] as

$$\|f\|_X \asymp \sup \left\{ \left(\int_{\mathsf{T}} |f(x)|^p g(x)\, dx \right)^{1/p} : g \ge 0, \ \|g\|_{Y'} \le 1 \right\} \tag{1.3}$$

where Y' is the associate space of the rearrangement invariant BFS $Y = X_{1/p}$. There are examples [9] of quasi BFS which are not *p*-convex for any $p > 0$.

$A(x) \asymp B(x)$ will be mean that there exist constants c and C such that $cA(x) \le B(x) \le CA(x)$ holds.

Let X be quasi BFS. A function $f \in X$ is said to have *absolutely continuous norm* if

$$\lim_{n \to \infty} \|f \chi_{A_n}\|_X = 0$$

for every decreasing sequence of measurable sets (A_n) with $\chi_{A_n} \to 0$ a.e. If every $f \in X$ has this property we will say X has *absolutely continuous norm*.

Hereafter throughout this work we will assume that $\mathbb{X} := X(AC, p)$ is a RIQBFS which has absolutely continuous norm and *p*-convex for some $p \in (0, 1]$. These assumptions on the function space are not very restrictive. For example Orlicz spaces, classical Lorentz spaces L^{pq}, $p, q \in (0, \infty)$ (in particular L^p spaces with $p \in (0, 1)$), Zygmund spaces $L^p (\log L)^\alpha$, $p \in (0, \infty)$, $\alpha \in \mathbb{R}$, Lorentz Λ spaces and Marcinkiewicz spaces satisfy [8] these conditions. For a complete treatise of rearrangement invariant BFS and RIQBFS we refer to [16, 5, 15, 7, 14, 19].

Remark 1.1. Let X be a RIQBFS. The following conditions are equivalent:

(i) The set of trigonometric polynomials is dense in X. (ii) The set of continuous functions is dense in X. (iii) Translation operator $T_h f(x) := f(x + h)$ is a bounded operator in X, namely,

$$\|T_h f\|_X \le c \|f\|_X$$

for every $f \in X$ and $h \in \mathbb{R}$.

(iv) X has absolutely continuous norm.

These properties are proved for rearrangement invariant BFS in [5, p. 157, Lemma 6.3] and they hold also for RIQBFS X which has absolutely continuous norm.

Let $x, h \in \mathbb{R}$, $\alpha \in \mathbb{R}^+ := (0, \infty)$, $f \in \mathbb{X}$ and we set

$$\Delta_h^\alpha f(x) := \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(x + (\alpha - k) h)$$

with Binomial coefficients $\begin{pmatrix} \alpha \\ k \end{pmatrix} := \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}$ for $k \geq 1$ and $\begin{pmatrix} \alpha \\ 0 \end{pmatrix} := 1$.

If $\frac{1}{\alpha+1} < p < 1$, then using [24, p.14]

$$\left| \begin{pmatrix} \alpha \\ k \end{pmatrix} \right| \leq \frac{c(\alpha)}{k^{\alpha+1}}, \quad k \in \mathbb{Z}^+$$

we obtain

$$\sum_{k=0}^{\infty} \left| \begin{pmatrix} \alpha \\ k \end{pmatrix} \right|^p \leq c(\alpha, p) \sum_{k=0}^{\infty} \frac{c(\alpha)}{k^{p(\alpha+1)}} < \infty. \tag{1.4}$$

On the other hand if nonnegative g belongs to Y', the associate space of the rearrangement invariant BFS $Y = X_{1/p}$, then using Levi's Monotone Convergence Theorem and Remark 1.1 (iii) we have

$$\int_{\mathsf{T}} \left| \sum_{k=0}^{\infty} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(x + (\alpha - k) h) \right|^p g(x) \, dx$$

$$\leq \lim_{j \to \infty} \left(\int_{\mathsf{T}} \sum_{k=0}^{j} \left| \begin{pmatrix} \alpha \\ k \end{pmatrix} f(x + (\alpha - k) h) \right|^p g(x) \, dx \right)$$

$$\leq c \left(\sum_{k=0}^{\infty} \left| \begin{pmatrix} \alpha \\ k \end{pmatrix} \right|^p \right) \left(\sup_{g \geq 0, \|g\|_{Y'} \leq 1} \int_{\mathsf{T}} |f(x)|^p g(x) \, dx \right)$$

and hence from (1.3) and (1.4)

$$\|\Delta_h^\alpha f\|_X \leq c \|f\|_X. \tag{1.5}$$

This last inequality signifies that if $f \in \mathbb{X}$, $\alpha \in \mathbb{R}^+$, $(\alpha + 1)^{-1} < p < 1$ and $h \in \mathbb{R}$, then $\Delta_h^\alpha f \in X$.

Now, if $\alpha \in \mathbb{R}^+$, $f \in \mathbb{X}$, $(\alpha + 1)^{-1} < p < 1$ and $h \in \mathbb{R}$, then we can define the α-th modulus of smoothness of a function f as

$$\omega_\alpha(f, \delta)_X := \sup_{0 < h \leq \delta} \|\Delta_h^\alpha f\|_X, \quad \delta \geq 0.$$

Remark 1.2. The α-th modulus of smoothness $\omega_\alpha(f, \delta)_X$, $\alpha \in \mathbb{R}^+$, $(\alpha + 1)^{-1} < p$, of $f \in \mathbb{X} = X(AC, p)$ has the following properties.

(i) $\omega_\alpha(f, \delta)_X$ is non-negative and non-decreasing function of $\delta \geq 0$. (ii) $\omega_\alpha^p(f_1 + f_2, \cdot)_X \leq \omega_\alpha^p(f_1, \cdot)_X + \omega_\alpha^p(f_2, \cdot)_X$. (iii) $\lim_{\delta \to 0^+} \omega_\alpha(f, \delta)_X = 0$.

By means of Remark 1.1 (ii) and (iv) let

$$E_n(f)_X := \inf_{T \in \mathcal{T}_n} \|f - T\|_X, \quad f \in \mathbb{X}, \, n = 0, 1, 2, \ldots,$$

where $\mathcal{T}_n$ be the class of trigonometric polynomials of degree not greater than n.

We denote by X^α, $\alpha \in \mathbb{R}^+$, the linear space of 2π-periodic real valued functions $f \in \mathbb{X}$ such that $f^{(\alpha)} \in X$ a.e.

We say that a function $g = f^{(\alpha)}$, $\alpha \in \mathbb{R}^+$, is the αth (Grunwald-Letnikov) derivative of $f \in X^\alpha$ if there is a function $g \in X$ such that

$$\lim_{h \to 0^+} \left\| \frac{\Delta_h^\alpha(f)}{h^\alpha} - g \right\|_X = 0.$$

If a.e. equal functions are identified, then the last condition determines αth derivative uniquely. Also αth derivative is additive with respect to finite number of functions.

