



BANACH FUNCTION ALGEBRAS AND CERTAIN POLYNOMIALLY NORM-PRESERVING MAPS

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Communicated by K. Jarosz

ABSTRACT. Let A and B be Banach function algebras on compact Hausdorff spaces X and Y , respectively. Given a non-zero scalar α and $s, t \in \mathbb{N}$ we characterize the general form of suitable powers of surjective maps $T, T' : A \rightarrow B$ satisfying $\|(Tf)^s(T'g)^t - \alpha\|_Y = \|f^s g^t - \alpha\|_X$, for all $f, g \in A$, where $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the supremum norms on X and Y , respectively. A similar result is given for the case where $T = T'$ and T is defined between certain subsets of A and B . We also show that if $T : A \rightarrow B$ is a surjective map satisfying the stronger condition $R_\pi((Tf)^s(Tg)^t - \alpha) \cap R_\pi(f^s g^t - \alpha) \neq \emptyset$ for all $f, g \in A$, where $R_\pi(\cdot)$ denotes the peripheral range of the algebra elements, then there exists a homeomorphism φ from the Choquet boundary $c(B)$ of B onto the Choquet boundary $c(A)$ of A such that $(Tf)^d(y) = (T1)^d(y) (f \circ \varphi(y))^d$ for all $f \in A$ and $y \in c(B)$, where d is the greatest common divisor of s and t .

1. INTRODUCTION

A generalization of the classical Banach-Stone theorem asserts that each surjective linear isometry between uniform algebras is a weighted composition operator [17, 13]. A related problem, is the study of certain maps, not assumed to be linear, between uniform algebras (or in general, semisimple commutative Banach algebras) preserving multiplicatively some structures such as norm, spectrum or certain subsets of the spectrum called the peripheral spectrum. The study of such maps was initiated by Molnar in [16] by proving that for a first countable compact

Date: Received: 24 October 2011; Accepted: 11 December 2011.

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2010 *Mathematics Subject Classification.* Primary 46J10; Secondary 47B48, 46J20.

Key words and phrases. Banach function algebras, polynomially norm-preserving maps, peripheral spectrum, peripheral range, Choquet boundary.

Hausdorff space X , a surjective map $T : C(X) \rightarrow C(X)$ preserving multiplicatively the spectrum of functions, i.e., $TfTg(Y) = fg(X)$, for all $f, g \in C(X)$, is a weighted composition operator. Then this result was extended in [5, 8, 10, 18, 19] for more general algebras of functions rather than $C(X)$. In particular, it is shown in [8] that the same result is valid for maps between unital semisimple commutative Banach algebras preserving multiplicatively the spectrum of the algebra elements. Introducing the peripheral range and the peripheral spectrum for the elements in a uniform algebra, surjective maps $T : A \rightarrow B$ between uniform algebras A and B preserving multiplicatively the peripheral ranges were studied by Luttmann and Tonev in [15]. A similar result was given in [12] for a mapping $T : A \rightarrow B$ between uniform algebras A and B , preserving peaking functions such that the peripheral range of fg has a nonempty intersection with the peripheral range of $TfTg$ for all $f, g \in A$. More results can also be found in [11, 24].

In [6] Hatori, Miura, Takagi characterized the general form of a surjective map $T : A \rightarrow B$ between uniform algebras A and B on compact Hausdorff spaces X and Y , respectively, satisfying $\|fg + \alpha\|_X = \|TfTg + \alpha\|_Y$ for all $f, g \in A$, where α is a non-zero complex number (i.e. $\alpha \in \mathbb{C} \setminus \{0\}$) and $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the supremum norms on X and Y , respectively. Then Luttmann and Lambert in [14] gave an alternative characterization for such maps. Similar results were obtained in [2] for the pointed Lipschitz algebras and also by the authors in [9] for certain subalgebras of $C(X)$ and $C(Y)$ (for compact Hausdorff spaces X and Y) which are Banach algebras under some norms. On the other hand, it was shown in [21] that if s, t are positive integers, i.e. $s, t \in \mathbb{N}$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $T : A \rightarrow B$ is a surjective map such that $\|(Tf)^s(Tg)^t - \alpha\|_Y = \|f^s g^t - \alpha\|_X$ for all $f, g \in A$, then there exist a homeomorphism φ from the Choquet boundary $c(B)$ of B onto the Choquet boundary $c(A)$ of A and a clopen subset K of $c(B)$ such that for each $f \in A$,

$$(Tf)^d(y) = (T1)^d(y) \begin{cases} f(\varphi(y))^d & y \in K, \\ \frac{f(\varphi(y))^d}{f(\varphi(y))^d} & y \in c(B) \setminus K, \end{cases}$$

where d is the greatest common divisor of s and t . Indeed, in [21], this result has been obtained for maps defined between arbitrary subsets A and B containing A^{-1} and B^{-1} , respectively, satisfying the same condition. For further results see also [3, 7, 22]. We also refer the interested reader to [4] for a recent survey on these topics.

In this paper we consider the case where A and B are certain subalgebras of continuous functions on compact Hausdorff spaces X and Y endowed with some Banach algebra norms (not necessarily supremum norm) and extend the latter result for surjective maps $T, T' : A \rightarrow B$ satisfying $\|(Tf)^s(T'g)^t - \alpha\|_Y = \|f^s g^t - \alpha\|_X$ for all $f, g \in A$ (Theorem 3.2). It should be noted that since the maps under consideration are not necessarily linear we can not simply extend them to the uniform closures of A and B and use the known results in uniform algebra case. Indeed, the extension will be defined between appropriate subsets of the uniform closures of A and B . We also show that for two-variables polynomial $p(z, w) = z^s w^t - \alpha$, where $s, t \in \mathbb{N}$, and $\alpha \in \mathbb{C} \setminus \{0\}$, if A_0 and B_0 are subsets of A and B containing all invertible elements and $T : A_0 \rightarrow B_0$ is a surjective map

satisfying

$$\|p(Tf, Tg)\|_Y = \|p(f, g)\|_X \quad (f, g \in A_0),$$

then there exist a homeomorphism φ from the Choquet boundary $c(B)$ of B onto the Choquet boundary $c(A)$ of A and a clopen subset K of $c(B)$ such that for each $f \in A_0$,

$$(Tf)^d(y) = (T1)^d(y) \begin{cases} f(\varphi(y))^d & y \in K, \\ f(\varphi(y)) & y \in c(B) \setminus K, \end{cases}$$

where d is the greatest common divisor of s and t (Theorem 3.7). In particular, this gives the description of T whenever $d = 1$. For $A_0 = A$ and $B_0 = B$ under the additional assumption $T(f + 1) = Tf + 1$, $f \in A$, we see that $Tf = f \circ \varphi$ on K and $Tf = \overline{f \circ \varphi}$ on $c(B) \setminus K$. Furthermore, we obtain the representation of a surjective map $T : A \rightarrow B$ satisfying the stronger condition

$$R_\pi(p(Tf, Tg)) \cap R_\pi(p(f, g)) \neq \emptyset \quad (f, g \in A),$$

where $p(z, w) = z^s w^t - \alpha$ and $R_\pi(\cdot)$ denotes the peripheral range of algebra elements and show that such maps are weighted composition operators if the the greatest common divisor of s and t is 1. This result gives a partial answer to the Question 7 in [4] for the case where the Banach algebra norm is not necessarily the supremum norm.

2. PRELIMINARIES

Let X be a compact Hausdorff space. By $C(X)$ we mean the algebra of all continuous complex-valued functions on X and $\|\cdot\|_X$ denotes the supremum norm on X . A subalgebra A of $C(X)$ is called a *function algebra* on X if A contains the constants and separates the points of X . A *Banach function algebra* on X is a function algebra on X which is a Banach algebra under a norm. A uniformly closed function algebra on X is called a *uniform algebra* on X . The group of invertible elements in a Banach function algebra A is denoted by A^{-1} .

