Banach J. Math. Anal. 6 (2012), no. 1, 147–154

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

BISHOP'S PROPERTY (β) AND RIESZ IDEMPOTENT FOR k-QUASI-PARANORMAL OPERATORS

SALAH MECHERI

Communicated by M. Fujii

Abstract. The study of operators satisfying Bishop's property (β) is of significant interest and is currently being done by a number of mathematicians around the world. Recently Uchiyama and Tanahashi [Oper. Matrices 4 (2009), 517–524] showed that a paranormal operator has Bishop's property (β). In this paper we introduce a new class of operators which we call the class of k-quasi-paranormal operators. An operator T is said to be a k-quasi-paranormal operator if it satisfies $||T^{k+1}x||^2 \leq ||T^{k+2}x||||T^kx||$ for all $x \in H$ where k is a natural number. This class of operators contains the class of paranormal operators and the class of quasi-class A operators. We prove basic properties and give a structure theorem of k-quasi-paranormal operators. We also show that Bishop's property (β) holds for this class of operators. Finally, we prove that if E is the Riesz idempotent for a nonzero isolated point λ_0 of the spectrum of a k-quasi-paranormal operator T, then E is self-adjoint if and only if the null space of $T - \lambda_0$, $\ker(T - \lambda_0) \subseteq \ker(T^* - \overline{\lambda_0})$.

1. Introduction

Let $B(H)$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space H. Let T be an operator in $B(H)$. An operator T is said to be positive (denoted $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. The operator T is said to be a p-hyponormal operator if and only if $(T^*T)^p \geq (TT^*)^p$ for a positive number p. In [8], the class of log-hyponormal operators is defined as follows: T is called log-hyponormal if it is invertible and

Date: Received: 6 August 2011; Accepted: 10 November 2011.

2010 *Mathematics Subject Classification.* Primary 47B47; Secondary 47B30, 47B20.

Key words and phrases. k-quasi-paranormal operator, paranormal operator, Riesz idempotent, Bishop's property (β), .

satisfies $\log T^*T \geq \log TT^*$. Class of p-hyponormal operators and class of log-hyponormal operators were defined as extension class of hyponormal operators, i.e., $T^*T \geq TT^*$. It is well known that every p-hyponormal operator is a q-hyponormal operator for $p \geq q > 0$, by the Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$", and every invertible p-hyponormal operator is a log-hyponormal operator since $\log(\cdot)$ is an operator monotone function. An operator T is called paranormal if $||Tx||^2 \leq ||T^2x||\|x\|$ for all $x \in H$. It is also well known that there exists a hyponormal operator T such that T^2 is not hyponormal (see [4]). In [2], Furuta, Ito and Yamazaki introduced the class $A(k)$ operators defined as follows: An operator T is called the $A(k)$ class operator if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2.$$

$A(1)$ is called the class A, which includes the class of log-hyponormal operators (see Theorem 2, in [2]) and is included in the class of paranormal operators (see Theorem 1 in [2]). $T \in B(H)$ is called the quasi-class A operator if $T^*|T^2|T \geq T^*|T|^2T$ [5]. In general the following implications hold:

$$\text{Hyponormal} \Rightarrow p-\text{Hyponormal} \Rightarrow \text{class}\, A \Rightarrow \text{paranormal};$$

$$\text{Hyponormal} \Rightarrow \text{class}\, A \Rightarrow \text{quasi-class}\, A.$$

It is shown [1] that T is paranormal if and only if

$$T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0 \ \ \text{for all}\, \lambda > 0.$$

In order to extend the class of paranormal operators and the class of quasi-class A operators we introduce a new class of operators which we call k-quasi-paranormal class of operators. An operator T is said to be a k-quasi-paranormal operator if it satisfies the following inequality:

$$||T^{k+1}x||^2 \leq ||T^{k+2}x||||T^kx||$$

for all $x \in H$ where k is a natural number. A 1-Quasi-paranormal operator is quasi-paranormal. It is shown that a quasi-class A operator is 1-quasi-paranormal (see Proposition 2.3). By this we get the following implications:

$$\text{Hyponormal} \Rightarrow p-\text{Hyponormal} \Rightarrow \text{class}\, A \Rightarrow \text{paranormal}$$

