Banach J. Math. Anal. 6 (2012), no. 1, 101–111

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
**www.emis.de/journals/BJMA/**

# COMMON FIXED POINT THEOREMS FOR CONTRACTIVE MAPPINGS SATISFYING Φ-MAPS IN $G$-METRIC SPACES

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Communicated by M. A. Japón Pineda

ABSTRACT. We prove the existence of the unique common fixed point theorems of a pair of weakly compatible mappings satisfying Φ-maps in $G$-metric spaces. These results generalize the well-known results in the literature.

## 1. Introduction and preliminaries

The metric fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for instant, variational inequalities, optimization, and approximation theory. There were many authors introduced the generalizations of metric spaces such as Gahler [8, 9] (called 2-metric spaces) and Dhage [6, 7] (called $D$-metric spaces). In 2003, Mustafa and Sims [17] found that most of the claims concerning the fundamental topological properties of $D$-metric spaces are incorrect. Consequently, they [18] introduced a generalization of metric spaces. Namely, $G$-metric spaces as the following:

**Definition 1.1.** Let $X$ be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying :

($G1$) $G(x, y, z) = 0$ if $x = y = z$,
($G2$) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,
($G3$) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $z \neq y$,
($G4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables), and

*Date*: Received: 3 June 2011; Revised: 4 October 2011; Accepted: 17 October 2011.
2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 47H09.
*Key words and phrases.* Common fixed points, $G$-metric spaces, Φ-maps, weakly compatible mappings.

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, for all $x,y,z,a \in X$ (rectangle inequality).

Then the function $G$ is called a generalized metric or more specifically a $G$-metric on $X$, and the pair $(X,G)$ is called a $G$-metric space.

Since then the fixed point theory in $G$-metric spaces has been studied and developed by many authors (see [1, 2, 3, 4, 15, 16, 18, 19, 21, 23, 24]). The common fixed point theorems for mappings satisfying certain contractive conditions in metric spaces have been continually studied for decade (see [5, 10, 11, 12, 13, 20] and references contained therein). In 2009, Abbas and Rhoades [3], proved the unique common fixed point theorems for a pair of weakly compatible mappings in $G$-metric spaces. Recently, Shatanawi [22], proved the unique fixed point theorems for contractive mappings satisfying $\Phi$-maps in $G$-metric spaces. In this paper, we prove the existence of the unique common fixed point theorems of a pair of weakly compatible mappings satisfying $\Phi$-maps in $G$-metric spaces. These results generalize the well-known results proved by Abbas and Rhoades [3] and Shatanawi [22].

We now recall some of the basic concepts and results in $G$-metric spaces that have been established in [18].

**Definition 1.2.** Let $(X,G)$ be a $G$-metric space. We say that a sequence $\{x_n\}$ in $X$ is:

(i) a $G$-convergent sequence if, for any $\varepsilon > 0$, there exist $x \in X$ and $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$,

(ii) a $G$-Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$.

**Theorem 1.3.** *Let $(X,G)$ be a $G$-metric space and $\{x_n\}$ be a sequence in $X$. Then the following are equivalent:*

(i) *$\{x_n\}$ is $G$-convergent to $x$,*

(ii) *$G(x_n, x_n, x) \to 0$, as $n \to \infty$,*

(iii) *$G(x_n, x, x) \to 0$, as $n \to \infty$,*

(iv) *$G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.*

**Theorem 1.4.** *Let $(X,G)$ be a $G$-metric space and $\{x_n\}$ be a sequence in $X$. Then the following are equivalent:*

*(i) $\{x_n\}$ is $G$-Cauchy.*

*(ii) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.*

A $G$-metric space $X$ is said to be complete if every $G$-Cauchy sequence in $X$ is a $G$-convergent sequence in $X$.

**Proposition 1.5.** *Let $(X,G)$ be a $G$-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

**Definition 1.6.** Let $f$ and $g$ be single-valued self mappings on a set $X$. If $w = fx = gx$ for some $x \in X$, then $x$ is called a coincidence point of $f$ and $g$, and $w$ is called a point of coincidence of $f$ and $g$.

