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CONVERGENCE THEOREMS BASED ON THE SHRINKING PROJECTION METHOD FOR HEMI-RELATIVELY NONEXPANSIVE MAPPINGS, VARIATIONAL INEQUALITIES AND EQUILIBRIUM PROBLEMS

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ABSTRACT. In this paper, we introduce a new hybrid projection algorithm based on the shrinking projection methods for two hemi-relatively nonexpansive mappings. Using the new algorithm, we prove some strong convergence theorems for finding a common element in the fixed points set of two hemi-relatively nonexpansive mappings, the solutions set of a variational inequality and the solutions set of an equilibrium problem in a uniformly convex and uniformly smooth Banach space. Furthermore, we apply our results to finding zeros of maximal monotone operators. Our results extend and improve the recent ones announced by Li [J. Math. Anal. Appl. 295 (2004) 115–126], Fan [J. Math. Anal. Appl. 337 (2008) 1041–1047], Liu [J. Glob. Optim. 46 (2010) 319–329], Kamraksa and Wangkeeree [J. Appl. Math. Comput. DOI: 10.1007/s12190-010-0427-2] and many others.

1. INTRODUCTION

Let E be a Banach space and E^* be the dual space of E. Let C be a nonempty closed convex subset of E. Let J be the normalized duality mapping from E into 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

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where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

The duality mapping J has the following properties:

(1) If E is smooth, then J is single-valued;

(2) If E is strictly convex, then J is one-to-one;

(3) If E is reflexive, then J is surjective;

(4) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E;

(5) If E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E and J is singe-valued and also one-to-one(see [6, 12, 23, 30]).

Let $A : C \to E^*$ be an operator. We consider the following variational inequality: Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

A point $x_0 \in C$ is called a solution of the variational inequality (1.1) if $\langle Ax_0, y - x_0\rangle \geq 0$. The solutions set of the variational inequality (1.1) is denoted by $VI(A, C)$. The variational inequality (1.1) has been intensively considered due to its various applications in operations research, economic equilibrium and engineering design. When A has some monotonicity, many iterative methods for solving the variational inequality (1.1) have been developed (see [1, 2, 3, 4, 7, 8]).

Let C is a nonempty closed and convex subset of a Hilbert space H and $P_C : H \to C$ be the metric projection of H onto C, then P_C is nonexpansive, that is,

$$\|P_C x - P_C y\| \leq \|x - y\|, \quad \forall x, y \in H.$$

This fact actually characterizes Hilbert spaces, however, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Recently, applying the generalized projection operator in uniformly convex and uniformly smooth Banach spaces, Li [16] established the following Mann type iterative scheme for solving some variational inequalities without assuming the monotonicity of A in compact subset of Banach spaces.

Theorem 1.1. [16] *Let E be a uniformly convex and uniformly smooth Banach space and C be a compact convex subset of E. Let $A : C \to E^*$ be a continuous mapping on C such that*

$$\langle Ax - \xi, J^{-1}(Jx - (Ax - \xi))\rangle \geq 0, \quad \forall x \in C,$$

where $\xi \in E^*$. For any $x_0 \in C$, define the Mann type iteration scheme as follows:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \Pi_C(Jx_n - (Ax_n - \xi)), \quad \forall n \geq 1,$$

where the sequence $\{\alpha_n\}$ satisfies the following conditions:

(a) $0 \leq \alpha_n \leq 1$ for all $n \in N$;

(*b*) $\Sigma_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$.
Then the variational inequality $\langle Ax-\xi, y-x\rangle \geq 0$ *for all* $y\in C$ (*when* $\xi=0$, *the variational inequality* (1.1) *has a solution* $x^*\in C$ *and there exists a subsequence* $\{n_i\}\subset\{n\}$ *such that*

$$x_{n_i}\to x^* \quad (i\to\infty).$$

In addition, Fan [11] established some existence results of solutions and the convergence of the Mann type iterative scheme for the variational inequality (1.1) in a noncompact subset of a Banach space and proved the following theorem.

Theorem 1.2. [11] *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a compact convex subset of* E. *Suppose that there exists a positive number* β *such that*

$$\langle Ax, J^{-1}(Jx-\beta Ax)\rangle \geq 0, \quad \forall x\in C,$$

and $J-\beta A : C\to E^*$ *is compact. if*

$$\langle Ax, y\rangle \leq 0, \quad \forall x\in C,\, y\in VI(A,C),$$

then the variational inequality (1.1) *has a solution* $x^*\in C$ *and the sequence* $\{x_n\}$ *defined by the following iteration scheme:*

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n\Pi_C(Jx_n-\beta Ax_n), \quad \forall n\geq 1,$$

where the sequence $\{\alpha_n\}$ *satisfies that* $0<a\leq\alpha_n\leq b<1$ *for all* $n\geq 1$ $(a,b\in(0,1]$ *with* $a<b)$, *converges strongly a point to* $x^*\in C$.

Motivated by Li [16] and Fan [11], Liu [17] introduced the iterative sequence for approximating a common element of the fixed points set of a relatively weak nonexpansive mapping defined by Kohasaka and Takahashi [15] and the solutions set of the variational inequality in a noncompact subset of Banach spaces without assuming the compactness of the operator $J-\beta A$. More precisely, Liu [17] proved the following theorems:

Theorem 1.3. [17] *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty, closed convex subset of* E. *Suppose that there exists a positive number* β *such that*

$$\langle Ax, J^{-1}(Jx-\beta Ax)\rangle \geq 0, \quad \forall x\in C, \tag{1.2}$$

and

$$\langle Ax, y\rangle \leq 0, \quad \forall x\in C,\, y\in VI(A,C), \tag{1.3}$$

then $VI(A,C)$ *is closed and convex.*

Theorem 1.4. [17] *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed convex subset of* E. *Assume that* A *is a continuous operator of* C *into* E^* *satisfying the conditions* (1.2) *and* (1.3) *and* S :

$C \to C$ is a relatively weak nonexpansive mapping with $F := F(S) \cap VI(A,C) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by the following iterative scheme:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)J\Pi_C(Jz_n - \beta Az_n)), \\ C_0 = \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_0 = C, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle Jx_0 - Jx_n, x_n - z\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} Jx_0, \quad \forall n \geq 1, \end{cases}$$

where the sequences $\{\alpha_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

$$0 \leq \delta_n < 1, \quad \limsup_{n\to\infty} \delta < 1, \quad 0 < \alpha_n < 1, \quad \liminf_{n\to\infty} \alpha_n(1-\alpha) > 0.$$

Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{F(S)\cap VI(A,C)}Jx_0$.

Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for f is as follows: Find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of the problem (1.4) is denoted by $EP(f)$.

Equilibrium problems, which were introduced in [5] in 1994, have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, physics, image reconstruction, ecology, transportation, network, elasticity and optimization. Numerous problems in physics, optimization and economics reduce to finding a solution of the problem (1.4). Some methods have been proposed to solve the equilibrium problem in a Hilbert space. See [5, 10, 20].

Very recently, Kamraksa and Wangkeeree [14] motivated and inspired by Li [16], Fan [11] and Liu [17] introduce a hybrid projection algorithm based on the shrinking projection method for two relatively weak nonexpansive mappings, a variational inequality and an equilibrium problem in Banach spaces as follows:

Theorem 1.5. [14] *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(B_1) - (B_4)$ in section 2. Assume that A is a continuous operator of C into E^* satisfying the conditions (1.2) and (1.3) and $S, T : C \to C$ are two relatively and weakly nonexpansive mappings with $F := F(S) \cap F(T) \cap VI(A,C) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the following* *iterative scheme:*

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ y_n = J^*(\delta_n Jx_n + (1-\delta_n)J\Pi_C(Jz_n - \beta A z_n)), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx, \quad \forall n \leq 0, \end{cases}$$

where the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *in* $[0, 1]$ *satisfy the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n = 1$;
(b) $0 \leq \delta_n < 1$ *and* $\limsup_{n\to\infty} \delta_n < 1$;
(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;
(d) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ *and* $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx$.

Motivated by the results mentioned above, we introduce a new hybrid projection algorithm based on the shrinking projection method for two hemi-relatively nonexpansive mappings. Using the new algorithm, we prove some strong convergence theorem which approximate a common element in the fixed points set of two hemi-relatively nonexpansive mappings, the solutions set of a variational inequality and the solutions set of the equilibrium problem in a uniformly convex and uniformly smooth Banach space. Our results extend and improve the recent ones announced by Li [16], Fan [11], Liu [17], Kamraksa and Wangkeeree [14] and many others.

2. Preliminaries

A Banach space E is said to be strictly convex if $\frac{x+y}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{\to\infty} \|\frac{x_n+y_n}{2}\| = 1$.

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U_E$.

It is well known that, if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E and, if E is uniformly smooth if and only if E^* is uniformly convex.

A Banach space E is said to have the Kadec-Klee property if, for a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, $x_n \to x$.

It is known that, if E is uniformly convex, then E has the Kadec-Klee property (see [30, 9, 31] for more details).

Let C be a closed convex subset of E and T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

A mapping T from C into itself is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The mapping T is said to be relatively nonexpansive [18, 19, 13] if

$$\widehat{F}(T) = F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\ p \in F(T).$$

The asymptotic behavior of a relatively nonexpansive mapping was studied in [18, 19, 13]. A point $p \in C$ is called a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of strong asymptotic fixed points of T is denoted by $\widetilde{F}(T)$.

A mapping T from C into itself is said to be relatively and weakly nonexpansive if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\ p \in F(T).$$

The mapping T is said to be hemi-relatively nonexpansive if

$$F(T) \neq \emptyset, \quad \phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C,\ p \in F(T).$$

It is obvious that a relatively nonexpansive mapping is a relatively and weakly nonexpansive mapping and, further, a relatively and weakly nonexpansive mapping is a hemi-relatively nonexpansive mapping, but the converses are not true as in the following example:

Example 2.1. [28] Let E be any smooth Banach space and $x_0 \neq 0$ be any element of E. We define a mapping $T : E \to E$ as follows: For all $n \geq 1$,

$$T(x) = \begin{cases} (\frac{1}{2} + \frac{1}{2^n})x_0, & \text{if } x = (\frac{1}{2} + \frac{1}{2^n})x_0, \\ -x, & \text{if } x \neq (\frac{1}{2} + \frac{1}{2^n})x_0. \end{cases}$$

Then T is a hemi-relatively nonexpansive mapping, but it is not relatively nonexpansive mapping.

Next, we give some important examples which are hemi-relatively nonexpansive.

Example 2.2. [21] Let E be a strictly convex reflexive smooth Banach space. Let A be a maximal monotone operator of E into E^* and J_r be the resolvent for A with $r > 0$. Then $J_r = (J + rA)^{-1}J$ is a hemi-relatively nonexpansive mapping from E onto $D(A)$ with $F(J_r) = A^{-1}0$.

Remark 2.3. There are other examples of hemi-relatively nonexpansive mappings and the generalized projections (or projections) and others (see [21]).

In [12, 4], Alber introduced the functional $V : E^* \times E \to \mathbb{R}$ defined by

$$V(\phi, x) = \|\phi\|^2 - 2\langle \phi, x\rangle + \|x\|^2,$$

where $\phi \in E^*$ and $x \in E$. It is easy to see that

$$V(\phi, x) \geq (\|\phi\| - \|x\|)^2$$

and so the functional $V : E^* \times E \to \mathbb{R}^+$ is nonnegative.

In order to prove our results in the next section, we present several definitions and lemmas.