Let A be a Banach function algebra on a compact Hausdorff space X . We denote the uniform closure of A in $C(X)$ by \overline{A} . A subset E of X is called a *boundary* for A if every $f \in A$ attains its maximum modulus at some point of E . The unique minimal closed boundary for A , denoted by ∂A , is called the *Šilov boundary* of A . The *Choquet boundary* of A which is denoted by $c(A)$ is the set of all $x \in X$ such that the evaluation homomorphism δ_x at x is an extreme point of the unit ball of the dual space of $(A, \|\cdot\|_X)$. It is well known that $c(A)$ is dense in ∂A (see [23, Corollary 7.24, Theorem 7.30] for uniform algebra case and [1, Theorem 1] for general case).

For an element f in a Banach function algebra A on X , $\sigma(f)$ and $r(f)$ denote the spectrum and the spectral radius of f , respectively. The *peripheral spectrum* and the *peripheral range* of $f \in A$ is defined by

$$\sigma_\pi(f) = \{\lambda \in \sigma(f) : |\lambda| = r(f)\}$$

and

$$R_\pi(f) = \{\lambda \in f(X) : |\lambda| = \|f\|_X\}.$$

We should note that the peripheral spectrum and the peripheral range of elements in a uniform algebra are the same by [15, Lemma 1].

Let A be a Banach function algebra on a compact Hausdorff space X . A function $f \in A$ is called a *peaking function* of A if $R_\pi(f) = \{1\}$. We call a subset K of X a *peak set* for A if there exists a peaking function $f \in A$ such that $K = \{x \in X : f(x) = 1\}$. For an arbitrary function $f \in A$ we set $M_f = \{t \in X : |f(t)| = \|f\|_X\}$. A point $x \in X$ is a *strong boundary point* for A if for every neighborhood V of x , there exists a function $f \in A$ such that $\|f\|_X = f(x) = 1$ and $|f| < 1$ on $X \setminus V$. It is easy to see that the function f in this definition can be chosen to be a peaking function in $\exp(A)$. It is well known that if A is a uniform algebra, then $c(A)$ is, indeed, the set of all strong boundary points for A (see [23, Theorem 7.30]).

The following generalizations of the classical Bishop's Lemma may be used frequently in this paper:

Lemma 2.1. [14, Corollary 1.1] *Let A be a uniform algebra on a compact Hausdorff space X . If $E \subseteq X$ is a peak set and $f \in A$ such that $f|_E \neq 0$, then there exists a peaking function $h \in \exp(A)$ such that $M_h = E$ and fh attains its maximum modulus exclusively on E .*

Lemma 2.2. [5, Lemma 2.1] *Let A be a uniform algebra on a compact Hausdorff space X . Let $f \in A$ and $x_0 \in c(A)$ such that $f(x_0) \neq 0$. Then there exists a peaking function $h \in A$ with $h(x_0) = 1$ and $R_\pi(fh) = \{f(x_0)\}$.*

We also use the following result, proved by Shindo, concerning the general form of certain preserving maps between some subgroups of invertible elements of uniform algebras:

Theorem 2.3. [21, Proposition 2.6] *Let A and B be uniform algebras on compact Hausdorff spaces X and Y and let A_0 and B_0 be subgroups of A^{-1} and B^{-1} which contain $\exp(A)$ and $\exp(B)$, respectively. Let $S : A_0 \rightarrow B_0$ be a surjection satisfying $S(1) = 1$ and*

$$\left\| \frac{S(f)}{S(g)} - 1 \right\|_Y = \left\| \frac{f}{g} - 1 \right\|_X$$

for all $f, g \in A_0$. Then there exist a homeomorphism $\varphi : c(B) \rightarrow c(A)$ and a clopen subset K of $c(B)$ such that for every $f \in A_0$,

$$Sf(y) = \begin{cases} f(\varphi(y)) & y \in K, \\ f(\varphi(y)) & y \in c(B) \setminus K. \end{cases}$$

We should note that the subset K given in the proof of the above theorem is defined by $K = \{y \in c(B) : S(i)(y) = i\}$.

3. MAIN RESULTS

Let A be a Banach function algebra on a compact Hausdorff space X and let \overline{A} be the closure of A in $(C(X), \|\cdot\|_X)$. For $s, t \in \mathbb{N}$ and a nonempty subset E of

X we set

$$E_t^s = \{f \in E : \text{there is an } f' \in E \text{ with } f^s f'^t = 1\}.$$

Clearly A_t^s is a subgroup of A^{-1} containing $\exp(A)$ and also $A_t^s = (A^{-1})_t^s$. Moreover, if E is a subgroup of A^{-1} , then $E_s^1 = \{f^s : f \in E\}$. The idea of considering such subsets comes from [21].

Given $s, t \in \mathbb{N}$ we use the notation A' for the set of all invertible elements $f \in \overline{A}$ for which there exists a sequence $\{f_n\}$ in $(A_t^s)_s^1$ such that $\lim_{n \rightarrow \infty} \|f_n - f\|_X = 0$, that is, $A' = (\overline{A})^{-1} \cap \overline{(A_t^s)_s^1}$. We note that since $\exp(A) \subseteq (A_t^s)_s^1$ it follows easily that $\exp(\overline{A}) \subseteq A'$. Moreover, it is obvious that A' contains the subset $(A_t^s)_s^1$ of A^{-1} . For each $x \in c(A)$, we also set $F_x = \{f \in A' : |f(x)| = 1 = \|f\|_X\}$.

Remark 3.1. a) If A is a Banach function algebra on a compact Hausdorff space X , then since \overline{A} is a uniform algebra on X and $c(A) = c(\overline{A})$ it follows that for each point $x \in c(A)$ and open neighborhood V of x there exists a peaking function $f \in \exp(\overline{A}) \subseteq A'$ such that $f(x) = 1 = \|f\|_X$ and $|f| < 1$ on $X \setminus V$.

b) If A is a Banach function algebra on a compact Hausdorff space X , then for each $f \in A$ and $x_0 \in c(A)$ with $f(x_0) \neq 0$ we can find a peaking function $h \in A'$ with $h(x_0) = 1$ and $R_\pi(fh) = \{f(x_0)\}$. For this, it suffices to choose, by Lemma 2.2, a peaking function $k \in \overline{A}$ with $k(x_0) = 1$ and $R_\pi(fk) = \{f(x_0)\}$ and then, using Lemma 2.1, choose a peaking function $h \in \exp(\overline{A}) \subseteq A'$ with $M_h = M_k$ and $M_{fh} \subseteq M_k$. Then it is easy to see that h has the desired properties. Therefore, the peaking function h given by Lemma 2.2 for the uniform algebra \overline{A} can be chosen to be an element of A' .

c) If f and g are elements of a Banach function algebra A on X such that $\|fh - 1\|_X = \|gh - 1\|_X$ for all $h \in \exp(A)$, then since this equality also holds for all $h \in \exp(\overline{A})$, it follows immediately from [21, Lemma 3.1] that $f = g$.

Theorem 3.2. *Let A and B be Banach function algebras on compact Hausdorff spaces X and Y , respectively. Let $s, t \in \mathbb{N}$ and $\alpha \in \mathbb{C} \setminus \{0\}$. If $T : A \rightarrow B$ and $T' : A \rightarrow B$ are surjective maps satisfying*

$$\|(Tf)^s (T'g)^t - \alpha\|_Y = \|f^s g^t - \alpha\|_X$$

for all $f, g \in A$, then there exist a homeomorphism φ from $c(B)$ onto $c(A)$ and a clopen subset K of $c(B)$ such that for each $f \in A$,

$$(Tf)^s(y) = (T1)^s(y) \begin{cases} f(\varphi(y))^s & y \in K, \\ f(\varphi(y))^s & y \in c(B) \setminus K, \end{cases}$$

and

$$(T'f)^t(y) = (T'1)^t(y) \begin{cases} f(\varphi(y))^t & y \in K, \\ f(\varphi(y))^t & y \in c(B) \setminus K. \end{cases}$$

Proof. Let β be a complex number with $\beta^t = \alpha$. We first show that $T(A_t^s) = B_t^s$. For suppose that $g \in A_t^s$, then there exists a function $g' \in A$ such that $g^s (g')^t = 1$. Hence

$$\|(Tg)^s T'(\beta g')^t - \alpha\|_Y = \|g^s (\beta g')^t - \alpha\|_X = 0,$$

and consequently $(Tg)^s(\beta^{-1}T'(\beta g'))^t = 1$. This shows that $T(A_t^s) \subseteq B_t^s$. Conversely, if $G \in B$ and $G^s(G')^t = 1$ for some $G' \in B$, then since there exist $g, g' \in A$ such that $G = Tg$, $T'g' = \beta G'$, a similar argument shows that $g^s(\beta^{-1}g')^t = 1$, that is $g \in A_t^s$. Therefore $B_t^s \subseteq T(A_t^s)$ and consequently $T(A_t^s) = B_t^s$.