Abbas and Rhoades [3] proved the existence of the unique common fixed points of a pair of weakly compatible mappings in $G$-metric spaces by using the following proposition as a main tool.

**Proposition 1.7.** *([3, Proposition 1.5]) Let $f$ and $g$ be weakly compatible self mappings on a set $X$. If $f$ and $g$ have a unique point of coincidence $w = fx = gx$, then $w$ is the unique common fixed point of $f$ and $g$.*

## 2. Common fixed point theorems

In 1977, Matkowski [14] introduced the $\Phi$-map as the following: let $\Phi$ be the set of all function $\phi$ such that $\phi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function satisfying $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. If $\phi \in \Phi$, then $\phi$ is called a $\Phi$-map. Furthermore, if $\phi$ is a $\Phi$-map, then

(i) $\phi(t) < t$ for all $t \in (0, +\infty)$,
(ii) $\phi(0) = 0$.

From now on, unless otherwise stated, $\phi$ is meant the $\Phi$-map.

**Theorem 2.1.** *Let $(X, G)$ be a $G$-metric space. Suppose that the mappings $f, g : X \to X$ satisfy either*

$$G(fx, fy, fz) \leq \phi(\max\{G(gx, fx, fx), G(gy, fy, fy), G(gz, fz, fz)\}), \qquad (2.1)$$

*or*

$$G(fx, fy, fz) \leq \phi(\max\{G(gx, gx, fx), G(gy, gy, fy), G(gz, gz, fz)\}) \qquad (2.2)$$

*for all $x, y, z \in X$. If the range of $g$ contains the range of $f$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

*Proof.* Assume that $f$ and $g$ satisfy the condition (2.1). Let $x_0$ be an arbitrary point in $X$. Since the range of $g$ contains the range of $f$, there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then $f$ and $g$ have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\ &\leq \phi(\max\{G(gx_{n-1}, fx_{n-1}, fx_{n-1}), G(gx_n, fx_n, fx_n), \\ &\quad G(gx_n, fx_n, fx_n)\}) \\ &\leq \phi(\max\{G(gx_{n-1}, fx_{n-1}, fx_{n-1}), G(gx_n, fx_n, fx_n)\} \\ &\leq \phi(\max\{G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1})\}. \end{aligned}$$

If $\max\{G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1})\} = G(gx_n, gx_{n+1}, gx_{n+1})$, then

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \phi(G(gx_n, gx_{n+1}, gx_{n+1})) < G(gx_n, gx_{n+1}, gx_{n+1}),$$

which leads to a contradiction. This implies that

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \phi(G(gx_{n-1}, gx_n, gx_n)).$$

That is, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\ &\leq \phi(G(gx_{n-1}, gx_n, gx_n)) \\ &\leq \phi^2(G(gx_{n-2}, gx_{n-1}, gx_{n-1})) \\ &\vdots \\ &\leq \phi^n(G(gx_0, gx_1, gx_1)). \end{aligned}$$

We will show that $\{gx_n\}$ is $G$-Cauchy. Let $\varepsilon > 0$.
Since $\lim_{n\to\infty} \phi^n(G(gx_0, gx_1, gx_1)) = 0$ and $\phi(\varepsilon) < \varepsilon$, there exists $N \in \mathbb{N}$ such that

$$\phi^n(G(gx_0, gx_1, gx_1)) < \varepsilon - \phi(\varepsilon) \quad \text{for all } n \geq N.$$

This implies that

$$G(gx_n, gx_{n+1}, gx_{n+1}) < \varepsilon - \phi(\varepsilon) \quad \text{for all } n \geq N. \tag{2.3}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$G(gx_n, gx_m, gx_m) < \varepsilon \quad \text{for all } m \geq n \geq N \tag{2.4}$$

by induction on $m$. Since $\varepsilon - \phi(\varepsilon) < \varepsilon$ and by (2.3), we obtain that (2.4) holds for $m = n + 1$. Suppose that (2.4) holds for $m = k$. Therefore, for $m = k + 1$, we have

$$\begin{aligned} G(gx_n, gx_{k+1}, gx_{k+1}) &\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{k+1}, gx_{k+1}) \\ &< \varepsilon - \phi(\varepsilon) + G(fx_n, fx_k, fx_k) \\ &\leq \varepsilon - \phi(\varepsilon) \\ &\quad + \phi(\max\{G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_k, gx_{k+1}, gx_{k+1})\}) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon. \end{aligned}$$