$$\Rightarrow \text{quasi-paranormal} \Rightarrow\ k-\text{quasi-paranormal};$$

$$\text{Hyponormal} \Rightarrow \text{class}\, A \Rightarrow \text{quasi-class}\, A \Rightarrow \text{quasi-paranormal}$$

$$\Rightarrow k-\text{quasi-paranormal}.$$

It is well known that a paranormal operator is normaloid. We give an example of a k-quasi-paranormal operator which is not normaloid (see Section 3). An operator $T \in B(H)$ is said to have the single-valued extension property (or SVEP) if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to H$ such that $(T-z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For $T \in B(H)$ and $x \in H$, the set $\rho_T(x)$ is defined to consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, which verifies $(T-z)f(z) = x$, and it is called the local resolvent set of T at x. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the local spectrum of T at x, and define the local spectral subspace of T , $H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$ for each subset F of $\mathbb{C}$. An operator $T \in B(H)$ is said to have the property (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to H$ of H-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in B(H)$ is said to have Dunford's property (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. It is well known that

$$\text{Property } (\beta) \Rightarrow \text{ Dunford's property}(C) \Rightarrow \text{ SVEP}.$$

Let $\mu \in \text{iso}\sigma(T)$. Then the Riesz idempotent E of T with respect to μ is defined by

$$E := \frac{1}{2\pi i}\int_{\partial D}(\mu I - T)^{-1}d\mu,$$

where D is a closed disk centered at μ which contains no other points of the spectrum of T. In [7], Stampfli showed that if T satisfies the growth condition G_1, then E is self-adjoint and $E(H) = N(T - \mu)$. Recently, Jeon and Kim [5] and Uchiyama [10] obtained Stampfli's result for quasi-class A operators and paranormal operators. In general even though T is a paranormal operator, the Riesz idempotent E of T with respect to $\mu \in \text{iso}\sigma(T)$ is not necessary self-adjoint. The study of operators satisfying Bishop's property (β) is of significant interest and is currently being done by a number of mathematicians around the world. Recently in [9] the authors showed that a paranormal operator has Bishop's property (β). In this paper we prove basic properties and give a structure theorem of k-quasi-paranormal operators and show that Bishop's property (β) holds for this class of operators. Finally we prove that if E is the Riesz idempotent for a nonzero isolated point λ_0 of the spectrum of a k-quasi-paranormal operator T, then E is self-adjoint if and only if the null space of $T - \lambda_0$, $\ker(T - \lambda_0) \subseteq \ker(T^* - \overline{\lambda_0})$. Throughout this paper, let k be some natural number.

2. Main results

It is well known that for any operators A, B and C,

$$A^*A - 2\lambda B^*B + \lambda^2 C^*C \geq 0 \quad \text{for all}\, \lambda > 0$$
$$\Longleftrightarrow ||Bx||^2 \leq ||Ax||||Cx|| \quad \text{for all}\, x \in H.$$

Thus we have the following proposition.

Proposition 2.1. *An operator $T \in B(H)$ is k-quasi-paranormal if and only if*

$$T^{*k+2}T^{k+2} - 2\lambda T^{*k+1}T^{k+1} + \lambda^2 T^{*k}T^k \geq 0, \;\; \textit{for all}\; \lambda > 0.$$

Proposition 2.2. *Let M be a closed T-invariant subspace of H. Then the restriction $T_{|M}$ of a k-quasi-paranormal operator T to M is a k-quasi-paranormal operator.*

Proof. Let

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \qquad \text{on} \;\; H = M \oplus M^{\perp}.$$

Since T is k-quasi-paranormal, we have

$$T^{*k+2}T^{k+2} - 2\lambda T^{*k+1}T^{k+1} + \lambda^2 T^{*k}T^k \geq 0 \quad \text{for all } \lambda > 0.$$

Hence

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{*k} \Big\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{*2} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{2} - 2\lambda \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{*} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} + \lambda^2 \Big\} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}^{k} \geq 0$$

for all $\lambda > 0$. Therefore

$$\begin{pmatrix} A^{*k}(A^{*2}A^2 - 2\lambda A^*A + \lambda^2)A^k & E \\ F & G \end{pmatrix} \geq 0,$$

for some operators E, F and G. Hence

$$A^{*k}(A^{*2}A^2 - 2\lambda A^*A + \lambda^2)A^k \geq 0,$$

for all $\lambda > 0$. This implies that $A = T_{|M}$ is k-quasi-paranormal. □

Proposition 2.3. *If $T \in B(H)$ belongs to the quasi-class A, then T is 1-quasi-paranormal.*