We should note that if $f \in A$ and $g \in A_t^s$, then letting $g' \in A$ such that $g^s \cdot (g')^t = 1$ it follows from the above argument that

$$\left\| \frac{(Tf)^s}{(Tg)^s} - 1 \right\|_Y = \frac{1}{|\alpha|} \|(Tf)^s(T'(\beta g'))^t - \alpha\|_Y = \frac{1}{|\alpha|} \|f^s(\beta g')^t - \alpha\|_Y = \left\| \frac{f^s}{g^s} - 1 \right\|_X.$$

Hence

$$\left\| \frac{(Tf)^s}{(Tg)^s} - 1 \right\|_Y = \left\| \frac{f^s}{g^s} - 1 \right\|_X \quad (f \in A, g \in A_t^s) \quad (3.1)$$

We prove the theorem through the following steps:

Step 1. *There is a surjective map $S : A' \rightarrow B'$ such that $S((A_t^s)_s^1) = (B_t^s)_s^1$ and $S(f^s) = \frac{(Tf)^s}{(T1)^s}$ for all $f \in A_t^s$. Furthermore, $S1 = 1$ and $\left\| \frac{Sf}{Sg} - 1 \right\|_Y = \left\| \frac{f}{g} - 1 \right\|_X$ for all $f, g \in A'$.*

Proof. We first define the map S on $(A_t^s)_s^1 = \{f^s : f \in A_t^s\}$ by

$$S(f^s) = \frac{(Tf)^s}{(T1)^s} \quad (f \in A_t^s).$$

Note that by (3.1), S is well-defined. It is clear that $S(f^s) = \frac{(Tf)^s}{(T1)^s} \in (B_t^s)_s^1 \subseteq B'$ for all $f \in A_t^s$. Since B_t^s is closed under multiplication and $T(A_t^s) = B_t^s$ we conclude that $S((A_t^s)_s^1) = (B_t^s)_s^1$. Furthermore, it follows from (3.1) that

$$\left\| \frac{Sf}{Sg} - 1 \right\|_Y = \left\| \frac{f}{g} - 1 \right\|_X \quad (f, g \in (A_t^s)_s^1) \quad (3.2)$$

We now extend S on A' as follows. Let $f \in A'$ be an arbitrary element and let $\{f_n\}$ be a sequence in $(A_t^s)_s^1$ converging uniformly on X to f . Since f is invertible in \bar{A} , it follows that there exist scalars $c, d \in \mathbb{R}^+$ such that $c < |f_n(x)| < d$ for all sufficiently large $n \in \mathbb{N}$ and $x \in X$. So we can assume that $c < |f_n(x)| < d$ for all $n \in \mathbb{N}$ and $x \in X$. Let $\epsilon > 0$. Then since $\{f_n\}$ is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\|_X < \epsilon$ for all $m, n \geq N$. Thus for all $m, n \geq N$ we have

$$\left| \frac{f_n(x)}{f_m(x)} - 1 \right| \leq \frac{\epsilon}{|f_m(x)|} < \frac{\epsilon}{c} \quad (x \in X),$$

and so $\left\| \frac{f_n}{f_m} - 1 \right\|_X < \frac{\epsilon}{c}$. Hence, by (3.2), $\left\| \frac{Sf_n}{Sf_m} - 1 \right\|_Y = \left\| \frac{f_n}{f_m} - 1 \right\|_X < \frac{\epsilon}{c}$ for all $n, m \geq N$. Therefore,

$$|Sf_n(y) - Sf_m(y)| < \frac{\epsilon}{c} |Sf_m(y)| \leq \frac{\epsilon}{c} \|Sf_m\|_Y \quad (n, m \geq N, y \in Y).$$

On the other hand, since for all $n \in \mathbb{N}$, $\|Sf_n\|_Y \leq \|Sf_n - 1\|_Y + 1 = \|f_n - 1\|_X + 1 \leq \|f_n\|_X + 2$, it follows that $\{\|Sf_n\|_Y\}$ is a bounded sequence. Hence $\{Sf_n\}$ is a Cauchy sequence in the uniform closure \bar{B} of B . Therefore, there exists an $F \in \bar{B}$ with $\lim_{n \rightarrow \infty} \|Sf_n - F\|_Y = 0$. It is easy to see that the function F , obtained in

this way, is independent of the choice of the sequence $\{f_n\}$ in $(A_t^s)_s^1$ converging uniformly to f . We claim that F is an element of B' . Note that since for each $n \in \mathbb{N}$, $Sf_n \in S((A_t^s)_s^1) = (B_t^s)_s^1$ and $\|Sf_n - F\|_Y \rightarrow 0$ it follows that F is in the uniform closure of $(B_t^s)_s^1$. Therefore it suffices to show that F is invertible in \overline{B} . Assume on the contrary that $F \notin (\overline{B})^{-1}$. Then since $\{Sf_n\}$ converges uniformly to F and for each $n \in \mathbb{N}$, $Sf_n \in B^{-1} \subseteq (\overline{B})^{-1}$ we conclude that $\|(Sf_n)^{-1}\|_Y \rightarrow \infty$. Since $S1 = 1$ it follows from (3.2) that $\|f_n^{-1} - 1\|_X \rightarrow \infty$ and consequently $\|f_n^{-1}\|_X \rightarrow \infty$ which is impossible, since $f_n \rightarrow f$ in \overline{A} and so $f_n^{-1} \rightarrow f^{-1}$ in \overline{A} . Therefore, $F \in B'$.

Using this fact we extend S on A' in such a way that for each $f \in A'$, Sf is the element $F \in B'$ given in the above argument. As we noted before, for each $f \in A'$, the function F is independent of the sequence in $(A_t^s)_s^1$ converging to f , that is S is well-defined. It is easy to see that $\|\frac{f}{g} - 1\|_X = \|\frac{Sf}{Sg} - 1\|_Y$ holds for all $f, g \in A'$. We shall show that $S : A' \rightarrow B'$ is surjective. For suppose that $F \in B'$. Then there exists a sequence $\{F_n\}$ in $(B_t^s)_s^1$ such that $\|F_n - F\|_Y \rightarrow 0$. Since $S((A_t^s)_s^1) = (B_t^s)_s^1$ it follows that for each $n \in \mathbb{N}$ there exists a function f_n in $(A_t^s)_s^1$ such that $F_n = Sf_n$. Hence, $\|\frac{F_n}{F_m} - 1\|_Y = \|\frac{f_n}{f_m} - 1\|_X$, by (3.2), and so using the same argument as above, we conclude that $\{f_n\}$ converges uniformly to a function f in \overline{A} . A similar argument concludes that $f \in A'$. Now by the definition of S we have $\lim_{n \rightarrow \infty} \|F_n - Sf\|_Y = \lim_{n \rightarrow \infty} \|Sf_n - Sf\|_Y = 0$ and so $Sf = F$. This shows that $S : A' \rightarrow B'$ is a surjective map with $S((A_t^s)_s^1) = (B_t^s)_s^1$.