The αth Weyl's derivative ($\alpha \in \mathbb{R}^+$) of a trigonometric polynomial

$$T_n(x) = \sum_{v=-n}^{n} \gamma_v e^{ivx}, \quad n \geq 1, \, x \in \mathbb{R}$$

of class $\mathcal{T}_n$ is defined as

$$T_n^{\{\alpha\}}(x) = \sum_{v \in \mathbb{Z}_n^*} \gamma_v \, (iv)^\alpha \, e^{ivx}$$

a.e. on $\mathbb{R}$, where $\mathbb{Z}_n^* := \{\pm 1, \pm 2, \ldots, \pm n\}$ and $(iv)^{-\alpha} := |v|^{-\alpha} e^{(-1/2)\pi i \alpha \, \mathrm{sign} v}$ as principal value.

Remark 1.3. Let

$$T_n(x) = \sum_{v=-n}^{n} \gamma_v e^{ivx}, \quad (n \geq 1)$$

be a trigonometric polynomial of class $\mathcal{T}_n$ with complex coefficients γ_v. Then for every $\alpha \in \mathbb{R}^+$ and $x \in \mathbb{R}$ we have

$$T_n^{\{\alpha\}}(x) = T_n^{(\alpha)}(x).$$

Using Dominated Convergence Theorem this follows from the following equalities in $\mathbb{X} = X(AC, p)$

$$\lim_{h \to 0^+} \left\| \frac{\Delta_h^\alpha T_n}{h^\alpha} - T_n^{\{\alpha\}} \right\|_X^p = \lim_{h \to 0^+} \left\| \sum_{v \in \mathbb{Z}_n} \gamma_v e^{ivx} \left(e^{iv\alpha h/2} \left(2i \frac{\sin(vh/2)}{h} \right)^\alpha - (iv)^\alpha \right) \right\|_X^p$$

$$= \left\| \sum_{v \in \mathbb{Z}_n} \gamma_v e^{ivx} \left(\lim_{h \to 0^+} \left(iv \frac{\sin(vh/2)}{vh/2} \right)^\alpha - (iv)^\alpha \right) \right\|_X^p = 0.$$

Throughout this work we will denote by $q_{\mathbb{X}}$ the upper Boyd's indice of $\mathbb{X}$. Let us denote by $[x]$ the integer part of a real number x and $\{x\} := x - [x]$.

For $\alpha, t \in \mathbb{R}^+$ and $f \in \mathbb{X}$ we define for $n = 1, 2, 3, \ldots$, the Polynomial K-functional

$$K_\alpha(f, 1/n, X, X^\alpha) := \inf_{T \in \mathcal{T}_n} \left\{ \|f - T\|_X + n^{-\alpha} \|T^{(\alpha)}\|_X \right\}.$$

The main results of this work are

Theorem 1.4. *If $\alpha \in \mathbb{R}^+$, $f \in \mathbb{X}$, $p^{-1} < \min\{\alpha + 1, 2 - \{\alpha\}\}$ and $q_{\mathbb{X}} < \infty$, then the equivalance*

$$\omega_\alpha\left(f, 1/n\right)_X \asymp K_\alpha\left(f, 1/n, X, X^\alpha\right)$$

holds.

The following Jackson type direct theorem of trigonometric approximation hold.

Corollary 1.5. *If $\alpha \in \mathbb{R}^+$, $f \in \mathbb{X}$, $p^{-1} < \min\{\alpha + 1, 2 - \{\alpha\}\}$ and $q_{\mathbb{X}} < \infty$, then there exists a constant $c > 0$ dependent only on α and $\mathbb{X}$ such that for $n = 1, 2, 3, \ldots$*

$$E_n\left(f\right)_X \le c\omega_\alpha\left(f, \frac{1}{n}\right)_X$$

holds.

The following converse estimate of trigonometric approximation holds.

Theorem 1.6. *If $\alpha \in \mathbb{R}^+$, $f \in \mathbb{X}$, $(\alpha + 1)^{-1} < p$ and $q_{\mathbb{X}} < \infty$, then for $n = 1, 2, 3, \ldots$*

$$\omega_\alpha\left(f, \frac{\pi}{n}\right)_X \le \frac{c}{n^\alpha}\left(\sum_{\nu=0}^n (\nu + 1)^{p\alpha - 1} E_\nu^p\left(f\right)_X\right)^{1/p} \tag{1.6}$$

hold, where the constant $c > 0$ dependent only on α and $\mathbb{X}$.

Corollary 1.7. *Under the conditions of Theorem 1.4 the estimate*

$$E_n\left(f\right)_X = \mathcal{O}\left(n^{-\sigma}\right), \; 1 > \sigma > 0, \; n = 1, 2, \ldots,$$

holds if and only if

$$\omega_\alpha\left(f, \delta\right)_X = \mathcal{O}\left(\delta^\sigma\right).$$

Corollary 1.8. *Under the conditions of Theorem 1.4 the converse inequality (1.6) is sharp in the sense that*

$$\sup_{E_n(f)_{L^p} \le 1/n} \omega_1\left(f, \delta\right)_{L^p} \asymp \beta\left(\ln\left(1/\delta\right)\right)^{1/p}, \;\; 0 < p < 1. \tag{1.7}$$

Exactness of (1.7) can be seen by 2π periodic function $f\left(x\right) = |x|^{1-(1/p)}$, $|x| \le \pi$.

Theorem 1.9. *Let $f \in \mathbb{X}$ and $q_{\mathbb{X}} < \infty$. If $\beta \in (0, \infty)$ and*

$$\sum_{\nu=1}^\infty \nu^{p\beta - 1} E_\nu^p\left(f\right)_X < \infty \tag{1.8}$$

then the derivative $f^{(\beta)} \in X$ exists. Further, denoting by $\mathrm{T}_n \in \mathcal{T}_n$, $n \ge 1$, the best approximating polynomial of f in $\|\cdot\|_X$ metric we have

$$\left\|f^{(\beta)} - \mathrm{T}_n^{(\beta)}\right\|_X \le c\left(n^\beta E_n\left(f\right)_X + \left(\sum_{\nu=n+1}^\infty \nu^{p\beta - 1} E_\nu^p\left(f\right)_X\right)^{1/p}\right)$$

where the constant $c > 0$ dependent only on β and $\mathbb{X}$.

