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AN EXTENSION OF KY FAN'S DOMINANCE THEOREM

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Abstract. We prove that for a separable Hilbert space $\mathcal{H}$ with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$, the equality $\|\cdot\| = \|\sum_{i=1}^{\infty} s_i(\cdot)e_i \otimes e_i\|$ holds for all unitarily invariant norms on $\mathbb{B}(\mathcal{H})$ and Ky Fan's dominance theorem remains valid on $\mathbb{B}(\mathcal{H})$.

1. Introduction

There has been a great interest in studying unitarily invariant norms and symmetric norm ideals on $\mathbb{B}(\mathcal{H})$ in the last few decades (see, e.g., [1]-[4],[5],[7],[9]-[12] and the references therein). A norm $\|\cdot\|$ on a non-zero ideal $\mathcal{J}$ of $\mathbb{B}(\mathcal{H})$ is called *unitarily invariant* if $\|UTV\| = \|T\|$ for all unitary operators $U, V \in \mathbb{B}(\mathcal{H})$ and $T \in \mathcal{J}$. The ith s-number of an operator T on $\mathcal{H}$ is displayed by $s_i(T)$ and is given by

$$s_i(T) = \inf\{\|T - F\|_{op} : F \in \mathbb{B}(\mathcal{H}) \text{ has rank} < i\},$$

where $\|\cdot\|_{op}$ denotes the usual operator norm on $\mathbb{B}(\mathcal{H})$. Note that every finite rank operator belongs to any non-zero ideal of $\mathbb{B}(\mathcal{H})$.

Typical examples of unitarily invariant norms on $\mathbb{B}(\mathcal{H})$ are Ky Fan k-norms that are defined by $N_k(\cdot) = s_1(\cdot) + \cdots + s_k(\cdot)$ [3], see also [10]. We say that a norm $\|\cdot\|$ on $\mathcal{J}$, satisfies Ky Fan's dominance theorem, if for every $T, R \in \mathcal{J}$, with $N_k(T) \le N_k(R)$ for all $k \in \mathbb{N}$, the inequality $\|T\| \le \|R\|$ holds. Ky Fan's dominance theorem holds for $\mathcal{J}$ if it holds for all unitarily invariant norms on $\mathcal{J}$.

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In the finite dimensional case, owing to the presence of the singular value decomposition (SVD), there is a nice representation of unitarily invariant norms as symmetric gauge functions [2, Theorem 3.5.18], which plays a major role in solving problems and proving theorems in the finite dimensional case. In fact using SVD, we conclude that for every matrix $A \in \mathbb{M}_n(\mathbb{C})$ the equality $\|A\| = \|diag(s_1(A), \cdots, s_n(A))\|$ is satisfied for all unitarily invariant norms on $\mathbb{M}_n(\mathbb{C})$, where $\mathbb{M}_n(\mathbb{C})$ is the algebra of all $n \times n$ matrices with the entries in $\mathbb{C}$.

In this paper we prove an alternative equality in the infinite dimensional case. In fact, we show if $\mathcal{H}$ is a separable Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^{\infty}$, the equality $\|\cdot\| = \|\sum_{i=1}^{\infty} s_i(\cdot)e_i \otimes e_i\|$ holds for all unitarily invariant norms on $\mathbb{B}(\mathcal{H})$, where $(s_i(\cdot)e_i \otimes e_i)(h) = s_i(\cdot) < h, e_i > e_i$, for all $h \in \mathcal{H}$. As a corollary we conclude that for a separable Hilbert space $\mathcal{H}$, Ky Fan's dominance theorem remains valid on $\mathbb{B}(\mathcal{H})$.

2. Unitarily invariant norms on $\mathbb{B}(\mathcal{H})$

Thought this section, when we say $\mathcal{J}$ is a non-zero ideal of $\mathbb{B}(\mathcal{H})$, it is possible for $\mathcal{J}$ to be equal the whole $\mathbb{B}(\mathcal{H})$. Also $\|\cdot\|$ will be an arbitrary unitarily invariant norm on $\mathbb{B}(\mathcal{H})$. We say that a norm $|||\cdot|||$ on $\mathcal{J}$ is symmetric if $|||T_1ST_2||| \leq \|T_1\|_{op}|||S|||\|T_2\|_{op}$ for all $T_1, T_2 \in \mathbb{B}(\mathcal{H})$ and $S \in \mathcal{J}$.