Definition 2.4. [13] If E be a uniformly convex and uniformly smooth Banach space, then the generalized projection $\Pi_C : E^* \to C$ is a mapping that assigns an arbitrary point $\phi \in E^*$ to the minimum point of the functional $V(\phi, x)$, i.e., a solution to the minimization problem

$$V(\phi, \Pi_C(\phi)) = \inf_{y \in C} V(\phi, y).$$

Li [16] proved that the generalized projection operator $\Pi_C : E^* \to C$ is continuous if E is a reflexive, strictly convex and smooth Banach space.

Consider the function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = V(Jy, x), \quad \forall x, y \in E.$$

The following properties of the operator Π_C and V are useful for our paper (see, for example, [1, 16]):

(A1) $V : E^* \times E \to \mathbb{R}$ is continuous;
(A2) $V(\phi, x) = 0$ if and only if $\phi = Jx$;
(A3) $V(J\Pi_C(\phi), x) \leq V(\phi, x)$ for all $\phi \in E^*$ and $x \in E$;
(A4) The operator Π_C is J fixed at each point $x \in E^*$ and $x \in E$;
(A5) If E is smooth, then, for any given $\phi \in E^*$ and $x \in C$, $x \in \Pi_C(\phi)$ if and only if

$$\langle \phi - Jx, x - y\rangle \geq 0, \quad \forall y \in C;$$

(A6) The operator $\Pi_C : E^* \to c$ is single valued if and only if E is strictly convex;
(A7) If E is smooth, then, for any given point $\phi \in E^*$ and $x \in \Pi_C(\phi)$, the following inequality holds:

$$V(Jx, y) \leq V(\phi, y) - V(\phi, x), \quad \forall y \in C;$$

(A8) $v(\phi, X)$ is convex with respect to ϕ when x is fixed and with respect to x when ϕ is fixed;
(A9) If E is reflexive, then, for any point $\phi \in E^*$, $\Pi_C(\phi)$ is a nonempty closed convex and bounded subset of C.

Using some properties of the generalized projection operator Π_C, Alber [1] proved the following theorem:

Lemma 2.5. [1] *Let E be a strictly convex reflexive smooth Banach space. Let A be an arbitrary operator from a Banach space E to E^* and β be an arbitrary* *fixed positive number. Then $x \in C \subset E$ is a solution of the variational inequality (1.1) if and only if x is a solution of the following operator equation in E:*

$$x = \Pi_C(Jx - \beta Ax).$$

Lemma 2.6. [13] *Let E be a uniformly convex smooth Banach space and $\{y_n\}$, $\{z_n\}$ be two sequences in E such that either $\{y_n\}$ or $\{z_n\}$ is bounded. If we have $\lim_{n\to\infty} \phi(y_n, z_n) = 0$, then $\lim_{n\to\infty} \|y_n - z_n\| = 0$.*

Lemma 2.7. [7] *Let E be a uniformly convex and uniformly smooth Banach space. We have*

$$\|\phi + \Phi\|^2 \le \|\phi\|^2 + 2\langle \Phi, J(\phi + \Phi)\rangle, \quad \forall \phi, \Phi \in E^*.$$

From Lemma 1.9 in Qin et al. [22], the following lemma can be obtained immediately:

Lemma 2.8. *Let E be a uniformly convex Banach space, $s > 0$ be a positive number and $B_s(0)$ be a closed ball of E. Then there exists a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\Sigma_{i=1}^N(\alpha_i x_i)\|^2 \le \Sigma_{i=1}^N(\alpha_i\|x_i\|^2) - \alpha_i\alpha_j g(\|x_i - x_j\|) \tag{2.2}$$

for all $x_1, x_2, \cdots, x_N \in B_s(0) = \{x \in E : \|x\| \le s\}$, $i \ne j$ for all $i, j \in \{1, 2, \cdots, N\}$ and $\alpha_1, \alpha_2, \cdots, \alpha_N \in [0, 1]$ such that $\Sigma_{i=1}^N \alpha_i = 1$.

For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(B1) $f(x, x) = 0$ for all $x \in C$;
(B2) f is monotone, that is, $f(x, y) + f(y, x) \le 0$ for all $x, y \in C$;
(B3) For all $x,\ y,\ z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \le f(x, y);$$

(B4) For all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x, y) = \langle Ax, y - x\rangle, \quad \forall x, y \in C.$$

Then f satisfies (B1)-(B4).

Lemma 2.9. [5] *Let C be a closed and convex subset of a smooth, strictly convex and reflexive Banach spaces E, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions $(B1)$-$(B4)$ and let $r > 0$, $x \in E$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.10. [32] *Let C be a closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach spaces E, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions $(B1)$-$(B4)$. For all $r > 0$ and $x \in E$, define the mapping*

$$T_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C\}.$$

Then the following hold:

(C1) T_r *is single-valued;*

(C2) T_r *is a firmly nonexpansive-type mapping, that is, for all* $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \le \langle T_r x - T_r y, Jx - Jy\rangle;$$

(C3) $F(T_r) = \hat{F}(T_r) = EP(f)$;

(C4) $EP(f)$ *is closed and convex.*

Lemma 2.11. [32] *Let* C *be a closed convex subset of a smooth, strictly convex and reflexive Banach space* E*, let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* $(B_1) - (B_4)$ *and let* $r > 0$*. Then, for* $x \in E$ *and* $q \in F(T_r)$,

$$\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x)$$

Lemma 2.12. [17] *If* E *is a reflexive, strictly convex and smooth Banach space, then* $\Pi_C = J^{-1}$.

Lemma 2.13. [28] *Let* E *be a strictly convex and smooth real Banach space,* C *be a closed convex subset of* E *and* T *be a hemi-relatively nonexpansive mapping from* C *into itself. Then* $F(T)$ *is closed and convex.*

Recall that an operator T in Banach space is said to be closed if $x_n \to x$ and $Tx_n \to y$ implies $Tx = y$.

3. Main results

Now, we give our mail results in this paper.