As a corollary of Theorems 1.9 and 1.6

Corollary 1.10. *Let* $f \in \mathbb{X}$, $\beta \in (0, \infty)$, $q_{\mathbb{X}} < \infty$ *and*

$$\sum_{\nu=1}^{\infty} \nu^{p\alpha-1} E_{\nu}^{p} (f)_X < \infty$$

for some $\alpha > 0$. *In this case for* $n = 1, 2, \ldots$, *there exists a constant* $c > 0$ *dependent only on* α, β *and* $\mathbb{X}$ *such that*

$$\omega_{\beta} \left(f^{(\alpha)}, \frac{1}{n} \right)_X \leq c \left\{ \frac{1}{n^{\beta}} \left(\sum_{\nu=0}^{n} (\nu + 1)^{p(\alpha+\beta)-1} E_{\nu}^{p} (f)_X \right)^{1/p} + \right.$$

$$\left. + \left(\sum_{\nu=n+1}^{\infty} \nu^{p\alpha-1} E_{\nu}^{p} (f)_X \right)^{1/p} \right\}$$

hold.

2. Linear Polynomial Operators in $\mathbb{X}$

Linear polynomial operators are powerful approximants in approximation theory and they are commonly used to solve various approximation problems. For example, among others, in [22] the linear operators $V_{n,\lambda}$ of a real parameter λ

$$V_{n,\lambda} (f, x) = \frac{2}{2N+1} \sum_{i=0}^{2N} f \left(t_N^i + \lambda \right) W_n^{(l)} \left(x - t_N^i - \lambda \right), \; f \in L^p, \; 0 < p < 1, \; x \in \mathbb{R}$$

are defined and used to prove a direct theorem of trigonometric polynomial approximation in L^p, $0 < p < 1$. Here $N := \begin{cases} 0 & ; \; n = 0, \\ n(l+1) - l & ; \; n = 1, 2, 3, \ldots, \end{cases}$,
$t_N^i = 2\pi i / (2N+1)$, $i = 0, 1, 2, \ldots, 2N$, $n = 0, 1, 2, 3, \ldots$ and $W_n^{(l)} \in \mathcal{T}_N$ is certain even trigonometric polynomial. This polynomial operator has the following useful properties for $\lambda \in \mathbb{R}$ and $n = 0, 1, 2, 3, \ldots$:

Remark 2.1. i) $V_{n,\lambda} \in \mathcal{T}_n$; ii) $V_{n,\lambda} (T, \cdot) = T (\cdot)$ for $T \in \mathcal{T}_n$; iii) $V_{n,\lambda+t_N^1} = V_{n,\lambda}$; iv) $T_h \circ V_{n,\lambda} = V_{n,\lambda-h} \circ T_h$ for $h \in \mathbb{R}$ and v) $T_{t_N^1} \circ V_{n,\lambda} = V_{n,\lambda} \circ T_{t_N^1}$, where $f_1 \circ f_2$ denotes composition of functions f_1 and f_2.

In this section, on the bases of these facts we will use the operators $V_{n,\lambda} (f, \cdot)$ as an apparatus of approximation to $f \in \mathbb{X}$ and prove some auxiliary lemmas.

Lemma 2.2. *If* $f \in \mathbb{X} = X (AC, p)$, $k = 1, 2, \ldots$ *and* $(\alpha + 1)^{-1} < p$, *then there exists a constant* $c > 0$ *such that*

$$\left(\int_{\mathbb{T}} \| V_{n,\lambda} (f, \cdot) \|_X^p \, d\lambda \right)^{1/p} \leq c \, \| f \|_X$$

holds where $n = 0, 1, 2, \ldots$.

Proof. Since [22, Theorem 1.2]

$$\int_0^\pi x \left| W_n^{(k)}(x) \right|^p dx \le c(n+1)^{p-2} \quad \text{and} \quad \int_0^\pi \left| W_n^{(k)}(x) \right|^p dx \le c(n+1)^{p-1}$$

we have

$$\int_{\mathsf{T}} \| V_{n,\lambda}(f,\cdot) \|_X^p \, d\lambda = \int_{\mathsf{T}} \left\| \frac{2}{2N+1} \sum_{i=0}^{2N} f\left(t_N^i + \lambda\right) W_n^{(l)} \left(\cdot - t_N^i - \lambda\right) \right\|_X^p d\lambda$$

$$\le c \int_{\mathsf{T}} \sup \left\{ \int_{\mathsf{T}} \left| \frac{2}{2N+1} \sum_{i=0}^{2N} f\left(t_N^i + \lambda\right) W_n^{(l)} \left(x - t_N^i - \lambda\right) \right|^p g(x) \, dx : \right.$$

$$: g \ge 0, \ \|g\|_{X'} \le 1 \} \, d\lambda$$

$$\le c \int_{\mathsf{T}} \left(\frac{2}{2N+1} \right)^p \sum_{i=0}^{2N} \sup \left\{ \int_{\mathsf{T}} \left| f\left(t_N^i + \lambda\right) W_n^{(l)} \left(x - t_N^i - \lambda\right) \right|^p g(x) : \right.$$

$$: g \ge 0, \ \|g\|_{X'} \le 1 \} \, d\lambda$$

$$\le \frac{4^p c}{(n+1)^p} \sum_{i=0}^{2N} \sup \left\{ \int_{\mathsf{T}} \left(\int_{\mathsf{T}} \left| f(x-u) W_n^{(k)}(u) \right|^p du \right) |g(x)| \, dx : \right.$$

$$: g \ge 0, \ \|g\|_{X'} \le 1 \}$$

$$\le 2 \frac{4^p c}{(n+1)^p} \|f\|_X^p \int_0^\pi \left| W_n^{(k)}(u) \right|^p du \le C \|f\|_X^p$$

and the result follows. $\square$

Theorem 2.3 (Extrapolation Theorem). [8, Theorem 2.1] *Let $0 < p_0 < \infty$ and let $\mathcal{F}$ be a family of couples of nonnegative functions such that*

$$\int_{\mathsf{T}} f(x)^{p_0} \, dx \le C \int_{\mathsf{T}} g(x)^{p_0} \, dx, \quad (f,g) \in \mathcal{F}$$

holds with the left hand side is finite. We suppose that X is a RIQBFS which is p-convex for some $p \in (0,1]$ and $q_{\mathbb{X}} < \infty$. Then

$$\|f\|_X \le C \|g\|_X, \quad (f,g) \in \mathcal{F}$$

holds when the left hand side is finite.