Proof. Since T belongs to the quasi-class A, we have $T^*|T|^2T \leq T^*|T^2|T$. Let $x \in H$. Then

$$\begin{aligned} ||T^2x||^2 &= \langle T^*T^2x, Tx\rangle = \langle T^*|T|^2Tx, x\rangle \\ &\leq \langle T^*|T^2|Tx, x\rangle \leq |||T^2|Tx||||Tx|| = ||T^3x||||Tx||. \end{aligned}$$

Therefore $||T^2x||^2 \leq ||T^3x|||Tx||$. Hence T is 1-quasi-paranormal. □

For an operator $T \in B(H)$, the closure of the range, the kernel and the spectrum of T are denoted by $\overline{\text{ran}T}$, ker T and $\sigma(T)$, respectively.

Lemma 2.4. *Let $T \in B(H)$ be a k-quasi-paranormal operator, the range of T^k be not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \textit{on} \quad H = \overline{\text{ran } T^k} \oplus \ker T^{*k}.$$

Then T_1 is paranormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Let

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{\text{ran } T^k} \oplus \ker T^{*k}$$

and let P be the orthogonal projection onto $\overline{\text{ran } T^k}$. Since T is k-quasi-paranormal, we have

$$P(T^{*2}T^2 - 2\lambda T^*T + \lambda^2)P \geq 0 \text{ for all } \lambda > 0.$$

Therefore

$$P(T^{*2}T^2)P - 2\lambda P(T^*T)P + \lambda^2 \geq 0 \text{ for all } \lambda > 0.$$

Hence $T_1^{*2}T_1^2 - 2\lambda T_1^*T_1 + \lambda^2 \geq 0$ for all $\lambda > 0$. This shows that T_1 is paranormal on $\overline{\text{ran } T^k}$. Further, we have

$$\langle T_3^k x_2, y_2\rangle = \langle T^k(I-P)x, (I-P)y\rangle = \langle (I-P)x, T^{*k}(I-P)y\rangle = 0,$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in H$. Thus $T_3^{*k} = 0$. Since $\sigma(T_3) = \{0\}$, we have

$$\sigma(T) = \sigma(T_1) \cup \{0\}.$$

□

As a consequence we obtain the following corollary.

Corollary 2.5. *Let $T \in B(H)$ be k-quasi-paranormal operator. If T_1 is invertible, then T is similar to a direct sum of a paranormal and a nilpotent operator.*

Proof. Since by assumption $0 \notin \sigma(T_1)$, we have $\sigma(T_1) \cap \sigma(T_3) = \varnothing$. Then there exists an operator S such that $T_1 S - S T_3 = T_2$ [6]. Since $\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix}$, hence

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}.$$

□

Theorem 2.6. *Let $T \in B(H)$ be a k-quasi-paranormal operator. Then T has Bishop's property (β). Hence T has the single valued extension property.*

Proof. If the range of T^k is dense, then T is paranormal. Hence, T has Bishop's property (β) by [9]. So, we assume that the range of T^k is not dense. By Lemma 2.1, we have

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad H = \overline{\text{ran } T^k} \oplus \ker T^{*k}.$$

Let D be an open subset of $\mathbb{C}$ and $f_n(z)$ be analytic functions on D to H. Assume $(T - z) f_n(z) \to 0$ uniformly on every compact subset of D. Put $f_n(z) = f_{n1}(z) \oplus f_{n2}(z)$ on $H = \overline{\text{ran } T^k} \oplus \ker T^{*k}$. Then we can write

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_{n1}(z) \\ f_{n2}(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z) f_{n1}(z) + T_2 f_{n2}(z) \\ (T_3 - z) f_{n2}(z) \end{pmatrix} \to 0.$$

Since T_3 is nilpotent, T_3 has Bishop's property (β). Hence $f_{n2}(z) \to 0$ uniformly on every compact subset of D. Then $(T_1 - z) f_{n1}(z) \to 0$. Since T_1 is paranormal, T_1 has Bishop's property (β) by [9]. Hence $f_{n1}(z) \to 0$ uniformly on every compact subset of D. Thus T has Bishop's property (β). □

Theorem 2.7. *Let $T \in B(H)$ be k-quasi-paranormal operator. Write*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = \overline{\text{ran}(T^k)} \oplus \ker(T^{k*})$. Then the following statements hold.