Theorem 3.1. *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed convex subset of* E*. Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying the conditions* (B_1)*-*(B_4)*. Assume that* A *is a continuous operator of* C *into* E^* *satisfying the conditions* (1.2) *and* (1.3) *and* $S, T : C \to C$ *are two closed hemi-relatively nonexpansive mappings with* $F := F(S) \cap F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$*. Let* $\{x_n\}$ *be a sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n), \\ y_n = J^{-1}(\lambda_n Jx_n + (1-\lambda_n)J\Pi_C(Jz_n - \beta Az_n)), \\ u_n \in C \ \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \le (1-\lambda_n)\alpha_n \phi(z, x_0) \\ \qquad\qquad + [1 - (1-\lambda_n)\alpha_n]\phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*,* $\{\delta_n\}$ *and* $\{\lambda_n\}$ *are the sequences in* $[0, 1]$ *with the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

(b) $0 \le \lambda_n < 1$ *and* $\limsup_{n\to\infty} \lambda_n < 1$;

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;

(d) $\lim_{n\to\infty}\alpha_n = 0$, $\liminf_{n\to\infty}\beta_n\gamma_n > 0$ *and* $\liminf_{n\to\infty}\beta_n\delta_n > 0$.
Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$, *where* Π_F *is the generalized projection from* C *onto* F.

Proof. We divide the proof into five steps.
Step (1): $\Pi_F Jx_0$ and $\Pi_{C_{n+1}}Jx_0$ are well defined.

From Lemma 2.13, we know that $F(T)$ and $F(S)$ are closed and convex and so $F(T)\cap F(S)$ is closed and convex. From Theorem 1.3, it follows that $VI(A,C)$ is closed and convex. From Lemma 2.10(C4), we also know that $EP(f)$ is closed and convex. Hence F is a nonempty closed and convex subset of C. Therefore, $\Pi_F Jx_0$ is well defined.

Next, we show that C_n is closed and convex for all $n \geq 0$. From the definitions of C_n, it is obvious that C_n is closed for all $n \geq 0$.

Next, we prove that C_n is convex for all $n \geq 0$. Since

$$\phi(z,u_n) \leq (1-\lambda_n)\alpha_n\phi(z,x_0) + [1-(1-\lambda_n)\alpha_n]\phi(z,x_n)$$

is equivalent to the following:

$$2\langle z, \theta_n Jx_0 + (1-\theta_n)Jx_n - Ju_n\rangle \leq (1-\theta_n)\|x_0\|^2 + (1-\theta_n)\|x_n\|^2,$$

where $\theta_n = (1-\lambda_n)\alpha_n$. It is easy to see that C_n is convex for all $n \geq 0$. Thus, for all $n \geq 0$, C_n is closed and convex and so $\Pi_{C_{n+1}}Jx_0$ is well defined.
Step (2): $F \subset C_n$ for all $n \geq 0$.

Observe that $F \subset C_0 = C$ is obvious. Suppose that $F \subset C_k$ for some $k \in \mathbb{N}$. Let $w \in F \subset C_k$. Then, from the definition of ϕ and V, the property $(A3)$ of V, Lemma 2.7, the conditions (1.2) and (1.3), it follows that

$$\begin{aligned}
\phi(w,\Pi_C(Jz_n-\beta Az_n)) &= V(J\Pi_C(Jz_n-\beta Az_n),w)\\
&\leq V(Jz_n-\beta Az_n,w)\\
&= \|Jz_n-\beta Az_n\|^2 - 2\langle Jz_n-\beta Az_n, w\rangle + \|w\|^2\\
&\leq \|Jz_n\|^2 - 2\beta\langle Az_n, J^{-1}(Jz_n-\beta Az_n)\rangle \\
&\quad - 2\langle Jz_n-\beta Az_n, w\rangle + \|w\|^2\\
&\leq \|Jz_n\|^2 - 2\langle Jz_n, w\rangle + \|w\|^2\\
&= \phi(w,z_n), \quad \forall n\geq 0.
\end{aligned} \tag{3.1}$$

From Lemma 2.10, we see that T_{r_n} is a hemi-relatively nonexpansive mapping. Therefore, by the properties (A3) and (A8) of the operator V and (3.1), we obtain

$$\begin{aligned}
\phi(w,u_k) &= \phi(w,T_{r_k}y_k)\\
&\leq \phi(w,y_k)\\
&= V(Jy_k,w)\\
&\leq \lambda_k V(Jx_k,w) + (1-\lambda_k)V(J\Pi_C(Jz_k-\beta Az_k),w)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_k\phi(w,x_k) + (1-\lambda_k)\phi(w,\Pi_C(Jz_k-\beta Az_k))\\
&= \lambda_k\phi(w,x_k) + (1-\lambda_k)\phi(w,z_k))\\
&= \lambda_k\phi(w,x_k) + (1-\lambda_k)V(Jz_k,w))\\
&= \lambda_k\phi(w,x_k) + (1-\lambda_k)V(\alpha_k Jx_0+\beta_k Jx_k+\gamma_k JTx_k+\delta_k JSx_k,w)\\
&= \lambda_k\phi(w,x_k) + (1-\lambda_k)\phi(w,J^{-1}(\alpha_k Jx_0+\beta_k Jx_k+\gamma_k JTx_k+\delta_k JSx_k))\\
&= \lambda_k\phi(w,x_k) + (1-\lambda_k)[\|w\|^2 - 2\alpha_k\langle w,Jx_0\rangle - 2\beta_k\langle w,Jx_k\rangle - 2\gamma_k\langle w,JTx_k\rangle\\
&\quad - 2\delta_k\langle w,JSx_k\rangle + \|\alpha_k Jx_0+\beta_k Jx_k+\gamma_k JTx_k+\delta_k JSx_k\|^2]\\
&\le \lambda_k\phi(w,x_k) + (1-\lambda_k)[\|w\|^2 - 2\alpha_k\langle w,Jx_0\rangle - 2\beta_k\langle w,Jx_k\rangle - 2\gamma_k\langle w,JTx_k\rangle\\
&\quad - 2\delta_k\langle w,JSx_k\rangle + \|\alpha_k Jx_0+\beta_k Jx_k+\gamma_k\|JTx_k\|^2+\delta_k\|JSx_k\|^2]\\
&= \lambda_k\phi(w,x_k) + (1-\lambda_k)[\alpha_k\phi(w,x_0)+\beta_k\phi(w,x_k) \qquad (3.2)\\
&\quad + \gamma_k\phi(w,Tx_k)+\delta_k\phi(w,Sx_k)]\\
&\le \lambda_k\phi(w,x_k) + (1-\lambda_k)[\alpha_k\phi(w,x_0)+\beta_k\phi(w,x_k)\\
&\quad + \gamma_k\phi(w,x_k)+\delta_k\phi(w,x_k)]\\
&= (1-\lambda_k)\alpha_k\phi(w,x_0) + \lambda_k\phi(w,x_n) + (1-\lambda_k)(1-\alpha_k)\phi(w,x_k)\\
&= (1-\lambda_k)\alpha_k\phi(w,x_0) + [1-(1-\lambda_k)\alpha_k]\phi(w,x_k)
\end{aligned}
$$