Thus (2.4) holds for all $m \geq n \geq N$. It follows that $\{gx_n\}$ is $G$-Cauchy. By the completeness of $g(X)$, we obtain that $\{gx_n\}$ is $G$-convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. Suppose that $gp \neq fp$. By (2.1), we have

$$\begin{aligned} G(gx_n, fp, fp) &= G(fx_{n-1}, fp, fp) \\ &\leq \phi(\max\{G(gx_{n-1}, gx_n, gx_n), G(gp, fp, fp)\}). \end{aligned}$$

Case 1.

$$\max\{G(gx_{n-1}, gx_n, gx_n), G(gp, fp, fp)\} = G(gx_{n-1}, gx_n, gx_n),$$

we obtain that

$$G(gx_n, fp, fp) \leq \phi(G(gx_{n-1}, gx_n, gx_n)) < G(gx_{n-1}, gx_n, gx_n).$$

By taking $n \to \infty$, we have $G(gp, fp, fp) = 0$ and so $gp = fp$.
Case 2.

$$\max\{G(gx_{n-1}, gx_n, gx_n), G(gp, fp, fp)\} = G(gp, fp, fp),$$

we obtain that

$$G(gx_n, fp, fp) \leq \phi(G(gp, fp, fp)).$$

By taking $n \to \infty$, we have $G(gp, fp, fp) \leq \phi(G(gp, fp, fp)) < G(gp, fp, fp)$, which leads to a contradiction. Therefore $gp = fp$. We now show that $f$ and $g$ have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in X$. By applying (2.1), it follows that

$$\begin{aligned} G(gp, gp, gq) &= G(fp, fp, fq) \\ &\leq \phi(\max\{G(gp, fp, fp), G(gp, fp, fp), G(gq, fq, fq)\}) \\ &= 0. \end{aligned}$$

Therefore $gp = gq$. This implies that $f$ and $g$ have a unique point of coincidence. By Proposition 1.7, we can conclude that $f$ and $g$ have a unique common fixed point. The proof using (2.2) is similar. □

**Corollary 2.2.** *([3, Theorem 2.5]) Let $(X, G)$ be a $G$-metric space. Suppose that the mappings $f, g : X \to X$ satisfy either*

$$G(fx, fy, fz) \leq k \max\{G(gx, fx, fx), G(gy, fy, fy), G(gz, fz, fz)\},$$

*or*

$$G(fx, fy, fz) \leq k \max\{G(gx, gx, fx), G(gy, gy, fy), G(gz, gz, fz)\}$$

*for all $x, y, z \in X$ where $0 \leq k < 1$. If the range of $g$ contains the range of $f$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

*Proof.* Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) = kt$. Therefore $\phi$ is a nondecreasing function and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0, +\infty)$. It follows that the contractive conditions in Theorem 2.1 are now satisfied. This completes the proof. □

**Example 2.3.** Let $X = [0, 2], G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$ and $\phi(t) = \frac{t}{2}$. Therefore $\phi$ is a $\Phi$-map. Define $f, g : X \to X$ by

$$fx = 1 \text{ and } gx = 2 - x.$$

We obtain that $f$ and $g$ satisfy (2.1) and (2.2) in Theorem 2.1. Indeed, we have

$$\begin{aligned} &G(fx, fy, fz) = 0, \\ &\phi(\max\{G(gx, fx, fx), G(gy, fy, fy), G(gz, fz, fz)\}) \\ &= \frac{1}{2}(\max\{|1 - x|, |1 - y|, |1 - z|\}), \end{aligned}$$

and

$$\begin{aligned} &\phi(\max\{G(gx, gx, fx), G(gy, gy, fy), G(gz, gz, fz)\}) \\ &= \frac{1}{2}(\max\{|1 - x|, |1 - y|, |1 - z|\}). \end{aligned}$$

It is obvious that the range of $g$ contains the range of $f$ and $g(X)$ is a complete subspace of $(X, G)$. Furthermore, $f$ and $g$ are weakly compatible. Thus all assumptions in Theorem 2.1 are satisfied. This implies that $f$ and $g$ have a unique common fixed point which is $x = 1$.