Lemma 2.4. *Let X be a RIQBFS which is p-convex for some $p \in (0,1]$. If $T_n \in \mathcal{T}_n$, $n \ge 1$, $\alpha \in \mathbb{R}^+$, $q_{\mathbb{X}} < \infty$, $0 < h \le 2\pi/n$, then there exist constants $c, C > 0$ such that*

$$\left\| T_n^{(\alpha)} \right\|_X \le c \left(\frac{n}{2 \sin(nh/2)} \right)^\alpha \left\| \Delta_h^\alpha T_n \right\|_X \quad , \tag{2.1}$$

and

$$\left\|\Delta_h^\alpha T_n\right\|_X \le Ch^\alpha \left\|T_n^{(\alpha)}\right\|_X. \tag{2.2}$$

Proof. Since the inequalities

$$\left\|T_n^{(\alpha)}\right\|_{L^q} \le c\left(\frac{n}{2\sin(nh/2)}\right)^\alpha \left\|\Delta_h^\alpha T_n\right\|_{L^q}$$

and

$$\left\|\Delta_h^\alpha T_n\right\|_{L^q} \le Ch^\alpha \left\|T_n^{(\alpha)}\right\|_{L^q}$$

are hold for every $q \in [1, \infty)$ we obtain from the Extrapolation Theorem that

$$\left\|T_n^{(\alpha)}\right\|_X \le c\left(\frac{n}{2\sin(nh/2)}\right)^\alpha \left\|\Delta_h^\alpha T_n\right\|_X,$$

$$\left\|\Delta_h^\alpha T_n\right\|_X \le Ch^\alpha \left\|T_n^{(\alpha)}\right\|_X.$$

$\square$

Corollary 2.5. *(i) Taking $h = \pi/n$ in (2.1) we have*

$$\left\|T_n^{(\alpha)}\right\|_X \le cn^\alpha \left\|\Delta_{\pi/n}^\alpha T_n\right\|_X \tag{2.3}$$

and hence fractional Bernstein Inequality

$$\left\|T_n^{(\alpha)}\right\|_X \le cn^\alpha \left\|T_n\right\|_X. \tag{2.4}$$

(ii) Combining (2.2) and (2.3) we have

$$\omega_\alpha\left(T_n, \pi/n\right)_X \le c\left\|\Delta_{\pi/n}^\alpha T_n\right\|_X. \tag{2.5}$$

Lemma 2.6. *If $f \in \mathbb{X}$, $q_{\mathbb{X}} < \infty$ and $k = 1, 2, \ldots$, then there exists a constant $c > 0$ such that*

$$I := \left(\int_{\mathbb{T}}\int_{\mathbb{T}} \left\|[V_{n,\lambda}, T_h](f, x)\right\|_X^p \, d\lambda dh\right)^{1/p} \le c\omega_k\left(f, \frac{1}{n+1}\right)_X$$

holds where $n = 0, 1, 2, \ldots$ and $[A, B](f, \cdot) = (A \circ B)(f, \cdot) - (B \circ A)(f, \cdot)$.

Proof. Since $[V_{n,\lambda}, T_h] = T_h \circ \Delta_{h;\lambda}^1 \circ V_{n,\lambda}$ for $h, \lambda \in \mathbb{R}$ we get

$$\left\|[V_{n,\lambda}, T_h](f, \cdot)\right\|_X = \left\|\Delta_{h;\lambda}^1 \circ V_{n,\lambda}(f, \cdot)\right\|_X \text{ for } h, \lambda \in \mathbb{R}.$$

Here and below $\Delta_{h;\lambda}^k$ will denote kth difference operator with step h in variable λ and $\|\cdot\|_{X,\lambda}$ will denote the quasinorm in the variable λ. Since

$$\int_{\mathbb{T}} \left\|\Delta_h^1 f(x)\right\|_X^p \, dh \le Cm \int_0^{(2\pi)/m} \left\|\Delta_h^k f(x)\right\|_X^p \, dh, \quad k, m = 1, 2, \ldots$$

we have by Lemma 2.2 and (iii) of Remark 2.1 that

$$I^p \leq c\,(2N+1) \int\limits_{\mathsf{T}} \int\limits_0^{t_N^1} \left\| \Delta_{h;\lambda}^{2k} \left(V_{n,\lambda}\left(f\left(x\right)\right)\right)\right\|_{X,\lambda}^p dh dx$$

$$= c\,(2N+1) \int\limits_{\mathsf{T}} \int\limits_0^{t_N^1} \left\| \frac{2}{2N+1} \sum_{i=0}^{2N} \Delta_{h;\lambda}^{2k} \left(f\left(t_N^i + \lambda\right) W_n^{(l)}\left(x - t_N^i - \lambda\right)\right)\right\|_{X,\lambda}^p dh dx.$$

Using

$$\Delta_h^k \left(f\left(x\right) g\left(x\right)\right) = \sum_{i=0}^k \binom{k}{i} \Delta_h^{k-i} f\left(x+ih\right) \Delta_h^i g\left(x\right), \quad x,h \in \mathbb{R},$$

we have

$$I^p \leq c\,(2N+1) \sum_{v=0}^{2k} \binom{2k}{v}^p \int\limits_{\mathsf{T}} \int\limits_0^{t_N^1} \left\| \frac{2}{2N+1} \sum_{i=0}^{2N} \Delta_{h;\lambda}^{2k-v} f\left(t_N^i + \lambda + vh\right) \times \right.$$

$$\left. \times \Delta_{h;\lambda}^v W_n^{(k)}\left(x - t_N^i - \lambda\right)\right\|_{X,k}^p dh dx$$

$$\leq c\,(2N+1) \sum_{v=0}^{2k} \binom{2k}{v}^p \int\limits_{\mathsf{T}} \int\limits_0^{t_N^1} \left\| \frac{2}{2N+1} \sum_{i=0}^{2N} \Delta_{h;\lambda}^{2k-v} f\left(t_N^i + \lambda + vh\right) \times \right.$$

$$\left. \times \Delta_{-h;x}^v W_n^{(k)}\left(x - t_N^i - \lambda\right)\right\|_{X,\lambda}^p dh dx$$

$$\leq c\,(2N+1) \sum_{v=0}^{2k} \binom{2k}{v}^p \int\limits_0^{t_N^1} \int\limits_{\mathsf{T}} \left\| \Delta_{-h}^v \circ V_{n,\lambda} \circ \Delta_h^{2k-v} \circ T_h^v \left(f,x\right)\right\|_X^p d\lambda dh.$$

Since $V_{n,\lambda} \circ \Delta_h^{2k-v}\left(f,x\right) \in \mathcal{T}_N$ in x, by (2.3) we have for $0 \leq h \leq t_N^1$, $v = 0,1,\ldots,k$ and $\lambda \in \mathbb{R}$ that

$$\left\| \Delta_{-h}^v \circ V_{n,\lambda} \circ \Delta_h^{2k-v} \circ T_h^v \right\|_X \leq c \left\| \Delta_{t_N^1}^v \circ V_{n,\lambda} \circ \Delta_h^{2k-v} \circ T_h^v \right\|_X$$

$$= c \left\| V_{n,\lambda} \circ T_h^v \circ \Delta_{t_N^1}^v \circ \Delta_h^{2k-v} \right\|_X.$$