which shows that $w \in C_{k+1}$. This implies that $F \subset C_n$ for all $n \geq 0$.

Step (3): $\{x_n\}$ is a Cauchy sequence.

Since $x_n = \Pi_{C_n} Jx_0$ and $F \subset C_n$, we have $V(Jx_0, x_n) \leq V(Jx_0, w)$ for all $w \in F$. Therefore, $\{V(Jx_0, x_n)\}$ is bounded and, moreover, from the definition of V, it follows that $\{x_n\}$ is bounded. Since $x_{n+1} = \Pi_{C_{n+1}} Jx_0 \in C_{n+1}$ and $x_n = \Pi_{C_n} Jx_0$, we have

$$V(Jx_0, x_n) \leq V(Jx_0, x_{n+1}), \quad \forall n \geq 0.$$

Hence it follows that $\{V(Jx_0, x_n)\}$ is nondecreasing and so $\lim_{n\to\infty} V(Jx_0, x_n)$ exists. By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m} Jx_0 \in C_n$ for any positive integer $m \geq n$. From the property (A3), we have

$$V(Jx_n, x_m) \leq V(Jx_0, x_m) - V(Jx_0, x_n)$$

for all $n \geq 0$ and any positive integer $m \geq n$. This implies that

$$V(Jx_n, x_m) \to 0 \quad (n, m \to \infty).$$

The definition of ϕ implies that

$$\phi(x_m, x_n) \to 0 \quad (n, m \to \infty).$$

Applying Lemma 2.6, we obtain

$$\|x_m - x_n\| \to 0 \quad (n, m \to \infty).$$

Hence $\{x_n\}$ is a Cauchy sequence. In view of the completeness of a Banach space E and the closeness of C, it follows that

$$\lim_{n\to\infty} x_n = p$$

for some $p \in C$.

Step (4): $p \in F$.

First, we show that $p \in F(S) \cap F(T)$. In fact, from (3.3), we obtain that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0 \tag{3.3}$$

and, since $\{x_n\}$ is a Cauchy sequence in E, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Note that $x_{n+1} = \Pi_{C_{n+1}} Jx_0 \in C_{n+1}$ and so

$$\phi(x_{n+1}, u_n) \leq (1-\lambda_n)\alpha_n \phi(x_{n+1}, x_0) + [1-(1-\lambda_n)\alpha_n]\phi(x_{n+1,x_n}).$$

By $\lim_{n\to\infty} \alpha_n = 0$ and (3.3), it follows that

$$\begin{aligned}\lim_{n\to\infty} \phi(x_{n+1}, u_n) &\leq \lim_{n\to\infty} \phi(x_{n+1}, x_n)\\ &= 0\end{aligned}$$

and so

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

Using Lemma 2.6, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.4}$$

Combining 2.12 and (3.4), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0 \tag{3.5}$$

and hence it follows that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} x_n = p. \tag{3.6}$$

On the other hand, since J is uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.7}$$

Since $\{x_n\}$ is bounded, $\{Jx_n\}$, $\{JTx_n\}$ and $\{JSx_n\}$ are also bounded. Since E is a uniformly smooth Banach space, one knows that E^* is a uniformly convex Banach space. Let $r = \sup_{n\geq 0}\{\|Jx_n\|, \|JTx_n\|, \|JSx_n\|\}$. Therefore, from Lemma 2.8, it follows that there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ satisfying $g(0) = 0$ and the inequality (2.2). It follows from the property (A3) of the operator V, (3.1) and the definition of S and T that

$$\begin{aligned}\phi(w, z_n) &= V(Jz_n, w)\\ &\le V(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n, w)\\ &= \phi(w, J^{-1}(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n))\\ &= \|w\|^2 - 2\alpha_n\langle w, Jx_0\rangle - 2\beta_n\langle w, Jx_n\rangle - 2\gamma_n\langle w, JTx_n\rangle - 2\delta_n\langle w, JSx_n\rangle\\ &\quad + \|\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n\|^2\\ &\le \|w\|^2 - 2\alpha_n\langle w, Jx_0\rangle - 2\beta_n\langle w, Jx_n\rangle - 2\gamma_n\langle w, JTx_n\rangle - 2\delta_n\langle w, JSx_n\rangle\\ &\quad + \alpha_n\|Jx_0\|^2 + \beta_n\|Jx_n\|^2 + \gamma_n\|JTx_n\|^2 + \delta_n\|JSx_n\|^2 \qquad (3.8)\\ &\quad - \beta_n\gamma_n g(\|JTx_n - Jx_n\|)\\ &= \alpha_n\phi(w, x_0) + \beta_n\phi(w, x_n) + \gamma_n\phi(w, Tx_n) + \delta_n\phi(w, Sx_n)\\ &\quad - \beta_n\gamma_n g(\|JTx_n - Jx_n\|)\\ &\le \alpha_n\phi(w, x_0) + \beta_n\phi(w, x_n) + \gamma_n\phi(w, x_n) + \delta_n\phi(w, x_n)\\ &\quad - \beta_n\gamma_n g(\|JTx_n - Jx_n\|)\\ &= \alpha_n\phi(w, x_0) + (1-\alpha_n)\phi(w, x_n) - \beta_n\gamma_n g(\|JTx_n - Jx_n\|).\end{aligned}$$