**Theorem 2.4.** *Let $(X, G)$ be a $G$-metric space. Suppose that the mappings $f, g : X \to X$ satisfy*

$$G(fx, fy, fz) \leq \phi(G(gx, gy, gz)), \tag{2.5}$$

*for all $x, y, z \in X$. If the range of $g$ contains the range of $f$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. Since the range of $g$ contains the range of $f$, there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then $f$ and $g$ have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\ &\leq \phi(G(gx_{n-1}, gx_n, gx_n)) \\ &\leq \phi^2(G(gx_{n-2}, gx_{n-1}, gx_{n-1})) \\ &\ \vdots \\ &\leq \phi^n(G(gx_0, gx_1, gx_1)). \end{aligned}$$

We will show that $\{gx_n\}$ is $G$-Cauchy. Let $\varepsilon > 0$.
Since $\lim_{n\to\infty} \phi^n(G(gx_0, gx_1, gx_1)) = 0$ and $\phi(\varepsilon) < \varepsilon$, there exists $N \in \mathbb{N}$ such that

$$\phi^n(G(gx_0, gx_1, gx_1)) < \varepsilon - \phi(\varepsilon) \quad \text{for all } n \geq N.$$

This implies that

$$G(gx_n, gx_{n+1}, gx_{n+1}) < \varepsilon - \phi(\varepsilon) \quad \text{for all } n \geq N. \tag{2.6}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$G(gx_n, gx_m, gx_m) < \varepsilon \quad \text{for all } m \geq n \geq N \tag{2.7}$$

by induction on $m$. Since $\varepsilon - \phi(\varepsilon) < \varepsilon$ and by (2.6), we obtain that (2.7) holds for $m = n + 1$. Suppose that (2.7) holds for $m = k$. Therefore, for $m = k + 1$, we have

$$\begin{aligned} G(gx_n, gx_{k+1}, gx_{k+1}) &\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{k+1}, gx_{k+1}) \\ &< \varepsilon - \phi(\varepsilon) + G(fx_n, fx_k, fx_k) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(G(gx_n, gx_k, gx_k)\}) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon. \end{aligned}$$

Thus (2.7) holds for all $m \geq n \geq N$. It follows that $\{gx_n\}$ is $G$-Cauchy. By the completeness of $g(X)$, we obtain that $\{gx_n\}$ is $G$-convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. By (2.5), we have

$$\begin{aligned} G(gp, gp, fp) &\leq G(gp, gp, gx_{n+1}) + G(gx_{n+1}, gx_{n+1}, fp) \\ &\leq G(gp, gp, gx_{n+1}) + \phi(G(gx_n, gx_n, gp)) \\ &< G(gp, gp, gx_{n+1}) + G(gx_n, gx_n, gp). \end{aligned}$$

By taking $n \to \infty$, we have $G(gp, gp, fp) = 0$ and so $gp = fp$. We now show that $f$ and $g$ have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in X$. Assume that $gp \neq gq$. By applying (2.5), it follows that

$$\begin{aligned} G(gp, gp, gq) &= G(fp, fp, fq) \\ &\leq \phi(G(gp, gp, gq)) \\ &< G(gp, gp, gq), \end{aligned}$$

which leads to a contradiction. Therefore $gp = gq$. This implies that $f$ and $g$ have a unique point of coincidence. By Proposition 1.7, we can conclude that $f$ and $g$ have a unique common fixed point. □

By setting $g$ to be the identity function on $X$, we immediately have the following corollary.