(1) $\sigma_{T_3}(x_2) \subset \sigma_T(x_1 \oplus x_2)$ and $\sigma_{T1}(x_1) = \sigma_T(x_1 \oplus 0)$ where $x_1 \oplus x_2 \in H$.

(2) $R_{T_1}(F) \oplus 0 \subset H_T(F)$ where $R_{T_1}(F) := \{y \in \overline{\text{ran}(T^k)} : \sigma_{T_1}(y) \subset F\}$ for any set $F \subset \mathbb{C}$.

Proof. Let $T \in B(H)$ be k-quasi-paranormal. Write

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

on $H = \overline{\text{ran}(T^k)} \oplus \ker(T^{k*})$, where $T^k = 0$ and T_1 is paranormal.

(1) Let $x_1 \oplus x_2 \in H = \overline{\text{ran}(T^k)} \oplus \ker(T^{k*})$. If $\lambda_0 \in \rho_T(x_1 \oplus x_2)$, then there is an H-valued analytic function f defined on a neighborhood U of λ_0 such that $(T-\lambda)f(\lambda) = x_1 \oplus x_2$ for all $\lambda \in U$. We can write $f = f_1 \oplus f_2$ where $f_1 \in O(U, \overline{\text{ran}(T^k)})$ and $f_2 \in O(U, \ker(T^{k*}))$, where $O(U, H)$ denotes the Fréchet space of H-valued analytic functions on U with respect to the uniform topology. Then we get

$$\begin{pmatrix} T_1 - \lambda & T_2 \\ 0 & T_3 - \lambda \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $(T_3 - \lambda) f_2(\lambda) \equiv x_2$. Hence $\lambda_0 \in \rho_{T_3}(x_2)$. On the other hand, if $\lambda_0 \in \rho_T(x_1 \oplus 0)$, then there is an H-valued analytic function g defined on a neighborhood U of λ_0 such that $(T-\lambda)g(\lambda) = x_1 \oplus 0$ for all $\lambda \in U$. If we set $g = g_1 \oplus g_2$ where $g_1 \in O(U, \text{ran}(T^k))$ and $g_2 \in O(U, \ker(T^{k*}))$, then we get

$$\begin{pmatrix} T_1 - \lambda & T_2 \\ 0 & T_3 - \lambda \end{pmatrix} \begin{pmatrix} g_1(\lambda) \\ g_2(\lambda) \end{pmatrix} \equiv \begin{pmatrix} x_1 \\ 0 \end{pmatrix}.$$

Thus $(T_1 - \lambda) g_1(\lambda) + T_2 g_2(\lambda) \equiv x_1$ and $(T_3 - \lambda) g_2(\lambda) \equiv 0$. Since T_3 is nilpotent of order k, it has the single-valued extension property, which implies that $g_2(\lambda) \equiv 0$. Thus $(T_1 - \lambda) g_1(\lambda) \equiv x_1$, and so $\lambda_0 \in \rho_{T_1}(x_1)$. Conversely, let $\lambda_0 \in \rho_{T_1}(x_1)$. Then there exists a function $g_1 \in O(U, \overline{\text{ran}(T^k)})$ for some neighborhood U of λ_0 such that $(T_1 - \lambda) g_1(\lambda) \equiv x_1$. Then $(T-\lambda) g_1(\lambda) \oplus 0 \equiv x_1 \oplus 0$. Hence $\lambda_0 \in \rho_T(x_1 \oplus 0)$.

(2) If $x_1 \in R_{T_1}(F)$, then $\sigma_{T_1}(x_1) \subset F$. Since $\sigma_{T_1}(x_1) = \sigma_T(x_1 \oplus 0)$ by (1), $\sigma_T(x_1 \oplus 0) \subset F$. Thus $x_1 \oplus 0 \in H_T(F)$, and hence $R_{T_1}(F) \oplus 0 \subset H_T(F)$. □

For $T \in B(H)$, the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$ is called the ascent of T and denoted by $p(T)$. If no such integer exists, we set $p(T) = \infty$. The smallest nonnegative integer q such that $\text{ran}(T^q) = \text{ran}(T^{q+1})$ is called the descent of T and denoted by $q(T)$. If no such integer exists, we set $q(T) = \infty$. In the following theorem we will give a necessary and sufficient condition for the Riesz idempotent E of a k-quasi-paranormal operator to be self-adjoint. For this we need the following lemma.