Then from (iv) of Remark 2.1 and Lemma 2.2

$$I^p \leq c\,(2N+1) \sum_{v=0}^{2k} \binom{2k}{v}^p \int\limits_0^{t_N^1} \int\limits_{\mathsf{T}} \left\| V_{n,\lambda} \circ T_h^v \circ \Delta_{t_N^1}^v \circ \Delta_h^{2k-v}\left(f,\cdot\right)\right\|_X^p d\lambda dh$$

$$\leq c\,(2N+1) \sum_{v=0}^{2k} \binom{2k}{v}^p \int\limits_0^{t_N^1} \left\| \Delta_{t_N^1}^v \circ \Delta_h^{2k-v}\left(f,\cdot\right)\right\|_X dh \leq c \left(\omega_k\left(f,t_N^1\right)_X\right)^p$$

and Lemma 2.6 is proved.   □

Lemma 2.7. *If $f \in \mathbb{X}$, $q_{\mathbb{X}} < \infty$ and $k = 1, 2, \ldots$, then there exists a constant $c > 0$ such that*

$$\left((2N+1) \int_0^{t_N^1} \int_{\mathsf{T}} \left\| \left[R_{n,\lambda}, \Delta_h^1 \right] (f, \cdot) \right\|_X^p \, d\lambda dh \right)^{1/p} \leq c\omega_k \left(f, \frac{1}{n+1} \right)_X$$

holds where $R_{n,\lambda} := V_{n,\lambda} - I$, $n = 0, 1, 2, \ldots$ and I is identity operator.

Proof. By (iv) of Remark 2.1 and any real λ we have

$$\int_{\mathsf{T}} \left\| [V_{n,\lambda}, T_h] (f, \cdot) \right\|_X^p \, dh = \sum_{i=0}^{2N} \int_{t_N^i}^{t_N^{i+1}} \left\| [V_{n,\lambda}, T_h] (f, \cdot) \right\|_X^p \, dh$$

$$= \sum_{i=0}^{2N} \int_0^{t_N^1} \left\| \left[V_{n,\lambda}, T_{h+t_N^i} \right] (f, \cdot) \right\|_X^p \, dh = \sum_{i=0}^{2N} \int_0^{t_N^1} \left\| T_{t_N^i} \circ [V_{n,\lambda}, T_h] (f, \cdot) \right\|_X^p \, dh$$

$$= (2N+1) \int_0^{t_N^1} \left\| [V_{n,\lambda}, T_h] (f, \cdot) \right\|_X^p \, dh$$

$$= (2N+1) \int_0^{t_N^1} \left\| ((V_{n,\lambda} - I) \circ (T_h - I) - (T_h - I) \circ (V_{n,\lambda} - I)) (f, \cdot) \right\|_X^p \, dh$$

$$= (2N+1) \int_0^{t_N^1} \left\| \left[R_{n,\lambda}, \Delta_h^1 \right] (f, \cdot) \right\|_X^p \, dh.$$

Hence Lemma 2.6 completes the proof. $\qquad\square$

We define the following two auxiliary functions

$$\Lambda_k (f, \delta)_X := \frac{1}{\delta} \int_0^{\delta} \left\| \Delta_h^k (f, \cdot) \right\|_X \, dh,$$

$$\Omega_k (f, \delta)_X := \frac{1}{\delta^k} \int_0^{\delta} \cdots \int_0^{\delta} \left\| \left(\Delta_{h_1}^1 \circ \cdots \circ \Delta_{h_k}^1 \right) (f, \cdot) \right\|_X \, dh_1 \cdots dh_k$$

and prove that these are equivalent to moduli of smoothness $\omega_k (f, \cdot)_X$ of $f \in \mathbb{X}$.

Lemma 2.8. *If $f \in \mathbb{X}$, $q_{\mathbb{X}} < \infty$ and $k = 1, 2, \ldots$, then*

$$\omega_k (f, \delta)_X \leq c_1 \Omega_k (f, \delta)_X \leq c_2 \Lambda_k (f, \delta)_X \leq c_3 \omega_k (f, \delta)_X . \tag{2.6}$$

Proof. First inequality can be obtained [6, p. 184] from

$$\Delta_h^k (f, x) = \sum_{j=1}^{r} (-1)^j \left[\Delta_{js}^r (f, x + jh) - \Delta_{h+js}^k (f, x) \right], \ s \in \mathbb{R}.$$

For the second inequality we obtain

$$\left\| \Delta_t^{k-1} \Delta_h^1 f(x) \right\|_X \leq c \left(\left\| \Delta_t^k f(x) \right\|_X^p + \sum_{v=0}^{k-1} \left\| \Delta_{h+vt}^k f(x) \right\|_X^p \right), \ k = 2, 3, \ldots.$$

These inequalities follow from

$$\Delta_t^{k-1} \circ \Delta_h^1 = -\frac{1}{\gamma_k} \sum_{i=1}^{\beta_k} \alpha_i^{(k)} D_{i-1} \circ \Delta_h^1 \circ \Delta_t^k + \frac{1}{k\gamma_k} T_{(1-k)h} \circ E$$

where $\beta_k = \sum_{v=0}^{k-2} v(v+1), k = 2, 3, \ldots, \alpha_i^{(k)}$ are nonnegative integers with $\alpha_{\beta_k}^{(k)} = 1, \gamma_k = \sum_{i=0}^{\beta_k} \alpha_i^{(k)}, D_i = \begin{pmatrix} k \\ i \end{pmatrix} T_{ih} \circ \Delta_h^{k-i} \circ \Delta_t^i, E = \sum_{i=0}^{\beta_k} T_{it} \circ k T_{(k-1)h} \circ \Delta_t^{k-1} \circ \Delta_h^1$.

Let us assume that the second inequality in (2.6) holds for $k = s - 1, s = 2, 3, \ldots$. Then

$$\Omega_s^p (f, \delta)_X = \frac{1}{\delta} \int_0^\delta \Omega_{s-1}^p \left(\Delta_h^1 f, \delta \right)_X dh \leq \frac{c}{\delta} \int_0^\delta \Lambda_{s-1}^p \left(\Delta_h^1 f, \delta \right)_X dh$$

$$= \frac{c}{\delta^2} \int_0^\delta \int_0^\delta \left\| \Delta_t^{s-1} \Delta_h^1 f(x) \right\|_X^p dt dh$$

$$\leq \frac{c}{\delta^2} \left(2\delta^2 \Lambda_s^p (f, \delta)_X + \sum_{v=0}^{s-1} \int_0^\delta \int_0^\delta \left\| \Delta_{h+vt}^s f(x) \right\|_X^p dt dh \right).$$

Using transformation $u = h$ and $\mu = (h + vt)/(v+1)$ we get for $v = 1, 2, 3, \ldots$

$$\int_0^\delta \int_0^\delta \left\| \Delta_{h+vt}^s f(x) \right\|_X^p dt dh \leq \frac{v+1}{v} \int_0^\delta \int_0^\delta \left\| \Delta_{(v+1)\mu}^s f(x) \right\|_X du d\mu$$

$$\leq \frac{v+1}{v} \delta \int_0^\delta \left\| [I + T_\mu + \ldots + T_{v\mu}]^s \circ \Delta_\mu^s f(x) \right\|_X^p d\mu \leq c\delta^2 \Lambda_s^p (f, \delta)_X$$

and hence required inequality follows. The last inequality in (2.6) is obvious. □

At this stage we need a Jackson type theorem for integer order moduli of smoothness to obtain fractional order ones (Corollary 1.5).