Since $\overline{A}, \overline{B}$ are uniform algebras and A', B' are subgroups of $(\overline{A})^{-1}$ and $(\overline{B})^{-1}$ containing $\exp(\overline{A})$ and $\exp(\overline{B})$, respectively, it follows from Theorem 2.3 that there exists a homeomorphism $\varphi : c(B) \rightarrow c(A)$ such that

$$Sf(y) = \begin{cases} f(\varphi(y)) & y \in K, \\ f(\varphi(y)) & y \in c(B) \setminus K, \end{cases}$$

for every $f \in A'$, where $K = \{y \in c(B) : S(i)(y) = i\}$. Put $\psi = \varphi^{-1}$. Then by the above description of S we have $|Sf(\psi(x))| = |f(x)|$ for all $f \in A'$ and $x \in c(A)$. In the next steps we extend S to a function from $A^s \cup A'$ onto $B^s \cup B'$ having similar properties and show that the above description is valid for every $f \in A^s \cup A'$.

Step 2. *The map $S : A' \rightarrow B'$ can be extended to a surjective map $\tilde{S} : A^s \cup A' \rightarrow B^s \cup B'$ such that $\|\frac{\tilde{S}f}{\tilde{S}g} - 1\|_Y = \|\frac{f}{g} - 1\|_X$ for all $f \in A^s \cup A'$ and $g \in A'$.*

Proof. For each $f \in A'$ we set $\tilde{S}f = Sf$ and for each $f \in A$ we set

$$\tilde{S}(f^s) = \frac{(Tf)^s}{(T1)^s}.$$

We first show that \tilde{S} is well-defined, i.e., $\frac{(Tf)^s}{(T1)^s} = \frac{(Tg)^s}{(T1)^s}$ for each $f, g \in A$ with $f^s = g^s$ and, moreover, the two definitions of \tilde{S} coincide on the intersection $A^s \cap A'$. For this, first assume that $f, g \in A$ such that $f^s = g^s$ and let $H \in B'$ be an arbitrary element. Then by Step 1, we can find a function $h \in A'$ such

that $Sh = H$. Let $\{h_n\}$ be a sequence in A_t^s such that $\lim_{n \rightarrow \infty} \|(h_n)^s - h\|_X = 0$. Then, using the definition of S and (3.1) we have

$$\left\| \frac{(Tf)^s}{(T1)^s} - 1 \right\|_Y = \lim_{n \rightarrow \infty} \left\| \frac{(Tf)^s}{(T1)^s} - 1 \right\|_Y = \lim_{n \rightarrow \infty} \left\| \frac{(Tf)^s}{(Th_n)^s} - 1 \right\|_Y = \lim_{n \rightarrow \infty} \left\| \frac{f^s}{(h_n)^s} - 1 \right\|_X.$$

Hence $\left\| \frac{(Tf)^s}{(T1)^s} - 1 \right\|_Y = \lim_{n \rightarrow \infty} \left\| \frac{f^s}{(h_n)^s} - 1 \right\|_X$. The same argument shows that

$$\left\| \frac{(Tg)^s}{(T1)^s} - 1 \right\|_Y = \lim_{n \rightarrow \infty} \left\| \frac{g^s}{(h_n)^s} - 1 \right\|_X$$

and consequently $\left\| \frac{(Tf)^s}{(T1)^s} - 1 \right\|_Y = \left\| \frac{(Tg)^s}{(T1)^s} - 1 \right\|_Y$, since $f^s = g^s$. Therefore, by

Remark 3.1(c), $\frac{(Tf)^s}{(T1)^s} = \frac{(Tg)^s}{(T1)^s}$, since $\exp(\bar{B}) \subseteq B'$. Now assume that $h \in A^s \cap A'$. Then there exists a sequence $\{h_n\}$ in $(A_t^s)_s^1$ such that $\|h_n - h\|_X \rightarrow 0$ and there exists a function $f \in A$ such that $h = f^s$. For each $n \in \mathbb{N}$, since $h_n \in (A_t^s)_s^1$ we can find a function $h'_n \in A_t^s$ such that $h_n = (h'_n)^s$. Hence by (3.1)

$$\left\| \frac{(Tf)^s}{(Th'_n)^s} - 1 \right\|_Y = \left\| \frac{f^s}{(h'_n)^s} - 1 \right\|_X \rightarrow 0.$$

Thus $(Tf)^s = \lim_{n \rightarrow \infty} (Th'_n)^s$ and consequently $\frac{(Tf)^s}{(T1)^s} = \lim_{n \rightarrow \infty} S(h_n) = Sh$ which shows that the two definitions for $\tilde{S}(h)$ are the same. This clearly implies that \tilde{S} is well-defined.

We shall show that \tilde{S} is surjective. For this let $G \in B$. Then by the surjectivity of T , there exists a function $f \in A$ such that $Tf = G \cdot T1$. Thus $\frac{(Tf)^s}{(T1)^s} = G^s$ and so $\tilde{S}(f^s) = G^s$, that is \tilde{S} is surjective. The above argument also shows that $\left\| \frac{\tilde{S}f}{\tilde{S}g} - 1 \right\|_Y = \left\| \frac{f}{g} - 1 \right\|_X$ holds for all $f \in A^s \cup A'$ and $g \in A'$.

Step 3. *The map \tilde{S} is injective and $\|fg^{-1}\|_X = \|\tilde{S}f(\tilde{S}g)^{-1}\|_Y$ for all $f \in A^s \cup A'$ and $g \in A'$. In particular, $\|\tilde{S}f\|_Y = \|f\|_X$ and $\|g^{-1}\|_X = \|(Sg)^{-1}\|_Y$ for all $f \in A^s \cup A'$ and $g \in A'$.*

Proof. Let $f, g \in A^s \cup A'$ with $\tilde{S}f = \tilde{S}g$. Then for each $h \in A'$,

$$\|fh^{-1} - 1\|_X = \|\tilde{S}f(\tilde{S}h)^{-1} - 1\|_Y = \|\tilde{S}g(\tilde{S}h)^{-1} - 1\|_Y = \|gh^{-1} - 1\|_X,$$

in particular, $\|fu - 1\|_X = \|gu - 1\|_X$ for all $u \in \exp(\bar{A})$. Hence, by Remark 3.1(c), $f = g$, that is \tilde{S} is injective.

Now we show that $\|fg^{-1}\|_X = \|\tilde{S}f(\tilde{S}g)^{-1}\|_Y$ for all $f \in A^s \cup A'$ and $g \in A'$. Let $f \in A^s \cup A'$ and $g \in A'$ and let $\{g_n\}$ be a sequence in $(A_t^s)_s^1$ converging uniformly to g . For each $n \in \mathbb{N}$, set $k_n = (S(n^{-1}g_n))^{-1}S(g_n)$. Then $k_n \in B$ and

$$\|k_n\|_Y - 1 \leq \|k_n - 1\|_Y = \|(S(n^{-1}g_n))^{-1}S(g_n) - 1\|_Y = \|ng_n^{-1}g_n - 1\|_X = n - 1.$$

Hence $\|k_n\|_Y \leq n$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} n\|g_n^{-1}f\|_X - 1 &\leq \|ng_n^{-1}f - 1\|_X = \|(S(n^{-1}g_n))^{-1}\tilde{S}f - 1\|_Y \\ &\leq \|k_n\|_Y \|(Sg_n)^{-1}\tilde{S}f\|_Y + 1 \leq n\|(Sg_n)^{-1}\tilde{S}f\|_Y + 1. \end{aligned}$$

Thus $\|g_n^{-1}f\|_X \leq \|(Sg_n)^{-1}\tilde{S}f\|_Y + \frac{2}{n}$ for all $n \in \mathbb{N}$. Letting $n \rightarrow \infty$, we get $\|g^{-1}f\|_X \leq \|(Sg)^{-1}\tilde{S}f\|_Y$. Since $(\tilde{S})^{-1}$ has the same properties as \tilde{S} , it follows that $\|(Sg)^{-1}\tilde{S}f\|_Y \leq \|g^{-1}f\|_X$. Therefore, $\|fg^{-1}\|_X = \|\tilde{S}f(\tilde{S}g)^{-1}\|_Y$.