Lemma 2.8. *Let $T \in B(H)$ be k-quasi-paranormal. If μ is a non-zero isolated point of $\sigma(T)$, then μ is a simple pole of the resolvent of T.*

Proof. Assume that $\text{ran}(T^k)$ is dense. Then T is paranormal by Proposition 2.1 and [10] implies that μ is a simple pole of the resolvent of T. So we may assume that T^k does not have dense range. Then by Lemma 2.1 the operator T can be decomposed as follows:

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \text{ on } H = \overline{\text{ran}(T^k)} \oplus \ker(T^{*k}),$$

where A is paranormal and $C^k = 0$. Now if μ is a non-zero isolated point of $\sigma(T)$, then $\mu \in \text{iso}\sigma(A)$ because $\sigma(T) = \sigma(A) \cup \{0\}$. Therefore μ is a simple pole of the resolvent of A and the paranormal operator A can be written as follows:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ on } \overline{\text{ran}(T^k)} = \ker(A-\mu) \oplus \text{ran}(A-\mu),$$

where $\sigma(A_1) = \{\mu\}$. Therefore

$$T-\mu = \begin{pmatrix} 0 & 0 & B_1 \\ 0 & A_2-\mu & B_2 \\ 0 & 0 & C-\mu \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & F \end{pmatrix} \text{ on } H = \ker(A-\mu) \oplus \text{ran}(A-\mu) \oplus \ker(T^{*k}),$$

where $F = \begin{pmatrix} A_2-\mu & B_2 \\ 0 & C-\mu \end{pmatrix}$. We claim that F is an invertible operator on $\text{ran}(A-\mu) \oplus \ker(T^{*k})$. Indeed,

(1) $A_2 - \mu I$ is invertible, In fact, If not, then μ will be an isolated point in $\sigma(A_2)$. Since A_2 is paranormal, μ is an eigenvalue of A_2 and so $A_2 x = \mu x$ for some non-zero vector x in $\text{ran}(A - \mu I)$. On the other hand, $Ax = A_2 x$ implying x is in $\ker(A-\mu I)$. Hence x must be a zero vector. This contradicts leads to (1).

(2) F is invertible. Indeed, note that $C - \mu I$ is invertible. Then by (1) and [4, Problem 71], $(A_2 - \mu I)(C - \mu I)$ is invertible. It is easy to show that $p(T-\mu) = q(T-\mu) = 1$. Hence μ is a simple pole of the resolvent of T. □

Theorem 2.9. *Let $T \in B(H)$ be k-quasi-paranormal. Assume $0 \neq \mu \in \text{iso}\sigma(T)$ and E is the Riesz idempotent of T with respect to μ. Then E is self-adjoint if and only if* $\ker(T-\mu) \subseteq \ker(T^* - \overline{\mu})$.

Proof. Since E is the Riesz idempotent of T with respect to μ and T is k-quasi-paranormal, it results from Lemma 2.2 that

$$\text{ran}(E) = \ker(T-\mu) \text{ and } \ker(E) = \text{ran}(T-\mu).$$

Assume that E is self-adjoint. Then E is an orthogonal projection. Hence $\text{ran}(E)^{\perp} = \ker(E)$. Therefore we get $\ker(T-\mu) \subseteq \ker(T^* - \overline{\mu})$ by using the equality $\text{ran}(T-\mu) = \ker(T^* - \overline{\mu})^{\perp}$. Conversely, assume that $\ker(T-\mu) \subseteq \ker(T^* - \overline{\mu})$. Then $\ker(T-\mu)$ and $\text{ran}(T-\mu)$ are orthogonal. Hence $\text{ran}(E)^{\perp} = \ker(E)$, and so E is self-adjoint. □