From the property (A8) of the operator V, (3.1) and (3.8), we obtain

$$\begin{aligned}\phi(w, u_n) = \phi(w, T_{r_n}y_n) \le \phi(w, y_n) &= V(Jy_n, w)\\ &\le \lambda_n V(Jx_n, w) + (1-\lambda_n)V(J\Pi_C(Jz_n - \beta Az_n), w)\\ &= \lambda_n\phi(w, x_n) + (1-\lambda_n)\phi(w, \Pi_C(Jz_n - \beta Az_n))\\ &= \lambda_n\phi(w, x_n) + (1-\lambda_n)\phi(w, z_n))\\ &\le \lambda_n\phi(w, x_n) + (1-\lambda_n)[\alpha_n\phi(w, x_0) + (1-\alpha_n)\phi(w, x_n)\\ &\quad - \beta_n\gamma_n g(\|JTx_n - Jx_n\|)]\\ &= \alpha_n(1-\lambda_n)\phi(w, x_0) + [1 - \alpha_n(1-\lambda_n)]\phi(w, x_n)\\ &\quad - (1-\lambda_n)\beta_n\gamma_n g(\|JTx_n - Jx_n\|).\end{aligned}$$

Therefore, we have

$$\begin{aligned}(1-\lambda_n)\beta_n\gamma_n g(\|JTx_n - Jx_n\|) \le \theta_n\phi(w, x_0) + (1-\theta_n)\phi(w, x_n)\\ - \phi(w, u_n),\end{aligned} \tag{3.9}$$

where $\theta_n = \alpha_n(1-\lambda_n)$.

On the other hand, we have

$$\begin{aligned}\phi(w, x_n) - \phi(w, u_n) &= 2\langle Ju_n - Jx_n, w\rangle + \|x_n\|^2 - \|u_n\|^2\\ &\le 2\langle Ju_n - Jx_n, p\rangle + (\|x_n\| - \|u_n\|)(\|x_n\| + \|u_n\|)\\ &\le 2\|Ju_n - Jx_n\|\|w\| + \|x_n - u_n\|(\|x_n\| + \|u_n\|)\end{aligned}$$

It follows from (3.4) and (3.7) that

$$\lim_{n\to\infty}(\phi(w, x_n) - \phi(w, u_n)) = 0. \tag{3.10}$$

By the assumptions $\limsup_{n\to\infty} \lambda_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, (3.8) and (3.9), we have

$$\lim_{n\to\infty} g(\|JTx_n - Jx_n\|) = 0.$$

It follows from the property of g that

$$\lim_{n\to\infty} \|JTx_n - Jx_n\| = 0. \tag{3.11}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|J^{-1}JTx_n - J^{-1}Jx_n\| = 0. \tag{3.12}$$

Similarly, we can apply the condition $\liminf_{n\to\infty} \beta_n\delta_n > 0$ to get

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.13}$$

Since $\lim_{n\to\infty} x_n = p$ and the mappings T, S are closed, we know that p is a fixed point of T and S, that is, $p = Tp$ and $p = Sp$.

Secondly, we show that $p \in EP(f)$. In fact, from (3.2), we know that

$$\phi(w, y_n) \le (1-\lambda_n)\alpha_n\phi(w, x_0) + [1-(1-\lambda_n)\alpha_n]\phi(w, x_n).$$

In view of $u_n = T_{r_n}y_n$ and Lemma 2.11, one has

$$\begin{aligned}
&\phi(u_n, y_n)\\
&= \phi(T_{r_n}y_n, y_n) \le \phi(w, y_n) - \phi(w, T_{r_n}y_n)\\
&\le (1-\lambda_n)\alpha_n\phi(w, x_0) + [1-(1-\lambda_n)\alpha_n]\phi(w, x_n) - \phi(w, T_{r_n}y_n)\\
&= (1-\lambda_n)\alpha_n\phi(w, x_0) + [1-(1-\lambda_n)\alpha_n]\phi(w, x_n) - \phi(w, u_n).
\end{aligned}$$

In view of $\lim_{n\to\infty} \alpha_n = 0$ and (3.10), we obtain

$$\lim_{n\to\infty} \phi(u_n, y_n) = 0.$$

Applying Lemma 2.6, we obtain

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.14}$$

Since J is a uniformly norm-to-norm continuous on bounded sets, one has

$$\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0.$$

From the assumption that $r_n \ge a$, one has

$$\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0.$$

Observing that $u_n = T_{r_n}y_n$, one obtains

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy\rangle \ge 0, \quad \forall y \in C.$$

From (B2), one get

$$\begin{aligned}
\|y - u_n\|\frac{\|Ju_n - Jy_n\|}{r_n} &\ge \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \ge -f(u_n, y)\\
&\ge f(y, u_n), \quad \forall y \in C.
\end{aligned}$$

Taking $n \to \infty$ in the above inequality, it follows from (B4) and (3.6) that

$$f(y,p) \le 0, \quad \forall y \in C.$$

For all $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Note that $y,\ p \in C$, one obtains $y_t \in C$, which yields that $f(y_t, p) \le 0$. It follows from $B1$ that

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1-t)f(y_t, p) \le tf(y_t, y),$$

that is

$$f(y_t, y) \ge 0.$$

Let $t \downarrow 0$. From (B3), we obtain $f(p, y) \ge 0$ for all $y \in C$, which imply that $p \in EP(f)$.