Step 4. *The equality $|\tilde{S}f(\psi(x))| = |f(x)|$ holds for all $f \in A^s \cup A'$ and $x \in c(A)$.*

Proof. Let $f \in A^s \cup A'$, $x \in c(A)$ and set $y = \psi(x)$. We shall show that $|f(x)| = |\tilde{S}f(y)|$. If $\tilde{S}f(y) = 0$, then for each $\epsilon > 0$ we can easily find a function $g \in A'$ such that $Sg \in F_y$ and $\|\tilde{S}fSg\|_Y < \epsilon$, where F_y is defined as in the beginning of this section for the Banach function algebra B . Take $g' \in A'$ such that $Sg' = (Sg)^{-1}$. Then, using Step 3, we have

$$|f(x)g'^{-1}(x)| \leq \|fg'^{-1}\|_X = \|\tilde{S}fSg\|_Y < \epsilon. \quad (3.3)$$

Now by Remark 3.1(b) we can choose $u \in F_x$ with $R_\pi(g'^{-1} \cdot u) = \{g'^{-1}(x)\}$. Then it follows from the description of S on A' that, $Su \in F_y$ and hence

$$|g'^{-1}(x)| = \|g'^{-1}u\|_X = \|SgSu\|_Y = |Sg(y)Su(y)| = 1.$$

Thus $|g'^{-1}(x)| = 1$ and consequently $|f(x)| < \epsilon$, by (3.3). Since ϵ is arbitrary it follows that $f(x) = 0 = \tilde{S}f(y)$, as desired.

Now suppose that $\tilde{S}f(y) \neq 0$. Choosing $G \in F_y$ with $R_\pi(\tilde{S}f \cdot G) = \{\tilde{S}f(y)\}$ a similar argument as above shows that there exists a function $g \in A'$ such that $|g^{-1}(x)| = 1$ and $Sg = G^{-1}$. Therefore, by Step 3, we have

$$|f(x)| = |fg^{-1}(x)| \leq \|fg^{-1}\|_X = \|\tilde{S}f \cdot G\|_Y = |\tilde{S}f(y)|,$$

that is $|f(x)| \leq |\tilde{S}f(y)|$. A similar argument shows that $|\tilde{S}f(y)| \leq |f(x)|$ and consequently $|\tilde{S}f(y)| = |f(x)|$.

Step 5. *Let $f \in A^s \cup A'$ and $g \in A'$. Then $-1 \in R_\pi(fg^{-1})$ if and only if $-1 \in R_\pi(\tilde{S}f(Sg)^{-1})$.*

Proof. Assume first that $-1 \in R_\pi(fg^{-1})$, then

$$\|\tilde{S}f(Sg)^{-1} - 1\|_Y = \|fg^{-1} - 1\|_X = 2$$

and

$$\|\tilde{S}f(Sg)^{-1}\|_Y = \|fg^{-1}\|_X = 1,$$

by Step 3. Thus $-1 \in R_\pi(\tilde{S}f(Sg)^{-1})$. The converse statement is obtained in a similar manner.

We now show that the description given earlier for the map S is valid for the extended map \tilde{S} . The proof of the next step is basically a modification of [9, Lemma 3.8].

Step 6. For each $f \in A^s \cup A'$ and $y_0 \in c(B)$ we have

$$\tilde{S}f(y_0) = \begin{cases} f(\varphi(y_0)) & y_0 \in K, \\ \frac{f(\varphi(y_0))}{f(\varphi(y_0))} & y_0 \in c(B) \setminus K. \end{cases}$$

Proof. Let $f \in A^s \cup A'$ and $y_0 \in c(B)$. Assume first that $y_0 \in K$ and set $x_0 = \varphi(y_0)$. If either $f(x_0) = 0$ or $\tilde{S}f(y_0) = 0$, then since by Step 4, $|f(x_0)| = |\tilde{S}f(y_0)|$ it follows that $f(x_0) = \tilde{S}f(y_0)$. So we may assume that $f(x_0) \neq 0$ and $\tilde{S}f(y_0) \neq 0$. Put $a = f(x_0)$ and $b = \tilde{S}f(y_0)$. Let U be an arbitrary neighborhood of x_0 and set $V = \varphi(K) \cap U$. Then clearly V is an open subset of $c(A)$ and consequently there exists a neighborhood V' of x_0 such that $V' \cap c(A) = V$. We now claim that there exists a function $h \in A'$ with $M_h \subseteq V'$ and $R_\pi(fh) = \{a\}$ and, moreover, fh attains its maximum modulus exclusively on M_h . For this we first choose, by Remark 3.1(b), a peaking function $h_1 \in A'$ such that $h_1(x_0) = 1$ and $R_\pi(fh_1) = \{a\}$. Then since the set $E = \{x \in X : fh_1(x) = a\}$ is a peak set of \bar{A} , using Lemma 2.1 for the peak set E and the function fh_1 , we can find another peaking function $h_2 \in \exp(\bar{A}) \subseteq A'$, with $M_{fh_1h_2} = E$ and $M_{h_2} = E$. Moreover, since V' is a neighborhood of the strong boundary point x_0 we can choose a peaking function $h_3 \in A'$ with $h_3(x_0) = 1$ and $|h_3| < 1$ on $X \setminus V'$. Setting $h = h_1h_2h_3$, it is easy to see that $h \in A'$, $M_h \subseteq E \cap V'$ with $R_\pi(fh) = \{a\}$ and, in addition, fh attains its maximum modulus exclusively on M_h , as we claimed. Hence $R_\pi(-a^{-1}fh) = \{-1\}$ and so by the previous step, $-1 \in R_\pi(\tilde{S}f(\tilde{S}(-ah^{-1}))^{-1})$. Therefore, there exists a point $y \in c(B)$ such that

$$\tilde{S}f(y) (\tilde{S}(-ah^{-1}))^{-1}(y) = -1. \quad (3.4)$$

Let $x = \varphi(y)$. Then $x \in c(A)$ and by the above equality $|f(x)a^{-1}h(x)| = |\tilde{S}f(y)(\tilde{S}(-ah^{-1}))^{-1}(y)| = 1$. Hence $\|fh\|_X = |a| = |f(x)h(x)|$. In particular, $x \in M_{fh} \subseteq M_h$ and consequently

$$|(\tilde{S}(-ah^{-1}))^{-1}(y)| = |a^{-1}h(x)| = |a^{-1}| = \|a^{-1}h\|_X = \|(\tilde{S}(-ah^{-1}))^{-1}\|_Y,$$

which implies that $-1 \in R_\pi(-\gamma(\tilde{S}(-ah^{-1}))^{-1})$, where $\gamma = \tilde{S}(-ah^{-1})(y)$. We note that since $x \in M_h \cap c(A) \subseteq E \cap V \subseteq \varphi(K)$, it follows that $y \in K$ and so by the description of S we have $S(-\gamma)(y) = -\gamma$. Hence $-1 \in R_\pi(S(-\gamma)(\tilde{S}(-ah^{-1}))^{-1})$ and by Step 5, $-1 \in R_\pi(\gamma a^{-1}h) = \{\gamma a^{-1}\}$. This clearly shows that $a = -\gamma$, that is, $f(x_0) = -\tilde{S}(-ah^{-1})(y)$. Therefore, $f(x_0) = \tilde{S}f(y) = \tilde{S}f(\psi(x))$, by (3.4). Since U is an arbitrary neighborhood of x_0 , the continuity of f , $\tilde{S}f$ and ψ conclude that $f(x_0) = \tilde{S}f(\psi(x_0))$, i.e., $a = b$. One can use a similar argument to show that for each $y_0 \in c(B) \setminus K$, $\tilde{S}f(y_0) = \overline{f(\varphi(y_0))}$.