Lemma 2.1. *Every unitarily invariant norm $\|\cdot\|$ on a non-zero ideal $\mathcal{J}$ of $\mathbb{B}(\mathcal{H})$ is symmetric.*

Proof. Let $T \in \mathbb{B}(\mathcal{H})$, $S \in \mathcal{J}$ and consider a real number $\alpha > 1$. By [8], there are unitary elements $U_1, \cdots, U_n$ in $\mathbb{B}(\mathcal{H})$ such that $\frac{T}{\alpha\|T\|_{op}} = \frac{U_1+\cdots+U_n}{n}$. Hence

$$\|TS\| = \alpha\|T\|_{op} \left\|\frac{T}{\alpha\|T\|_{op}}S\right\| = \alpha\|T\|_{op} \left\|\frac{U_1 + \cdots + U_n}{n}S\right\| \leq \alpha\|T\|_{op} \|S\|.$$

Since $\alpha > 1$ is arbitrary, the inequality $\|TS\| \leq \|T\|_{op}\|S\|$ holds. Similarly we can show that $\|ST\| \leq \|S\| \|T\|_{op}$. □

Corollary 2.2. *Let $\|\cdot\|$ be a unitarily invariant norm on a non-zero ideal $\mathcal{J}$ of $\mathbb{B}(\mathcal{H})$ and $T, S \in \mathcal{J}$. Then*
(i) $\|T\| = \|\ |T|\ \|$.
(ii) $\|T\| = \|T^*\|$.
(iii) $\|p\| = \|q\|$, for any equivalent projections p and q in $\mathcal{J}$.
(iv) If $T \geq S \geq 0$, then $\|T\| \geq \|S\|$.

Proof. The polar decomposition $T = u|T|$ of T implies that $|T| = u^*T$, $T^* = |T|u^*$ and $|T| = T^*u$. Also, if p and q are equivalent then $p = vv^*$ and $q = v^*v$, for some partial isometry v in $\mathbb{B}(\mathcal{H})$ and hence we have $v^*pv = q$ and $vqv^* = p$. If $T \geq S \geq 0$, there is an operator R with $\|R\|_{op} \leq 1$ such that $S = RT$. These arguments together with Lemma 2.1 imply (i)-(iv). □

Lemma 2.3. *Let $\mathcal{J}$ be a non-zero ideal of $\mathbb{B}(\mathcal{H})$, $T \in \mathcal{J}$ and P be a projection of rank one. For every unitarily invariant norm $\|\cdot\|$ on $\mathcal{J}$ the inequality $\|T\| \geq \|P\| \|T\|_{op}$ holds.*

Proof. Suppose $T \neq 0$ and consider a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $\|x_n\| = 1$, for all n and $\lim_{n\to\infty} \|T(x_n)\| = \|T\|_{\text{op}}$. Without loss of generality we can suppose that $T(x_n) \neq 0$, for all $n \in \mathbb{N}$. Let U_n be a unitary operator that $U_n(\frac{T(x_n)}{\|T(x_n)\|}) = x_n$. Setting $P_n = x_n \otimes x_n$ we have

$$\begin{aligned}\|T\| = \|P_n\|_{\text{op}}\|T\| \geq \|TP_n\| &= \left\| \frac{T(x_n)}{\|T(x_n)\|} \otimes x_n \right\| \ \|T(x_n)\| \\ &= \left\| U_n\left(\frac{T(x_n)}{\|T(x_n)\|} \otimes x_n \right) \right\| \ \|T(x_n)\| \\ &= \|x_n \otimes x_n\| \ \|T(x_n)\| \\ &= \|P_n\| \ \|T(x_n)\| \\ &= \|P\| \ \|T(x_n)\|,\end{aligned}$$

where the last equality has resulted from (iii) of Corollary 2.2. Now if $n \to \infty$ we get the desired result. □