Finally, we show that $p \in VI(A, C)$. In fact, by (3.5) and (3.14), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jy_n - Jx_n\| = 0.$$

Since $\|Jy_n - Jx_n\| = (1-\lambda_n)\|J\Pi_C(Jz_n - \beta Az_n) - Jx_n\|$ and $\limsup_{n\to\infty} \lambda_n < 1$, we obtain

$$\lim_{n\to\infty} \|J\Pi_C(Jz_n - \beta Az_n) - Jx_n\| = 0. \tag{3.15}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\begin{aligned}\lim_{n\to\infty} \|\Pi_C(Jz_n - \beta Az_n) - x_n\| &= \lim_{n\to\infty} \|J^{-1}J\Pi_C(Jz_n - \beta Az_n) - J^{-1}Jx_n\| \\ &= 0.\end{aligned}$$

On the other hand, from Lemma 2.11, we compute that

$$\begin{aligned}\phi(x_n, Tx_n) &\le \phi(w, x_n) - \phi(w, Tx_n) \\ &= 2\langle Jx_n - JTx_n, w\rangle + \|x_n\|^2 - \|Tx_n\|^2 \\ &\le 2\langle Jx_n - JTx_n, w\rangle + (\|x_n\| - \|Tx_n\|)(\|x_n\| + \|Tx_n\|) \\ &\le 2\|Jx_n - JTx_n\|\|w\| + (\|x_n - Tx_n\|)(\|x_n\| + \|Tx_n\|).\end{aligned}$$

By (3.11) and (3.12), take $n \to \infty$ in the above inequality, we have

$$\lim_{n\to\infty} \phi(x_n, Tx_n) = 0.$$

Similarly, we can also obtain

$$\lim_{n\to\infty} \phi(x_n, Sx_n) = 0. \tag{3.16}$$

From the properties (A2) and (A3) of the operator V, we derive that

$$\begin{aligned}\phi(x_n, z_n) &= V(Jz_n, x_n)\\ &\le V(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n, x_n)\\ &= \|x_n\|^2 - 2\alpha_n\langle x_n, Jx_0\rangle - 2\beta_n\langle x_n, Jx_n\rangle\\ &\quad - 2\gamma_n\langle x_n, JTx_n\rangle - 2\delta_n\langle x_n, JSx_n\rangle\\ &\quad + \|\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n\|^2\\ &\le \|x_n\|^2 - 2\alpha_n\langle x_n, Jx_0\rangle - 2\beta_n\langle x_n, Jx_n\rangle\\ &\quad - 2\gamma_n\langle x_n, JTx_n\rangle - 2\delta_n\langle x_n, JSx_n\rangle\\ &\quad + \alpha_n\|Jx_0\|^2 + \beta_n\|Jx_n\|^2 + \gamma_n\|JTx_n\|^2 + \delta_n\|JSx_n\|^2\\ &= \alpha_n\phi(x_n, x_0) + \beta_n\phi(x_n, x_n) + \gamma_n\phi(x_n, Tx_n) + \delta_n\phi(x_n, Sx_n).\end{aligned}$$

By the continuity of the function ϕ, $\lim_{n\to\infty}\alpha_n = 0$, (3.12), (3.13) and the closeness property of the mappings S and T, we have

$$\lim_{n\to\infty}\phi(x_n, z_n) = 0.$$

From Lemma 2.6, we have

$$\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

In view of (3.15) and (3.16), we get

$$\begin{aligned}\|\Pi_C(Jz_n - \beta Az_n) - z_n\| &\le \|\Pi_C(Jz_n - \beta Az_n) - x_n\| + \|x_n - z_n\|\\ &\to 0 \quad (n\to\infty).\end{aligned}$$

Since $\lim_{n\to\infty} x_n = p$ and (3.16), it follows that $\lim_{n\to\infty} z_n = p$. By the continuity of the operator J, A and Π_C, we obtain

$$\lim_{n\to\infty}\|\Pi_C(Jz_n - \beta Az_n) - \Pi_C(Jp - \beta Ap)\| = 0.$$

Note that

$$\begin{aligned}\|\Pi_C(Jz_n - \beta Az_n) - p)\| &\le \|\Pi_C(Jz_n - \beta Az_n) - z_n\| + \|z_n - p\|\\ &\to 0 \quad (n\to\infty).\end{aligned}$$

Hence it follows from the uniqueness of the limit that $p = \Pi_C(Jp - \beta Ap)$. From Lemma 2.5, we have $p \in VI(A, C)$ and so $p \in F$.

Step (5): $p = \Pi_F Jx_0$.

Since $p \in F$, from the property (A3) of the operator Π_C, we have

$$V(J\Pi_F Jx_0, p) + V(Jx_0, \Pi_F Jx_0) \le V(Jx_0, p). \tag{3.17}$$

On the other hand, since $x_{n+1} = \Pi_{C_{n+1}} Jx_0$ and $F \subset C_{n+1}$ for all $n \ge 0$, it follows from the property (A7) of the operator Π_C that

$$V(Jx_{x+1}, \Pi_F Jx_0) + V(Jx_0, x_{n+1}) \le V(Jx_0, \Pi_F Jx_0). \tag{3.18}$$

Furthermore, by the continuity of the operator V, we get

$$\lim_{n\to\infty} V(Jx_0, x_{n+1}) = V(Jx_0, p). \tag{3.19}$$

Combining (3.17), (3.18) with (3.19), we obtain

$$V(Jx_0, p) = V(Jx_0, \Pi_F Jx_0).$$

Therefore, it follows from the uniqueness of $\Pi_F Jx_0$ that $p = \Pi_F Jx_0$. This completes the proof. □

Remark 3.2. Theorem 3.1 improves Theorem 3.1 of Liu [17], Theorem 3.1 of Kamraksa and Wangkeeree [14] in the following senses:

(1) The iteration algorithm (3.1) of Theorem 3.1 is more general than the one given in Liu [17], Kamraksa and Wangkeeree [14] and, further, the algorithm (3.1) of Theorem 3.1 in Liu [17] is related to two problems, that is, the fixed point and variational inequality problems, but our algorithm in Theorem 3.1 is related to 3 problems, that is, the fixed point, variational inequality and equilibrium problems.