**Corollary 2.5.** *([22, Theorem 3.1]) Let $(X, G)$ be a complete $G$-metric space. Suppose that the mapping $f : X \to X$ satisfies*

$$G(fx, fy, fz) \leq \phi(G(x, y, z)),$$

*for all $x, y, z \in X$. Then $f$ has a unique fixed point.*

**Theorem 2.6.** *Let $(X, G)$ be a $G$-metric space. Suppose that the mappings $f, g : X \to X$ satisfy*

$$G(fx, fy, fz)$$

$$\leq \phi(\max\{G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy), G(fx, gy, gz)\}) \tag{2.8}$$

*for all $x, y, z \in X$. If the range of $g$ contains the range of $f$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

*Proof.* Let $x_0$ be an arbitrary point in $X$. Since the range of $g$ contains the range of $f$, there is $x_1 \in X$ such that $gx_1 = fx_0$. By continuing the process as before, we can construct a sequence $\{gx_n\}$ such that $gx_{n+1} = fx_n$ for all $n \in \mathbb{N}$. If there is $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$, then $f$ and $g$ have a point of coincidence. Thus we can suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\ &\leq \phi(\max\{G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, fx_{n-1}, fx_{n-1}), \\ &\quad G(gx_n, fx_n, fx_n), G(fx_{n-1}, gx_n, gx_n)\}) \\ &\leq \phi(\max\{G(gx_{n-1}, gx_n, gx_n), G(gx_{n-1}, gx_n, gx_n), \\ &\quad G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_n, gx_n, gx_n)\}) \\ &\leq \phi(\max\{G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1})\}). \end{aligned}$$

If $\max\{G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n+1}, gx_{n+1})\} = G(gx_n, gx_{n+1}, gx_{n+1})$, then

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \phi(G(gx_n, gx_{n+1}, gx_{n+1})) < G(gx_n, gx_{n+1}, gx_{n+1}),$$

which leads to a contradiction. This implies that

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \phi(G(gx_{n-1}, gx_n, gx_n)).$$

That is, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &= G(fx_{n-1}, fx_n, fx_n) \\ &\leq \phi(G(gx_{n-1}, gx_n, gx_n)) \\ &\leq \phi^2(G(gx_{n-2}, gx_{n-1}, gx_{n-1})) \\ &\vdots \\ &\leq \phi^n(G(gx_0, gx_1, gx_1)). \end{aligned}$$

We will show that $\{gx_n\}$ is $G$-Cauchy. Let $\varepsilon > 0$.
Since $\lim_{n\to\infty} \phi^n(G(gx_0, gx_1, gx_1)) = 0$ and $\phi(\varepsilon) < \varepsilon$, there exists $N \in \mathbb{N}$ such that

$$\phi^n(G(gx_0, gx_1, gx_1)) < \varepsilon - \phi(\varepsilon) \quad \text{for all } n \geq N.$$

This implies that

$$G(gx_n, gx_{n+1}, gx_{n+1}) < \varepsilon - \phi(\varepsilon) \quad \text{for all } n \geq N. \tag{2.9}$$

Let $m, n \in \mathbb{N}$ with $m > n$. We will prove that

$$G(gx_n, gx_m, gx_m) < \varepsilon \quad \text{for all } m \geq n \geq N \tag{2.10}$$

by induction on $m$. Since $\varepsilon - \phi(\varepsilon) < \varepsilon$ and by (2.9), we obtain that (2.10) holds for $m = n + 1$. Suppose that (2.10) holds for $m = k$. Therefore, for $m = k + 1$, we have

$$\begin{aligned} G(gx_n, gx_{k+1}, gx_{k+1}) &\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{k+1}, gx_{k+1}) \\ &< \varepsilon - \phi(\varepsilon) + G(fx_n, fx_k, fx_k) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(\max\{G(gx_n, gx_k, gx_k), G(gx_n, fx_n, fx_n), \\ &\quad G(gx_k, fx_k, fx_k), G(fx_n, gx_k, gx_k)\}) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(\max\{G(gx_n, gx_k, gx_k), G(gx_n, gx_{n+1}, gx_{n+1}), \\ &\quad G(gx_k, fx_k, fx_k), G(fx_n, gx_k, gx_k)\}) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon. \end{aligned}$$