Lemma 2.9. *If $f \in \mathbb{X}$, $q_{\mathbb{X}} < \infty$ and $k = 1, 2, \ldots$, then there exists a constant $c > 0$ such that*

$$E_n(f)_X \leq c\omega_k\left(f, \frac{1}{n+1}\right)_X$$

holds where $n = 0, 1, 2, \ldots$.

Proof. It is easily seen that

$$\Delta^1_{h_1} \circ \cdots \circ \Delta^1_{h_s} \circ R_{n;\lambda} = \tag{2.7}$$

$$= R_{n;\lambda} \circ \Delta^1_{h_1} \circ \cdots \circ \Delta^1_{h_s} + \sum_{i=0}^{s} \Delta^1_{h_1} \circ \cdots \circ \left[\Delta^1_{h_i}, R_{v;\lambda}\right] \circ \cdots \circ \Delta^1_{h_s}$$

for $s = 1, 2, 3, \ldots$ and $\lambda, h_1, \ldots, h_s \in \mathbb{R}$. We prove that if $k = 1, 2, \ldots$, then

$$I_k \equiv \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \left\|R_{v;\lambda_1} \circ \cdots \circ R_{v;\lambda_k}(f, x)\right\|^p_X \, d\lambda_1 \cdots d\lambda_k \leq c\Omega^p_k\left(f, t^1_N\right)_X. \tag{2.8}$$

We suppose that $k = 1$. By (ii) of Remark 2.1 and Lemma 2.2 we find that

$$\int_{\mathbb{T}} \left\|f(x) - V_{n,\lambda}(f, x)\right\|^p_X \, d\lambda = \int_{\mathbb{T}} \left\|(f(x) - t^*_n(x)) - V_{n,\lambda}(f - t^*_n, x)\right\|^p_X \, d\lambda$$

$$\leq cE_n(f)_X + \int_{\mathbb{T}} \left\|V_{n,\lambda}(f - t^*_n, x)\right\|^p_X \, d\lambda \leq c\left(E_n(f)_X\right)^p \tag{2.9}$$

where t^*_n is the best approximating trigonometric polynomial to $f \in \mathbb{X}$.

On the other hand

$$2\pi\left(E_N(f)_X\right)^p \leq \int_{\mathbb{T}} \left\|f(x) - V_{n,\lambda}(f, x)\right\|^p_X \, d\lambda$$

$$\leq c(n+1)^{1-p} \int_{-\pi}^{\pi} \left\|f(\cdot + h) - f(\cdot)\right\|^p_X \left|W^{(k)}_n(h)\right|^p \, dh \leq$$

$$\leq c(n+1)^{1-p} \omega^p_1\left(f, \frac{1}{n+1}\right)_X \int_{-\pi}^{\pi} (1 + n|h|)\left|W^{(k)}_n(h)\right|^p \, dh$$

$$\leq c\left(\omega_1\left(f, \frac{1}{n+1}\right)\right)^p. \tag{2.10}$$

Hence from (2.9), (2.10) and Lemma 2.8 we get

$$I_1 \leq c\left(E_n(f)_X\right)^p \leq c\left(\omega_1\left(f, \frac{1}{n+1}\right)_X\right)^p \leq c\left(\omega_1\left(f, t^1_N\right)_X\right)^p \tag{2.11}$$

$$\leq c\left(\Omega_1\left(f, t^1_N\right)_X\right)^p.$$

This is (2.8) for $k = 1$. Now we suppose that (2.8) holds for $k = s$ Then from (2.7), (2.11) and Lemmas 2.7 and 2.8

$$I_{s+1} \leq c \int_0^{2\pi} \left(\Omega_s\left(R_{n;\lambda}(f), t^1_N\right)_X\right)^p \, d\lambda$$

$$\leq c\left(2N+1\right)^{s}\left\{\int_{0}^{t_{N}^{1}}\cdots\int_{0}^{t_{N}^{1}}\int_{\mathsf{T}}\left\|R_{n;\lambda}\circ\Delta_{h_{1}}^{1}\circ\cdots\Delta_{h_{s}}^{1}\left(f,x\right)\right\|_{X}^{p}d\lambda dh_{1}\cdots dh_{s}\right\}$$

$$+\sum_{i=1}^{s}\int_{0}^{t_{N}^{1}}\cdots\int_{0}^{t_{N}^{1}}\int_{\mathsf{T}}\left\|\Delta_{h_{1}}^{1}\circ\cdots\circ\left[\Delta_{h_{i}}^{1},R_{v;\lambda}\right]\circ\cdots\circ\Delta_{h_{s}}^{1}\left(f,x\right)\right\|_{X}^{p}d\lambda dh_{1}\cdots dh_{s}$$

$$\leq c\left(2N+1\right)^{s}\left\{\int_{0}^{t_{N}^{1}}\cdots\int_{0}^{t_{N}^{1}}\Omega_{1}^{p}\left(\Delta_{h_{1}}^{1}\circ\cdots\Delta_{h_{s}}^{1}\left(f\right),t_{N}^{1}\right)dh_{1}\cdots dh_{s}\right.$$

$$\left.+\sum_{i=1}^{s}\left(2N+1\right)^{1-i}\int_{0}^{t_{N}^{1}}\cdots\int_{0}^{t_{N}^{1}}\left\|\left[\Delta_{h_{i}}^{1}\circ R_{v;\lambda}\right]\circ\cdots\circ\Delta_{h_{s}}^{1}\left(f,x\right)\right\|_{X}^{p}d\lambda dh_{i+1}\cdots dh_{s}\right\}$$

$$\leq c\Omega_{s+1}\left(f,t_{N}^{1}\right)_{X}+c\left(2N+1\right)^{s}\sum_{i=1}^{s}\left(2N+1\right)^{-i}\times$$

$$\times\int_{0}^{t_{N}^{1}}\cdots\int_{0}^{t_{N}^{1}}\Omega_{s+1}^{p}\left(\Delta_{h_{i+1}}^{1}\circ\cdots\Delta_{h_{s}}^{1}\left(f\right),t_{N}^{1}\right)dh_{i+1}\cdots dh_{s}\leq c\Omega_{s+1}^{p}\left(f,t_{N}^{1}\right)_{X}.$$