Step 7. For each $f \in A$,

$$(Tf)^s(y) = (T1)^s(y) \begin{cases} f(\varphi(y))^s & y \in K, \\ \frac{f(\varphi(y))^s}{f(\varphi(y))^s} & y \in c(B) \setminus K, \end{cases}$$

and

$$(T'f)^t(y) = (T'1)^t(y) \begin{cases} f(\varphi(y))^t & y \in K, \\ \frac{f(\varphi(y))^t}{f(\varphi(y))^t} & y \in c(B) \setminus K. \end{cases}$$

Proof. Let $f \in A$. Then by the definition of \tilde{S} , $\tilde{S}(f^s) = \frac{(Tf)^s}{(T1)^s}$. Hence, by Step 6,

$$(Tf)^s(y) = (T1)^s(y) \begin{cases} f(\varphi(y))^s & y \in K, \\ \overline{f(\varphi(y))^s} & y \in c(B) \setminus K. \end{cases}$$

Interchanging the role of s and t , the same argument shows that there exist a homeomorphism $\varphi' : c(B) \rightarrow c(A)$ and a clopen subset K' of $c(B)$ such that for each $f \in A$,

$$(T'f)^t(y) = (T'1)^t(y) \begin{cases} f(\varphi'(y))^t & y \in K', \\ \overline{f(\varphi'(y))^t} & y \in c(B) \setminus K'. \end{cases}$$

We now show that $K = K'$ and $\varphi = \varphi'$. Let $y \in K$ and $y \in c(B) \setminus K'$. Then for each $f \in A$,

$$(Tf)^s(y) = (T1)^s(y)f(\varphi(y))^s$$

and

$$(T'f)^t(y) = (T'1)^t(y)\overline{f(\varphi'(y))^t}.$$

Now let $f \in A_t^s$. Then there exists $g \in A$ with $f^s g^t = 1$ and since

$$\|(Tf)^s(T'(\beta g))^t - \alpha\|_Y = \|f^s(\beta g)^t - \alpha\|_X = 0$$

it follows that $(Tf)^s(T'(\beta g))^t = \alpha$. Therefore

$$(T1)^s(y)(T'1)^t(y)f(\varphi(y))^s\overline{g(\varphi'(y))^t} = \alpha.$$

Obviously $(T1)^s(T'\beta)^t = \alpha$ which concludes that

$$T(1)^s(y)(T'1)^t(y)\bar{\alpha} = \alpha.$$

Consequently $f(\varphi(y))^s\overline{g(\varphi'(y))^t} = 1$. Since by assumption $f(\varphi'(y))^s g(\varphi'(y))^t = 1$ we get $f(\varphi(y))^s = \overline{f(\varphi'(y))^s}$ for all $f \in A_t^s$ which is impossible since A_t^s contains the constant functions. This shows that $K \subseteq K'$. In the same way $K' \subseteq K$, that is $K = K'$. Now assume that there exists $y \in K$ with $\varphi(y) \neq \varphi'(y)$ and let $f \in A_t^s$. Then since $(Tf)^s(y) = (T1)^s(y)f(\varphi(y))^s$ and $(T'g)^t(y) = (T'1)^t(y)g(\varphi'(y))^t$ where $g \in A$ is such that $f^s g^t = 1$ we get easily

$$f(\varphi(y))^s g(\varphi'(y))^t = 1.$$

Therefore, $f(\varphi(y))^s = f(\varphi'(y))^s$ for all $f \in A_t^s$. Now since A separates the point of X we can choose an element $h \in A$ with $h(\varphi(y)) = 0$ and $h(\varphi'(y)) = 1$. Then $f = \exp(h) \in A_t^s$ while $f(\varphi(y))^s \neq f(\varphi'(y))^s$, a contradiction. Hence $\varphi = \varphi'$ on K and a similar argument shows that $\varphi = \varphi'$ on $c(B) \setminus K$ as desired. This completes the proof of the theorem. \square

Corollary 3.3. *Let A and B be Banach function algebras on compact Hausdorff spaces X and Y and let $\alpha \in \mathbb{C} \setminus \{0\}$.*

(i) *If $\rho : B \rightarrow B$ and $T : A \rightarrow B$ are surjective maps satisfying $\|Tf \cdot \rho(Tg) - \alpha\|_Y = \|fg - \alpha\|_X$ for all $f, g \in A$, then there exist a homeomorphism φ from $c(B)$ onto $c(A)$ and a clopen subset K of $c(B)$ such that for each $f \in A$,*

$$Tf(y) = T1(y) \begin{cases} f(\varphi(y)) & y \in K, \\ \overline{f(\varphi(y))} & y \in c(B) \setminus K. \end{cases}$$

(ii) For two variables polynomial $p(z, w) = az^s w + bz^s - \alpha$, where $s \in \mathbb{N}$ and a, b are complex numbers with $a \neq 0$, any surjective map $T : A \rightarrow B$ satisfying $\|p(Tf, Tg)\|_Y = \|p(f, g)\|_X$ has the following representation:

$$Tf(y) = \begin{cases} w(y) \cdot \frac{(af(\varphi(y)) + b) - \frac{b}{a}}{w(y) \cdot \frac{(af(\varphi(y)) + b) - \frac{b}{a}}} & y \in K, \\ w(y) \cdot \frac{(af(\varphi(y)) + b) - \frac{b}{a}}{w(y) \cdot \frac{(af(\varphi(y)) + b) - \frac{b}{a}}} & y \in c(B) \setminus K, \end{cases}$$

where φ and K are as in (i) and $w = T(\frac{1-b}{a}) + \frac{b}{a}$.

Proof. (i) It is immediate from Theorem 3.2 by considering the case where $s = t = 1$ and $T' = \rho \circ T$.

(ii) It suffices to apply Theorem 3.2 for $t = 1$ and the surjective map $T' : A \rightarrow B$ defined by $T'(g) = aT(\frac{g-b}{a}) + b$. Then by this theorem there exist a homeomorphism $\varphi : c(B) \rightarrow c(A)$ and a clopen subset K of $c(B)$ such that

$$T'g(y) = T'1(y) \begin{cases} \frac{g(\varphi(y))}{g(\varphi(y))} & y \in K, \\ \frac{g(\varphi(y))}{g(\varphi(y))} & y \in c(B) \setminus K. \end{cases}$$

Let $w(y) = \frac{T'1(y)}{a}$, for all $y \in Y$. Then $w \in B$ and for each $f \in A$ by considering the above description for $g = af + b$ it follows that

$$Tf(y) = \begin{cases} w(y) \cdot \frac{(af(\varphi(y)) + b) - \frac{b}{a}}{w(y) \cdot \frac{(af(\varphi(y)) + b) - \frac{b}{a}}} & y \in K, \\ w(y) \cdot \frac{(af(\varphi(y)) + b) - \frac{b}{a}}{w(y) \cdot \frac{(af(\varphi(y)) + b) - \frac{b}{a}}} & y \in c(B) \setminus K. \end{cases}$$

□

Corollary 3.4. *Let A and B be Banach function algebras on compact Hausdorff spaces X and Y . Let $s, t \in \mathbb{N}$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $T : A \rightarrow B$ be a surjective map satisfying $\|(Tf)^s(Tg)^t - \alpha\|_Y = \|f^s g^t - \alpha\|_X$ for all $f, g \in A$, then there exist a homeomorphism φ from $c(B)$ onto $c(A)$ and a clopen subset K of $c(B)$ such that for each $f \in A$,*

$$(Tf)^d(y) = (T1)^d(y) \begin{cases} \frac{f(\varphi(y))^d}{f(\varphi(y))^d} & y \in K, \\ \frac{f(\varphi(y))^d}{f(\varphi(y))^d} & y \in c(B) \setminus K, \end{cases}$$

where d is the greatest common divisor of s and t .

Proof. It is immediate from Theorem 3.2 since $d = sc + tc'$ for some integers c and c' . □

Remark 3.5. We should note that a similar description to the one given for the power d of T in Corollary 3.4 may not be valid, in general, for T itself. For example, if A is a Banach function algebra on a compact Hausdorff space X and $\alpha \in \mathbb{C} \setminus \{0\}$, then a map $\delta : A \rightarrow \{1, -1\}$ can be chosen such that the self map $T : A \rightarrow A$ defined by $Tf = \delta(f)f$, $f \in A$, is a surjective map which is neither \mathbb{R} -linear nor multiplicative while it clearly satisfies the condition

$$\|(Tf)^2(Tg)^2 - \alpha\|_Y = \|f^2 g^2 - \alpha\|_X \quad (f, g \in A)$$

or even the stronger quality $R_\pi((Tf)^2(Tg)^2 - \alpha) \cap R_\pi(f^2 g^2 - \alpha) \neq \emptyset$ for all $f, g \in A$. Hence T does not have the representation given in this corollary for the power one instead of d . However, the next result shows that this is the case, if we impose an additional assumption on T . In particular, in this case T is injective

and \mathbb{R} -linear. The proof of the next theorem is a modification of [3, Corollary 4.1].