Remark 2.10. It is well known that a paranormal operator is normaloid, that is, $||T|| = r(T)$ (spectral radius of T). But a k-quasi-paranormal operator is not normaloid. Indeed, Let T be the unilateral weighted shift operator defined on l^2 by

$$Te_n = \alpha_n e_{n+1} \quad \text{for all } n \geq 0,$$

where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis for l^2. It is easy to see that T is paranormal if and only if $|\alpha_0| \leq |\alpha_1| \leq \cdots$ and by a simple calculation we show that T is a k-quasi-paranormal operator if and only if $|\alpha_{k+1}| \leq |\alpha_{k+2}| \leq |\alpha_{k+3}| \leq \cdots$, where $\alpha_0, \alpha_1, ..., \alpha_k$ are arbitrary. Hence, if we take $\alpha_0 = \alpha_1 = \cdots = \alpha_k = 2$ and $\alpha_i = \frac{1}{2}$ for $i \geq k$, then T is k-quasi-paranormal and $||T|| = 2 \neq 1 = r(T)$. Thus T is not normaloid.

It is also well known that there exists a hyponormal operator T such that T^2 is not a hyponormal operator (see [4]). In [3], Furuta showed that if T is paranormal, then T^n is also paranormal for every $n \in \mathbb{N}$. The same property remains true for k-quasi-paranormal operators. Indeed, Since T is k-quasi-paranormal, we have

$$||T^{k+1}x||^2 \leq ||T^{k+2}x||||T^k x||.$$

Hence

$$\frac{||T^{k+1}x||}{||T^k x||} \leq \frac{||T^{k+2}x||}{||T^{k+1}x||} \leq \frac{||T^{k+3}x||}{||T^{k+2}x||} \leq \cdots .$$

Then

$$\frac{||T^{nk+1}x||}{||T^{nk}x||} \leq \frac{||T^{nk+2}x||}{||T^{nk+1}x||} \leq \cdots \leq \frac{||T^{nk+n}x||}{||T^{nk+n-1}x||}$$
$$\leq \frac{||T^{nk+n+1}x||}{||T^{nk+n}x||} \leq \frac{||T^{nk+n+2}x||}{||T^{nk+n+1}x||} \leq \cdots \leq \frac{||T^{nk+2n}x||}{||T^{nk+2n-1}x||}.$$

Hence

$$\frac{||T^{nk+1}x||}{||T^{nk}x||} \times \frac{||T^{nk+2}x||}{||T^{nk+1}x||} \times \cdots \times \frac{||T^{nk+n}x||}{||T^{nk+n-1}x||}$$
$$\leq \frac{||T^{nk+n+1}x||}{||T^{nk+n}x||} \times \frac{||T^{nk+n+2}x||}{||T^{nk+n+1}x||} \times \cdots \times \frac{||T^{nk+2n}x||}{||T^{nk+2n-1}x||}$$

and

$$\frac{||T^{nk+n}x||}{||T^{nk}x||} \leq \frac{||T^{nk+2n}x||}{||T^{nk+n}x||}.$$

This implies that T^n is k-quasi-paranormal.

Acknowledgments. The authors wish to thank the referee for a careful reading and valuable comments for the original draft.

References

1. T. Ando,*Operators with a norm conditions*, Acta. Sci. Math (Szeged). **33** (1972), 169–178.
2. T. Furuta, M. Ito and T. Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math. **1** (1998), 389–403.
3. T. Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594–598.
4. P.R. Halmos, textit A Hilbert space problem book, Van Nostrand, Princeton. 1967.
5. In Ho Jean and In Hyoum Kim, *On operators satisfying* $T^*|T^2|T \geq T^*|T|^2T$, Linear Algebra Appl. **418**(2006), 854–862.
6. M.A. Rosenblum, *On the operator equation* $BX - XA = Q$, Duke Math. J. **23** (1956), 263–269.
7. J. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc. **117** (1965), 469–476.
8. K. Tanahashi, *Putnam's Inequality for* log-*hyponormal operators*, Integral Equations Operator Theory **43** (2004), 364–372.
9. A. Uchiyama and K. Tanahashi, *Bishop's property* (β) *for paranormal operators*, Oper. Matrices **4** (2009), 517–524.
10. A. Uchiyama, *On isolated points of the spectrum of paranomal operators*, Integral Equations and Operator Theory. **55**(2006), 145–151.

Department of Mathematics, Faculty of Science, Taibah University, P. O. Box 30002, Al Madinah Al Munawarah, Saudi Arabia.

E-mail address: mecherisalah@hotmail.com