Since for any real $\lambda_{1},\cdots,\lambda_{k}$

$$R_{v;\lambda_{1}}\circ\cdots\circ R_{v;\lambda_{k}}\left(f,x\right)=T\left(x\right)+\left(-1\right)^{k}f\left(x\right)$$

we obtain

$$E_{N}^{p}\left(f\right)_{X}\leq I_{k}\leq\Omega_{k}^{p}\left(f,t_{N}^{1}\right)_{X}\leq c\omega_{k}^{p}\left(f,t_{N}^{1}\right)_{X}$$

and this completes the proof. □

3. Proofs of the Main Results

Proof of Theorem 1.4. Starting with upper inequality we take a $t\in\left(0,2\pi\right)$. Then there exists $n\in\mathbb{Z}^{+}$ such that $\pi/n<t\leq2\pi/n$. Let t_{n}^{*} be the best approximating trigonometric polynomial to $f\in\mathbb{X}$.

Using Lemma 2.9 we get

$$E_{n}\left(f\right)_{X}=\left\|f-t_{n}^{*}\right\|_{X}\leq c\omega_{\left[\alpha\right]+1}\left(f,\frac{1}{n}\right)_{X}=c\sup_{0<h\leq\left(1/n\right)}\left\|\Delta_{h}^{\left[\alpha\right]+1}f\right\|_{X}$$

$$=c\sup_{0<h\leq\left(1/n\right)}\left\|\Delta_{h}^{1-\left\{\alpha\right\}}\left(\Delta_{h}^{\alpha}f\right)\right\|_{X}\leq c\sup_{0<h\leq\left(1/n\right)}\left\|\Delta_{h}^{\alpha}f\right\|_{X}$$

$$\leq c\sup_{0<h\leq\left(\pi/n\right)}\left\|\Delta_{h}^{\alpha}f\right\|_{X}=c\omega_{\alpha}\left(f,\frac{\pi}{n}\right)_{X}. \qquad (3.1)$$

From (2.3) and (3.1) we have

$$\left\|t_{n}^{*\left(\alpha\right)}\right\|_{X}\leq c2^{-\alpha}n^{\alpha}\left\|\Delta_{\pi/n}^{\alpha}t_{n}^{*}\right\|_{X}\leq$$

$$\leq c\left(\pi/t\right)^{\alpha}\left\{c\left(\alpha,\mathbb{X}\right)\left\|f-t_{n}^{*}\right\|_{X}+\left\|\Delta_{\pi/n}^{\alpha}f\right\|_{X}\right\}\leq c\left(\alpha,\mathbb{X}\right)t^{-\alpha}\omega_{\alpha}\left(f,\frac{\pi}{n}\right)_{X}$$

and therefore
$$K_\alpha\left(f,t,X,X^\alpha\right) \le \|f - t_n^*\|_X + t^\alpha \left\|t_n^{*(\alpha)}\right\|_X \le c\left(\alpha,\mathbb{X}\right)\omega_\alpha\left(f,t\right)_X.$$
The lower inequality is easy. □

Proof of Theorem 1.6. Let $T_n \in \mathcal{T}_n$ be the best approximating polynomial of f and let $m \in \mathbb{Z}^+$. We write
$$U_0\left(x\right) = T_1\left(x\right) - T_0\left(x\right), \ U_\upsilon\left(x\right) = T_{2^\upsilon}\left(x\right) - T_{2^{\upsilon-1}}\left(x\right), \ \upsilon \ge 1. \qquad (3.2)$$
Then
$$T_{2^m}\left(x\right) = T_0\left(x\right) + \sum_{\nu=0}^{m} U_\upsilon\left(x\right).$$
In this case
$$\omega_\alpha^p\left(f,\pi/n\right)_X \le \omega_\alpha^p\left(f - T_{2^m},\pi/n\right)_X + \omega_\alpha^p\left(T_{2^m},\pi/n\right)_X,$$
$$\omega_\alpha^p\left(f - T_{2^m},\pi/n\right)_X \le cE_{2^m}^p\left(f\right)_X$$
and
$$\omega_\alpha^p\left(T_{2^m},\pi/n\right)_X \le \omega_\alpha^p\left(U_0,\pi/n\right)_X + \sum_{\nu=1}^{m} \omega_\alpha^p\left(U_\upsilon,\pi/n\right)_X$$
$$\le c\left(\frac{\pi}{n}\right)^{p\alpha}\left(\|U_0\|_X^p + \sum_{\nu=1}^{m} 2^{\upsilon p\alpha}\|U_\upsilon\|_X^p\right).$$
On the other hand
$$\|U_0\|_X^p \le cE_0^p\left(f\right)_X$$
and
$$\|U_\upsilon\|_X^p \le 2E_{2^{\upsilon-1}}^p\left(f\right)_X.$$
It is easily seen that
$$2^{\upsilon p\alpha}E_{2^{\upsilon-1}}^p\left(f\right)_X \le c\sum_{\mu=2^{\upsilon-2}+1}^{2^{\upsilon-1}} \mu^{p\alpha-1}E_\mu^p\left(f\right)_X, \quad \nu = 2,3,\dots \qquad (3.3)$$
and therefore
$$\omega_\alpha^p\left(T_{2^m},\pi/n\right)_X \le c\left(\frac{\pi}{n}\right)^{p\alpha}\left\{E_0^p\left(f\right)_X + \sum_{\upsilon=1}^{m} 2^{\upsilon p\alpha}E_{2^{\upsilon-1}}^p\left(f\right)_X\right\}.$$
If we choose $2^{m-1} \le n < 2^m$, then
$$\omega_\alpha^p\left(T_{2^m},\pi/\left(n+1\right)\right)_X \le \frac{c\left(\alpha,\mathbb{X}\right)}{n^{p\alpha}}\sum_{\upsilon=0}^{n}\left(\nu+1\right)^{p\alpha-1}E_\upsilon^p\left(f\right)_X$$
and
$$E_{2^m}^p\left(f\right)_X \le E_{2^{m-1}}^p\left(f\right)_X \le \frac{c}{n^{\alpha p}}\sum_{\upsilon=0}^{n}\left(\nu+1\right)^{p\alpha-1}E_\upsilon^p\left(f\right)_X.$$
Last two inequalities complete the proof. □