Corollary 2.4. *If P is a projection of rank one in $\mathbb{B}(\mathcal{H})$ then*

$$\|P\| \ \|T\|_{op} \leq \|T\| \leq \|I\| \ \|T\|_{op} \quad (T \in \mathbb{B}(\mathcal{H})),$$

where I is the identity operator on $\mathcal{H}$. Therefore, all unitarily invariant norms on $\mathbb{B}(\mathcal{H})$ are equivalent to the operator norm.

Corollary 2.5. *If P is a projection of rank one in $\mathbb{B}(\mathcal{H})$ and $\|P\| = \|I\|$, then $\|\cdot\|$ is a multiple of the operator norm.*

Lemma 2.6. *Suppose $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence in $\mathcal{H}$. For positive diagonal operator $T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$ $(\lambda_i \geq 0)$, let $E = \{\lambda_i \mid i = 1, 2, \cdots\}$. Then*
(i) $s_1(T) = \|T\|_{op} = sup_{j\in\mathbb{N}}\lambda_j$.
(ii) If there exist $k-1$ distinct positive integers $n_1, \cdots, n_{k-1}$ such that $s_i(T) = \lambda_{n_i}$ $(1 \leq i \leq k-1)$, then $s_k(T) = sup_{j\notin\{n_1,\cdots,n_{k-1}\}}\lambda_j$. Also, if $s_k(T)$ is a limit point of E, then for every $i \in \mathbb{N}$, we have $s_{k+i}(T) = s_k(T)$.
(iii) If there is no s-number of T that is a limit point of E, then there are distinct positive integers $n_1, n_2, \ldots$ such that $s_i(T) = \lambda_{n_i}$, $i \in \mathbb{N}$. Otherwise there is positive integer k and $k-1$ distinct natural numbers $n_1, \ldots, n_{k-1}$ such that $s_i(T) = \lambda_{n_i}$, $1 \leq i \leq k-1$ and $s_k(T) = s_{k+1}(T) = \cdots$. In both cases, for every positive integer i, we have $s_i(T) = sup_{\lambda_j \leq s_i(T)}\lambda_j$.

Proof. (i) This is a well known fact [6, Problem 63].

(ii) If $k = 1$ the equality is resulted from (i). Otherwise setting

$$F = \sum_{i=1}^{k-1} \lambda_{n_i} e_{n_i} \otimes e_{n_i},$$

we have $rank(F) < k$ and so

$$s_k(T) \leq \|T - F\|_{op} = \sup_{j\notin\{n_1,\cdots,n_{k-1}\}}\lambda_j.$$

On the other hand if for every $j \in \mathbb{N} \setminus \{n_1, \cdots, n_{k-1}\}$, setting

$$R_j = \sum_{i=1}^{k-1} \lambda_{n_i} e_{n_i} \otimes e_{n_i} + \lambda_j e_j \otimes e_j,$$

we have $R_j \leq T$. Since Ky Fan norms are unitarily invariant, using (iv) of Corollary 2.2, we have $N_k(R_j) \leq N_k(T)$. Therefore $\lambda_j \leq s_k(T)$ and hence $s_k(T) = \sup_{j \notin \{n_1, \cdots, n_{k-1}\}} \lambda_j$.

If $s_k(T)$ is a limit point of E, then for every $\epsilon > 0$ and $i \in \mathbb{N}$, there exist distinct positive integers $m_1, \cdots, m_{i+1} \in \mathbb{N} \setminus \{n_1, \cdots, n_{k-1}\}$ such that

$$s_k(T) - \frac{\epsilon}{i+1} < \lambda_{m_j} \leq s_k(T), \ (1 \leq j \leq i+1).$$