Theorem 3.6. *Under the hypotheses of Corollary 3.4, if, furthermore, T satisfies the equality $T(f + 1) = Tf + T1$ for all $f \in A$, then for each $f \in A$,*

$$Tf = T1 \begin{cases} \frac{f \circ \varphi}{f \circ \varphi} & \text{on } K, \\ \frac{f \circ \varphi}{f \circ \varphi} & \text{on } c(B) \setminus K. \end{cases}$$

Proof. Let $f \in A$. Since $T(f + 1) = T(f) + T1$, it follows, by induction, that $T(f + k) = Tf + kT1$ for each $k \in \mathbb{N}$. For each $y \in K$, by Corollary 3.4 we have

$$\begin{aligned} (Tf + kT1)^d(y) &= ((T(f + k)(y))^d = (T1)(y)((f + k) \circ \varphi)(y))^d \\ &= (T1)^d(y)((f \circ \varphi)(y) + k)^d, \end{aligned}$$

i.e., $(Tf + kT1)^d(y) = (T1)^d(y)((f \circ \varphi)(y) + k)^d$ for all $f \in A$. Similarly for each $y \in c(B) \setminus K$, $(Tf + kT1)^d(y) = (T1)^d(y)((f \circ \varphi)(y) + k)^d$ holds for all $f \in A$. Let a_0, \dots, a_m be real numbers such that

$$ab^{d-1} = \sum_{k=0}^d a_k(a + kb)^d,$$

holds for all real numbers a and b (cf. [3, The proof of Theorem 1.3]). Then for each $y \in K$ and $f \in A$,

$$\begin{aligned} Tf(y)(T1)^{d-1}(y) &= \sum_{k=0}^d a_k((Tf + kT1)(y))^d \\ &= \sum_{k=0}^d a_k((T1)^d(y) \cdot ((f \circ \varphi)(y) + k)^d) \\ &= (T1)^d(y)(f \circ \varphi)(y). \end{aligned}$$

Hence $Tf(y) = (T1)(y)(f \circ \varphi)(y)$. Similarly for each $y \in c(B) \setminus K$, $Tf(y) = (T1)(y)(f \circ \varphi)(y)$ for all $f \in A$, that is

$$Tf = T1 \begin{cases} \frac{f \circ \varphi}{f \circ \varphi} & \text{on } K, \\ \frac{f \circ \varphi}{f \circ \varphi} & \text{on } c(B) \setminus K, \end{cases}$$

for all $f \in A$. □

The next theorem shows that similar descriptions can be obtained for the case where the maps are defined between certain subsets A_0 and B_0 of A and B rather than whole A and B .

Theorem 3.7. *Let A and B be Banach function algebras on compact Hausdorff spaces X and Y and A_0 and B_0 be subsets of A and B containing A^{-1} and B^{-1} , respectively. Let $\alpha \in \mathbb{C} \setminus \{0\}$ and $T : A_0 \rightarrow B_0$ be a surjective map satisfying*

$$\|(Tf)^s(Tg)^t - \alpha\|_Y = \|f^s g^t - \alpha\|_X$$

for all $f, g \in A_0$. Then there exist a homeomorphism $\varphi : c(B) \longrightarrow c(A)$ and a clopen subset K of $c(B)$ such that for each $f \in A_0$,

$$(Tf)^d(y) = (T1)^d(y) \begin{cases} f(\varphi(y))^d & y \in K, \\ \frac{f(\varphi(y))^d}{f(\varphi(y))^d} & y \in c(B) \setminus K, \end{cases}$$

where d is the greatest common divisor of s and t .

Proof. We first note that $(A_0)_t^s = (A^{-1})_t^s = A_t^s$ and $(B_0)_t^s = (B^{-1})_t^s = B_t^s$ and, moreover, $(A_0)_t^s, (B_0)_t^s$ are subgroups of A^{-1} and B^{-1} containing $\exp(A)$ and $\exp(B)$, respectively. An argument as in Theorem 3.2 implies that $T(A_t^s) = B_t^s$ and

$$\left\| \frac{(Tf)^s}{(Tg)^s} - 1 \right\|_Y = \left\| \frac{f^s}{g^s} - 1 \right\|_X \quad (f \in A_0, g \in A_t^s). \quad (3.5)$$

Moreover, similar arguments show that there is a surjective map $S : A' \longrightarrow B'$ such that $S((A_t^s)_s^1) = (B_t^s)_s^1$ and $S(f^s) = \frac{(Tf)^s}{(T1)^s}$ for all $f \in A_t^s$.

Set $B'' = \left\{ \frac{(Tf)^s}{(T1)^s} : f \in A_0 \right\} = \left\{ \frac{G^s}{(T1)^s} : G \in B_0 \right\}$. Since, by assumption, A_0 and B_0 contain A^{-1} and B^{-1} , respectively we can use the proof of Step 2 in Theorem 3.2 to extend S to a surjective map $\tilde{S} : (A_0)^s \cup A' \longrightarrow B'' \cup B'$ satisfying $\left\| \frac{\tilde{S}f}{\tilde{S}g} - 1 \right\|_Y = \left\| \frac{f}{g} - 1 \right\|_X$ for all $f \in (A_0)^s \cup A'$ and $g \in A'$. Indeed, for each $f \in A_0$ it suffices to define $\tilde{S}(f^s) = \frac{(Tf)^s}{(T1)^s}$. Then it is easy to see that \tilde{S} is well-defined and surjective. For this latter, let $G \in B_0$, then, by the surjectivity of T , there exists $f \in A_0$ such that $Tf = G$. Thus $\tilde{S}(f^s) = \frac{(Tf)^s}{(T1)^s} = \frac{G^s}{(T1)^s}$. Furthermore, $\tilde{S}1 = S1 = 1$ and it follows easily from (3.5) that

$$\left\| \frac{\tilde{S}f}{\tilde{S}g} - 1 \right\|_Y = \left\| \frac{f}{g} - 1 \right\|_X$$

for all $f, g \in (A_0)^s \cup A'$ and $g \in A'$.

Now one can apply the same argument as in Theorem 3.2 to define a homeomorphism $\varphi : c(B) \longrightarrow c(A)$ and a clopen subset K of $c(B)$ such that for each $f \in A_0$,

$$(Tf)^d(y) = (T1)^d(y) \begin{cases} f(\varphi(y))^d & y \in K, \\ \frac{f(\varphi(y))^d}{f(\varphi(y))^d} & y \in c(B) \setminus K, \end{cases}$$

where d is the greatest common divisor of s and t . □

Most recently, some studies have been done to analyze a polynomially spectrum (respectively, peripheral spectrum) preserving map, i.e. a map T satisfying $\sigma(p(Tf, Tg)) = \sigma(p(f, g))$ (respectively, $\sigma_\pi(p(Tf, Tg)) = \sigma_\pi(p(f, g))$) for some polynomial $p(z, w)$. For instance, in [25] (respectively [20]) surjective maps $T : A \longrightarrow B$ between uniform algebras A and B on X and Y satisfying $\|Tf + Tg\|_Y = \|f + g\|_X$ (respectively $\sigma_\pi(Tf + Tg) = \sigma_\pi(f + g)$) for all $f, g \in A$, are