Proof of Theorem 1.9. By Levi's theorem and (3.2)

$$\left\| T_0\left(x\right) + \sum_{v=0}^{\infty} U_v\left(x\right) \right\|_X^p = \lim_{r\to\infty} \left\| T_0\left(x\right) + \sum_{v=0}^{r} U_v\left(x\right) \right\|_X^p$$

$$\leq c\left\| T_0\left(x\right) \right\|_X^p + c\lim_{r\to\infty} \sum_{v=0}^{r} \left\| U_v\left(x\right) \right\|_X^p \leq cE_0^p\left(f\right)_X + c\sum_{v=1}^{\infty} E_{2^{v-1}}^p\left(f\right)_X < \infty.$$

From (1.8) and (3.3) the last series converges and therefore

$$f\left(x\right) = \lim_{r\to\infty} T_{2^r}\left(x\right) = T_0\left(x\right) + \sum_{v=0}^{\infty} U_v\left(x\right) \ a.e.$$

Analogously using Levi's Theorem

$$\left\| \sum_{v=0}^{\infty} U_v^{(\beta)}\left(x\right) \right\|_X^p \leq c\sum_{v=0}^{\infty} \left\| U_v^{(\beta)}\left(x\right) \right\|_X^p \leq c\sum_{v=0}^{\infty} 2^{vp\beta} \left\| U_v\left(x\right) \right\|_X^p$$

$$\leq c\left(E_0^p\left(f\right)_X + \sum_{v=1}^{\infty} 2^{vp\beta} E_{2^{v-1}}^p\left(f\right)_X \right) < \infty$$

and the series

$$\sum_{v=0}^{\infty} U_v^{(\beta)}\left(x\right)$$

converges a.e., its sum g is of class X. Now we prove that $g = f^{(\beta)}$ a.e.

For $0 \neq h \in \mathbb{R}$ we have

$$\left\| \frac{\Delta_h^{\beta} f\left(x\right)}{h^{\beta}} - g\left(x\right) \right\|_X^p \leq c\left\| \frac{1}{h^{\beta}} \sum_{k=0}^{N} \left(-1\right)^k \binom{\beta}{k} f\left(x + \left(\beta - k\right)h\right) - g\left(x\right) \right\|_X^p$$

$$+ c\left\| \frac{1}{h^{\beta}} \sum_{k=N+1}^{\infty} \left(-1\right)^k \binom{\beta}{k} f\left(x + \left(\beta - k\right)h\right) \right\|_X^p := c\left(I_1 + I_2 \right)$$

In this case

$$I_2 \leq \frac{1}{\left|h\right|^{\beta p}} \sum_{k=N+1}^{\infty} \left| \binom{\beta}{k} \right|^p \left\|f\right\|_X^p$$

and hence

$$\lim_{N\to\infty} I_2 = 0.$$

Now by Levi's theorem

$$I_1 = \left\| \frac{1}{h^{\beta}} \sum_{v=0}^{\infty} \sum_{k=0}^{N} \left(-1\right)^k \binom{\beta}{k} U_v\left(x + \left(\beta - k\right)h\right) - g\left(x\right) \right\|_X$$

$$\leq c\sum_{v=0}^{\infty} \left\| \frac{1}{h^{\beta}} \sum_{k=0}^{N} \left(-1\right)^k \binom{\beta}{k} U_v\left(x + \left(\beta - k\right)h\right) - U_v^{(\beta)}\left(x\right) \right\|_X^p =: Y_N.$$

The last series converges uniformly in $N \geq 1$, because its υth term doesn't exceed

$$\frac{1}{|h|^{\beta p}} \sum_{k=0}^{\infty} \left| \binom{\beta}{k} \right|^p \left(\|U_\upsilon\|_X^p + \|U_\upsilon^{(\beta)}\|_X^p \right) \leq c \left(\frac{1}{|h|^{\beta p}} + 1 \right) 2^{\upsilon p \beta} E_{2^{\upsilon-1}}^p (f)_X .$$

From Lebesgue Dominated convergence theorem we have

$$\lim_{N \to \infty} Y_N = \sum_{\upsilon=0}^{\infty} \left\| \frac{\Delta_h^\beta U_\upsilon (x)}{h^\beta} - U_\upsilon^{(\beta)} (x) \right\|_X^p$$

and then

$$\left\| \frac{\Delta_h^\beta f}{h^\beta} - g \right\|_X^p \leq c \sum_{\upsilon=0}^{\infty} \left\| \frac{\Delta_h^\beta U_\upsilon}{h^\beta} - U_\upsilon^{(\beta)} \right\|_X^p$$

$$\leq c \sum_{\upsilon=0}^{s} \left\| \frac{\Delta_h^\beta U_\upsilon}{h^\beta} - U_\upsilon^{(\beta)} \right\|_X^p + c \sum_{\upsilon=s+1}^{\infty} \left\| \frac{\Delta_h^\beta U_\upsilon}{h^\beta} \right\|_X^p + c \sum_{\upsilon=s+1}^{\infty} \|U_\upsilon^{(\beta)}\|_X^p$$

$$\leq c \sum_{\upsilon=0}^{s} \left\| \frac{\Delta_h^\beta U_\upsilon}{h^\beta} - U_\upsilon^{(\beta)} \right\|_X^p + c \sum_{\upsilon=s+1}^{\infty} 2^{\upsilon p \beta} E_{2^{\upsilon-1}}^p (f)_X .$$

For given positive ε the last term is less than ε for sufficiently large s. By Remark 1.3 we get

$$\lim_{h \to 0^+} \left\| \frac{\Delta_h^\beta U_\upsilon}{h^\beta} - U_\upsilon^{(\beta)} \right\|_X = 0$$

and therefore

$$\lim_{h \to 0^+} \left\| \frac{\Delta_h^\beta f}{h^\beta} - g \right\|_X < \varepsilon.$$

This implies that $g = f^{(\beta)}$ a.e.

Let $m \in \mathbb{Z}^+$ be such that $2^{m-1} \leq n < 2^m$. We have

$$\left\| T_n^{(\beta)} - f^{(\beta)} \right\|_X^p = \left\| T_n^{(\beta)} - \sum_{\upsilon=0}^{\infty} U_\upsilon^{(\beta)} \right\|_X^p \leq c \left\| T_n^{(\beta)} - T_{2^m}^{(\beta)} \right\|_X^p + c \sum_{\upsilon=m+1}^{\infty} \|U_\upsilon^{(\beta)}\|_X^p$$

$$\leq c \left(2^{mp\beta} E_n^p (f)_X + \sum_{\upsilon=m+1}^{\infty} 2^{\upsilon p \beta} E_{2^{\upsilon-1}}^p (f)_X \right)$$

$$\leq c \left(n^{\beta p} E_n^p (f)_X + \sum_{\mu=n+1}^{\infty} \mu^{p\beta-1} E_\mu^p (f)_X \right)$$

and the result is proved. $\qquad \square$

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