Setting

$$S = \sum_{j=1}^{i+1} \lambda_{m_j} e_{m_j} \otimes e_{m_j} + \sum_{j=1}^{k-1} \lambda_{n_j} e_{n_j} \otimes e_{n_j},$$

we have $0 \leq S \leq T$ and so $N_{k+i}(S) \leq N_{k+i}(T)$. This implies that

$$\begin{aligned} \sum_{j=1}^{i+1} (s_k(T) - \frac{\epsilon}{i+1}) &\leq s_k(T) + \cdots + s_{k+i}(T) \\ &\leq \underbrace{s_k(T) + \cdots + s_k(T)}_{i \ times} + s_{k+i}(T). \end{aligned}$$

Therefore $s_k(T) - \epsilon \leq s_{k+i}(T)$ and so $s_k(T) \leq s_{k+i}(T)$. Hence, $s_k(T) = s_{k+i}(T)$.
(iii) By (i) we have $s_1(T) = \sup_{j \in \mathbb{N}} \lambda_j$. If $s_1(T)$ is a limit point of E, then by the second part of (ii), we have $s_1(T) = s_2(T) = \cdots$. Otherwise there is $n_1 \in \mathbb{N}$ such that $s_1(T) = \lambda_{n_1}$ and by the first part of (ii) we have $s_2(T) = \sup_{j \notin \{n_1\}} \lambda_j$. Now if $s_2(T)$ is a limit point of E, then again by the second part of (ii), we have $s_2(T) = s_3(T) = \cdots$. Otherwise there is $n_2 \in \mathbb{N} - \{n_1\}$ such that $s_2(T) = \lambda_{n_2}$ and by the first part of (ii), we have $s_3(T) = \sup_{j \notin \{n_1, n_2\}} \lambda_j$. Continuing this process we get desired results.

□

In particular, the following corollary follows from the previous lemma.

Corollary 2.7. *Suppose $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence in $\mathcal{H}$. For positive diagonal operator $T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$, let $s = \inf\{s_i(T) \mid i \in \mathbb{N}\}$, $E = \{\lambda_i \mid i = 1, 2, \cdots\}$. Then*

(i) for every $\epsilon > 0$, there exist distinct positive integers $n_1, n_2, \cdots$ such that $0 \leq s_i(T) < \lambda_{n_i} + \epsilon$.

(ii) for every $\epsilon > 0$, $A = \{i \mid \lambda_i > s + \epsilon\}$ is a finite set. In fact, A is empty or there exist distinct positive integers $n_1, \cdots n_{N_0}$, such that $A = \{n_i : \ 1 \leq i \leq N_0\}$ and $\lambda_{n_i} = s_i(T)$ $(1 \leq i \leq N_0)$.

Lemma 2.8. *Suppose $\{e_i\}_{i=1}^{\infty}$ is an orthonormal sequence in $\mathcal{H}$ and $T = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$ $(\lambda_i \geq 0)$ is a positive diagonal operator in $\mathbb{B}(\mathcal{H})$. Then* *(i) for every* $\epsilon > 0$ *there exists a unitary element* $U \in \mathbb{B}(\mathcal{H})$ *such that*

$$UTU^* \leq \sum_{i=1}^{\infty}(s_i(T) + \epsilon)e_i \otimes e_i,$$

(ii) for every $\epsilon > 0$ *there exists a partial isometry* $U \in \mathbb{B}(\mathcal{H})$ *such that*

$$\sum_{i=1}^{\infty}(s_i(T) - \epsilon)e_i \otimes e_i \leq UTU^*,$$

(iii) $\|T\| = \|\sum_{i=1}^{\infty} s_i(T)e_i \otimes e_i\|$.

Proof. Let $\epsilon > 0$, $s = \inf\{s_i(T) \mid i \in \mathbb{N}\}$ and $A = \{i \mid \lambda_i > s + \epsilon\}$. The previous corollary implies that A is empty or there exist distinct positive integers $n_1, \cdots n_{N_0}$, such that $A = \{n_i : \ 1 \leq i \leq N_0\}$ and $\lambda_{n_i} = s_i(T)$ $(1 \leq i \leq N_0)$.