discussed. Hatori, Miura and Takagi in [7] proved that for certain two-variables polynomials p , every surjective map T between unital semisimple commutative Banach algebras A and B is an algebra isomorphism if $\sigma(p(Tf, Tg)) = \sigma(p(f, g))$ holds for all $f, g \in A$. They also look for two-variables polynomials p for which the above equality implies that T is linear and multiplicative. Moreover, for two variables polynomial $p(z, w) = z^s w^t$, $s, t \in \mathbb{N}$, surjective maps $T : A \rightarrow B$ between uniform algebras A and B satisfying $R_\pi(p(Tf, Tg)) \subseteq R_\pi(p(f, g))$ for all $f, g \in A$ have been characterized in [3]. In particular, it was shown that if the greatest common divisor of s and t is 1, then T is linear and is, in fact, a weighted composition operator. The case where $t = s = 1$ was proved by Lambert, Luttmann and Tonev in [12]. In the next result, we consider a surjective map T between Banach function algebras A and B satisfying the weaker condition

$$R_\pi(p(Tf, Tg)) \cap R_\pi(p(f, g)) \neq \emptyset \quad (f, g \in A),$$

where $p(z, w) = z^s w^t - \alpha$ for some $s, t \in \mathbb{N}$ and $\alpha \in \mathbb{C} \setminus \{0\}$. It should be noted that in general such a map need not be injective, linear nor multiplicative (see [3, Example 3.3]). But the next result shows that if the greatest common divisor of s and t is 1, then T is a weighted composition operator. Hence, in particular, complex-linearity and injectivity of T are concluded. In particular, for the polynomial $p(z, w) = z^s w^t - \alpha$, $\alpha \neq 0$, if the greatest common divisor of s and t is 1, then every surjective map $T : A \rightarrow B$ between uniform algebras A and B satisfying $T(1) = 1$ and $\sigma(p(Tf, Tg)) = \sigma(p(f, g))$ is an algebra isomorphism. This gives a partial answer to the Question 7 in [4] for Banach function algebras.

Theorem 3.8. *Let A and B be Banach function algebras on compact Hausdorff spaces X and Y , respectively. Let $s, t \in \mathbb{N}$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $T : A \rightarrow B$ be a surjective map satisfying*

$$R_\pi((Tf)^s (Tg)^t - \alpha) \cap R_\pi(f^s g^t - \alpha) \neq \emptyset$$

for all $f, g \in A$. Then there exists a homeomorphism $\varphi : c(B) \rightarrow c(A)$ such that for each $f \in A$, $(Tf)^d(y) = (T1)^d(y) (f \circ \varphi(y))^d$, where d is the greatest common divisor of s and t .

Proof. Clearly the assumption implies

$$\|(Tf)^s (Tg)^t - \alpha\|_Y = \|f^s g^t - \alpha\|_X \quad (f, g \in A).$$

Therefore, by Theorem 3.2, there exists a homeomorphism $\varphi : c(B) \rightarrow c(A)$ and a clopen subset K of $c(B)$ such that for each $f \in A$,

$$(Tf)^s(y) = (T1)^s(y) \begin{cases} f(\varphi(y))^s & y \in K, \\ \frac{f(\varphi(y))^s}{f(\varphi(y))} & y \in c(B) \setminus K. \end{cases}$$

As it was shown in the proof of this theorem, there exists a surjective map $S : A' \rightarrow B'$ such that $S((A'_t)^1_s) = (B'_t)^1_s$ and $\|\frac{Sf}{Sg} - 1\|_Y = \|\frac{f}{g} - 1\|_X$ for all $f, g \in A'$ and also

$$Sf(y) = \begin{cases} f(\varphi(y)) & y \in K, \\ \frac{f(\varphi(y))}{f(\varphi(y))} & y \in c(B) \setminus K. \end{cases}$$

Let $\beta^t = \alpha$ and let $h_1, h_2 \in (A_t^s)_s^1$. Then $h_1 = k_1^s$ and $h_2 = k_2^s$ for some $k_1, k_2 \in A_t^s$. Since $k_2 \in A_t^s$ there exists $k \in A$ with $k_2^s k^t = 1$. Hence by the argument at the beginning of the proof of Theorem 3.2 and the definition of S we have

$$R_\pi\left(\frac{Sh_1}{Sh_2} - 1\right) = R_\pi\left(\frac{(Tk_1)^s}{(Tk_2)^s} - 1\right) = \frac{1}{\alpha} R_\pi((Tk_1)^s (T(\beta k))^t - \alpha)$$

and also $R_\pi\left(\frac{h_1}{h_2} - 1\right) = R_\pi\left(\frac{(k_1)^s}{(k_2)^s} - 1\right) = \frac{1}{\alpha} R_\pi(k_1^s (\beta k)^t - \alpha)$. Therefore, for each $h_1, h_2 \in (A_t^s)_s^1$,

$$R_\pi\left(\frac{Sh_1}{Sh_2} - 1\right) \cap R_\pi\left(\frac{h_1}{h_2} - 1\right) \neq \emptyset,$$

and, in particular, since $S1 = 1$,

$$R_\pi(Sh_1 - 1) \cap R_\pi(h_1 - 1) \neq \emptyset. \quad (3.6)$$

A minor modification of the proof of [9, Theorem 3.9] can be applied to show that $K = c(B)$. For the sake of completeness we state it here. Assume on the contrary that $K \neq c(B)$ and let $y_0 \in c(B) \setminus K$. Then we can find easily a function $F \in B'$ such that $F(y_0) = i$, $|F| < \frac{1}{3}$ on K , and $\text{Im}(F) > 0$ (see the proof of [9, Theorem 3.9]).

Since $S(A') = B'$, there exists a function $f \in A'$ with $Sf = F$. Now let $\{f_n\}$ be a sequence in $(A_t^s)_s^1$ such that $\|f_n - f\|_X \rightarrow 0$. Then $\|Sf_n - F\|_Y \rightarrow 0$, by the definition of S . Clearly $F(y) = f(\varphi(y))$ for each $y \in K$ and $F(y) = \overline{f(\varphi(y))}$ for each $y \in c(B) \setminus K$. Since $\text{Im}(F) > 0$, it follows that $a < \text{Im}(F) < b$ for some $a, b > 0$. Choose now $0 < \epsilon < a$ small enough and let $c \in (\frac{4}{3} + \epsilon, \sqrt{2} - \epsilon)$. Then we can find a sufficiently large $N \in \mathbb{N}$ such that for each $n \geq N$, $\|Sf_n - F\|_Y = \sup\{|f_n(\varphi(y)) - f(\varphi(y))| : y \in c(B)\} \leq \epsilon$. Since $\text{Im}(F) > a$ it is easy to see that

$$R_\pi(Sf_n - 1) \subseteq \{z \in \mathbb{C} : \text{Im}(z) > 0\},$$

for all $n \geq N$.

Since $|f \circ \varphi| = |F| < 1/3$ on K and $|Sf_n - F| = |f_n \circ \varphi - f \circ \varphi| < \epsilon$ for all $n \geq N$ it follows that $|f_n(\varphi(y)) - 1| < c$ for all $y \in K$ and $n \geq N$. On the other hand, it is easy to see that $|f_n(\varphi(y_0)) - 1| > c$ for all $n \geq N$. Thus $R_\pi(f_n - 1) \subseteq (f_n - 1)(\varphi(c(B) \setminus K))$ for all $n \geq N$. Hence for each $z_0 \in R_\pi(f_n - 1)$ there exists a point $x_0 \in \varphi(c(B) \setminus K)$ such that $z_0 = f_n(x_0) - 1$. Therefore, $z_0 = \overline{Sf_n(\varphi^{-1}(x_0))} - 1$ which implies that

$$\text{Im}(f_n(x_0)) = -\text{Im}(Sf_n(\varphi^{-1}(x_0))) < \epsilon - \text{Im}(F(\varphi^{-1}(x_0))) < \epsilon - a < 0.$$

Consequently, $\text{Im}(f_n(x_0)) - 1 = \text{Im}(f_n(x_0)) < 0$. This shows that $R_\pi(f_n - 1) \subseteq \{z \in \mathbb{C} : \text{Im}(z) < 0\}$ which is a contradiction to (3.6). Therefore, $K = c(B)$ and consequently for each $f \in A$ and $y \in c(B)$,

$$(Tf)^d(y) = (T1)^d(y) f(\varphi(y))^d.$$

□

Acknowledgement. The authors would like to thank the referee for his/her helpful comments.

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