Thus (2.10) holds for all $m \geq n \geq N$. It follows that $\{gx_n\}$ is $G$-Cauchy. By the completeness of $g(X)$, we obtain that $\{gx_n\}$ is $G$-convergent to some $q \in g(X)$. So there exists $p \in X$ such that $gp = q$. We will show that $gp = fp$. By (2.8), we have

$$\begin{aligned} &G(gp, gp, fp) \\ \leq\ &G(gp, gp, gx_n) + G(gx_n, gx_n, fp) \\ \leq\ &G(gp, gp, gx_n) + G(fx_{n-1}, fx_{n-1}, fp) \\ \leq\ &G(gp, gp, gx_n) + \phi(\max\{G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, fx_{n-1}, fx_{n-1}), \\ &G(gx_{n-1}, fx_{n-1}, fx_{n-1}), G(fx_{n-1}, gx_{n-1}, gp)\}) \\ \leq\ &G(gp, gp, gx_n) + \phi(\max\{G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), \\ &G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n-1}, gp)\}). \end{aligned}$$

Case 1.

$$\max\{G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n-1}, gp)\}$$
$$= G(gx_{n-1}, gx_{n-1}, gp),$$

we obtain that

$$G(gp, gp, fp) < G(gp, gp, gx_n) + G(gx_{n-1}, gx_{n-1}, gp).$$

By taking $n \to \infty$, we have $G(gp, fp, fp) = 0$ and so $gp = fp$.
Case 2.

$$\max\{G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n-1}, gp)\}$$
$$= G(gx_{n-1}, gx_n, gx_n),$$

we obtain that

$$G(gp, gp, fp) < G(gp, gp, gx_n) + G(gx_{n-1}, gx_n, gx_n).$$

By taking $n \to \infty$, we have $G(gp, fp, fp) = 0$ and so $gp = fp$.
Case 3.

$$\max\{G(gx_{n-1}, gx_{n-1}, gp), G(gx_{n-1}, gx_n, gx_n), G(gx_n, gx_{n-1}, gp)\}$$
$$= G(gx_n, gx_{n-1}, gp),$$

we obtain that

$$G(gp, gp, fp) < G(gp, gp, gx_n) + G(gx_n, gx_{n-1}, gp).$$

By taking $n \to \infty$, we have $G(gp, fp, fp) = 0$ and so $gp = fp$. We now show that $f$ and $g$ have a unique point of coincidence. Suppose that $fq = gq$ for some $q \in X$. Assume that $gp \neq gq$. By applying (2.8), it follows that

$$\begin{aligned} G(gp, gp, gq) &= G(fp, fp, fq) \\ &\leq \phi(\max\{G(gp, gp, gq), G(gp, fp, fp), G(gp, fp, fp), G(fp, gp, gq)\}) \\ &\leq \phi(G(gp, gp, gq)) < G(gp, gp, gq), \end{aligned}$$

which leads to a contradiction. Therefore $gp = gq$. This implies that $f$ and $g$ have a unique point of coincidence. By Proposition 1.7, we can conclude that $f$ and $g$ have a unique common fixed point. □

Consequently, if we suppose that $g$ is the identity function on $X$, then we obtain the following corollary.

**Corollary 2.7.** *([22, Theorem 3.2]) Let $(X, G)$ be a complete G-metric space. Suppose that the mapping $f : X \to X$ satisfies*

$$G(fx, fy, fz)$$
$$\leq \phi(\max\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(fx, y, z)\})$$

*for all $x, y, z \in X$. Then $f$ has a unique fixed point.*

**Acknowledgement.** This research is supported by the Commission on Higher Education and the Thailand Research Fund under grant MRG5380208. The author would like to express her deep thanks to Professor Sompong Dhompongsa for his advices during the preparation of the paper and the referee for comments which lead to the improvement of this paper.

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