If A is empty, we set $U = I$, otherwise we consider U as a unitary operator that maps $\{e_n\}_{n=1}^{\infty}$ onto $\{e_n\}_{n=1}^{\infty}$ and $U(e_{n_i}) = e_i$ for $i = 1, \cdots, N_0$. Therefore

$$UTU^* = \sum_{i=1}^{N_0} s_i(T)e_i \otimes e_i + \sum_{i=N_0+1}^{\infty} \mu_i e_i \otimes e_i,$$

where $\mu_i \in \{\lambda_j : \ j \neq n_i, \text{ for all } 1 \leq i \leq N_0\}$. Since for all $i \geq N_0$ we have $\mu_i \leq s + \epsilon$, then $UTU^* \leq \sum_{i=1}^{\infty}(s_i(T) + \epsilon)e_i \otimes e_i$.

For proving (ii), we recall that there exist distinct positive integers $n_1, n_2, \cdots$ such that $0 \leq s_i(T) < \lambda_{n_i} + \epsilon$. Now consider a partial isometry U which satisfies

$$U(e_j) = \begin{cases} e_i & j = n_i, \text{ for some } i \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} UTU^* &= \sum_{i=1}^{\infty} \lambda_{n_i} e_i \otimes e_i \\ &\geq \sum_{i=1}^{\infty}(s_i(T) - \epsilon)e_i \otimes e_i. \end{aligned}$$

Finally (iii) follows from (i),(ii) and (iv) of Corollary 2.2. □

Lemma 2.9. *Let* $\mathcal{H}$ *be a separable Hilbert space and* T *be a positive operator in* $\mathbb{B}(\mathcal{H})$. *For every* $\epsilon > 0$, *there exists a diagonal operator* T_ϵ *such that*

$$|s_i(T) - s_i(T_\epsilon)| < \epsilon, \quad \textit{for all } i \in \mathbb{N}.$$

Proof. By [11] there exist a diagonal operator T_ϵ and a compact operator K_ϵ such that $T = T_\epsilon + K_\epsilon$ and the Hilbert-Schmidt norm of K_ϵ is less than ϵ. Hence for every finite rank operator F, the following inequalities hold

$$\|T_\epsilon - F\|_{\text{op}} - \epsilon \leq \|T - F\|_{\text{op}} \leq \|T_\epsilon - F\|_{\text{op}} + \epsilon.$$

Taking infimum over F with rank$(F) < i$, we get the desired result. □

Theorem 2.10. *Let* $\mathcal{H}$ *be a separable Hilbert space and* $T \in \mathbb{B}(\mathcal{H})$. *The equality* $\|T\| = \|\sum_{i=1}^{\infty} s_i(T)e_i \otimes e_i\|$ *holds, for all orthonormal sequence* $\{e_i\}_{i\in\mathbb{N}}$ *in* $\mathcal{H}$.

Proof. Since $\|T\| = \|\ |T|\ \|$ and $s_i(T) = s_i(|T|)$, we can suppose that T is a positive operator. Let $\epsilon > 0$ and $T_\epsilon = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$ be the diagonal operator of the previous lemma. We have by Lemmas 2.1 and 2.8

$$\begin{aligned} \|T\| \leq \|T_\epsilon\| + \|K_\epsilon\| &\leq \|T_\epsilon\| + \|I\| \|K_\epsilon\|_{\text{op}} \\ &\leq \left\| \sum_{i=1}^{\infty} s_i(T_\epsilon) e_i \otimes e_i \right\| + \epsilon \|I\| \\ &\leq \left\| \sum_{i=1}^{\infty} s_i(T) e_i \otimes e_i \right\| + 2\epsilon \|I\|. \end{aligned}$$

Similarly We have

$$\begin{aligned} \left\| \sum_{i=1}^{\infty} s_i(T) e_i \otimes e_i \right\| &\leq \left\| \sum_{i=1}^{\infty} s_i(T_\epsilon) e_i \otimes e_i \right\| + \epsilon \|I\| \\ &= \|T_\epsilon\| + \epsilon \|I\| \\ &\leq \|T\| + 2\epsilon \|I\|. \end{aligned}$$

□

Lemma 2.11. *Suppose $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal sequence in $\mathcal{H}$ and $D_1 = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$, $D_2 = \sum_{i=1}^{\infty} \mu_i e_i \otimes e_i$ $(\lambda_i, \mu_i \geq 0)$, are positive diagonal operators in $\mathbb{B}(\mathcal{H})$. Assume moreover that there exists $N \in \mathbb{N}$ such that $\lambda_k = \mu_k$ for all $k > N$ and $s_k(D_1) = \lambda_k$, $s_k(D_2) = \mu_k$ for all $1 \leq k \leq N$. If $N_k(D_1) \leq N_k(D_2)$ for all $1 \leq k \leq N$, then $\|D_1\| \leq \|D_2\|$.*

Proof. Let $X_1 = \sum_{i=1}^{N} \lambda_i e_i \otimes e_i$ and $X_2 = \sum_{i=1}^{N} \mu_i e_i \otimes e_i$. We have $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$, for every $1 \leq k \leq N$ and so, there are unitary matrices $U_1, \cdots, U_{2^N N!}$ in $M_N(\mathbb{C})$ and non-negative numbers $c_1, \cdots, c_{2^N N!}$ such that $X_1 = \sum_{j=1}^{2^N N!} c_j U_j X_2 U_j^*$ and $\sum_{j=1}^{2^N N!} c_j = 1$ [2, II.2.10]. Now, we can choose unitary operators $\tilde{U}_1, \cdots, \tilde{U}_{2^N N!}$ in $\mathbb{B}(\mathcal{H})$ such that

$$\sum_{i=1}^{N} \lambda_i e_i \otimes e_i = \sum_{j=1}^{2^N N!} c_j \tilde{U}_j \left(\sum_{i=1}^{N} \mu_i e_i \otimes e_i \right) \tilde{U}_j^*,$$

and $\tilde{U}_j(e_i) = e_i$, for every $1 \leq j \leq 2^N N!$ and $i > N$. A direct computation shows that $D_1 = \sum_{j=1}^{2^N N!} c_j \tilde{U}_j D_2 \tilde{U}_j^*$, and so $\|D_1\| \leq \|D_2\|$. □

Now, we can show that Ky Fan's dominance theorem is valid on $\mathbb{B}(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space.

Theorem 2.12. *Let $\mathcal{H}$ be a separable Hilbert space and $T, R \in \mathbb{B}(\mathcal{H})$. If $N_k(T) \leq N_k(R)$ for all $k \in \mathbb{N}$, then $\|T\| \leq \|R\|$.*

Proof. For every $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that:
i) $s_N(T) \leq s_N(R) + \epsilon$,

ii)$s_N(R) \leq s_i(R) + \epsilon$, for every $i \geq N$.
Using (iv) of Corollary 2.2 and Lemma 2.10 together with Lemma 2.11, we have

$$\begin{aligned}
\|T\| &= \left\| \sum_{i=1}^{\infty} s_i(T) e_i \otimes e_i \right\| \\
&\leq \left\| \sum_{i=1}^{N} s_i(T) e_i \otimes e_i + \sum_{i=N+1}^{\infty} s_N(T) e_i \otimes e_i \right\| \\
&\leq \left\| \sum_{i=1}^{N} s_i(R) e_i \otimes e_i + \sum_{i=N+1}^{\infty} s_N(T) e_i \otimes e_i \right\| \\
&\leq \left\| \sum_{i=1}^{N} s_i(R) e_i \otimes e_i + \sum_{i=N+1}^{\infty} (s_N(R) + \epsilon) e_i \otimes e_i \right\| \\
&\leq \left\| \sum_{i=1}^{N} s_i(R) e_i \otimes e_i + \sum_{i=N+1}^{\infty} (s_i(R) + 2\epsilon) e_i \otimes e_i \right\| \\
&\leq \left\| \sum_{i=1}^{\infty} s_i(R) e_i \otimes e_i \right\| + 2\epsilon \|I\| = \|R\| + 2\epsilon \|I\|.
\end{aligned}$$

□

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