(2) If The class of hemi-relatively nonexpansive mappings is more general than the class of relatively weak nonexpansive mappings used in Kamraksa and Wangkeeree [14].

Remark 3.3. As in Remark 3.1 of Liu [17], Theorem 3.1 also improve Theorem 3.3 in Li [16] and Theorem 3.1 in Fan [11].

If we only consider one hemi-relatively nonexpansive mapping, then the following result is obtained directly by Theorem 3.1:

Corollary 3.4. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (B_1)-(B_4). Assume that A is a continuous operator of C into E^* satisfying the conditions (1.2) and (1.3) and $T : C \to C$ is closed hemi-relatively nonexpansive mapping with $F := F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JTx_n), \\ y_n = J^{-1}(\lambda_n Jx_n + (1-\lambda_n)J\Pi_C(Jz_n - \beta Az_n)), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq (1-\lambda_n)\alpha_n \phi(z, x_0) \\ \qquad\qquad\qquad\qquad + [1 - (1-\lambda_n)\alpha_n]\phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases} \tag{3.20}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are the sequences in $[0, 1]$ with the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$;
(b) $0 \leq \lambda_n < 1$ *and* $\limsup_{n\to\infty} \lambda_n < 1$;
(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;
(d) $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

When $\alpha_n \equiv 0$ in (3.20), The following result can be directly obtained by Corollary 3.4:

Corollary 3.5. *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed convex subset of* E*. Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying the conditions* (B_1)*-*(B_4)*. Assume that* A *is a continuous operator of* C *into* E^* *satisfying the conditions* (1.2) *and* (1.3) *and* $T : C \to C$ *is closed hemi-relatively nonexpansive mapping with* $F : F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\beta_n Jx_n + \gamma_n JTx_n), \\ y_n = J^{-1}(\lambda_n Jx_n + (1-\lambda_n)J\Pi_C(Jz_n - \beta A z_n)), \\ u_n \in C \ \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\beta_n\}$*,* $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *are the sequences in* $[0, 1]$ *with the following restrictions:*

(a) $\beta_n + \gamma_n = 1$;
(b) $0 \leq \lambda_n < 1$ *and* $\limsup_{n\to\infty} \lambda_n < 1$;
(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;
(d) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

If we consider two relatively weak nonexpansive mappings, then the following result can be also obtained by Theorem 3.1:

Corollary 3.6. *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed convex subset of* E*. Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying the conditions* (B_1)*-*(B_4)*. Assume that* A *is a continuous operator of* C *into* E^* *satisfying the conditions* (1.2) *and* (1.3) *and* $S, T : C \to C$ *are two relatively and weakly nonexpansive mappings with* $F := F(S) \cap F(T) \cap VI(A, C) \cap EP(f) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n), \\ y_n = J^{-1}(\lambda_n Jx_n + (1-\lambda_n)J\Pi_C(Jz_n - \beta A z_n)), \\ u_n \in C \ \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq (1-\lambda_n)\alpha_n\phi(z, x_0) + [1-(1-\lambda_n)\alpha_n]\phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ *and* $\{\lambda_n\}$ *are the sequences in* $[0,1]$ *with the following restrictions:*

(a) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

(b) $0 \leq \lambda_n < 1$ *and* $\limsup_{n\to\infty} \lambda_n < 1$;

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;

(d) $\lim_{n\to\infty} \alpha_n = 0$, $\liminf_{n\to\infty} \beta_n\gamma_n > 0$ *and* $\liminf_{n\to\infty} \beta_n\delta_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

When $\alpha_n \equiv 0$ in the Theorem 3.1, we obtain the following modified Mann type hybrid projection algorithm:

Corollary 3.7. *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed convex subset of* E*. Let* f *be a bifunction from* $C\times C$ *to* $\mathbb{R}$ *satisfying the conditions* (B_1)*-*(B_4)*. Assume that* A *is a continuous operator of* C *into* E^* *satisfying the conditions* (1.2) *and* (1.3) *and* $S, T : C \to C$ *are two closed hemi-relatively nonexpansive mappings with* $F := F(S)\cap F(T)\cap VI(A,C)\cap EP(f) \neq \emptyset$*. Let* $\{x_n\}$ *be the sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\beta_n Jx_n + \gamma_n JTx_n + \delta_n JSx_n), \\ y_n = J^{-1}(\lambda_n Jx_n + (1-\lambda_n)J\Pi_C(Jz_n - \beta Az_n)), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ *and* $\{\lambda_n\}$ *are the sequences in* $[0,1]$ *with the following restrictions:*

(a) $\beta_n + \gamma_n + \delta_n = 1$;

(b) $0 \leq \lambda_n < 1$ *and* $\limsup_{n\to\infty} \lambda_n < 1$;

(c) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;

(d) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$ *and* $\liminf_{n\to\infty} \beta_n\delta_n > 0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

4. Applications to maximal monotone operators

In this section, we apply the our above results to prove some strong convergence theorem concerning maximal monotone operators in a Banach space E.

Let $\bar{\mathcal{B}}$ be a multi-valued operator from E to E^* with domain $D(\bar{\mathcal{B}}) = \{z \in E : \bar{\mathcal{B}}z \neq \emptyset\}$ and range $R(\bar{\mathcal{B}}) = \{z \in E : z \in D(\bar{\mathcal{B}})\}$. An operator $\bar{\mathcal{B}}$ is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$$

for all $x_1, x_2 \in D(\bar{\mathcal{B}})$ and $y_1 \in \bar{\mathcal{B}}x_1$, $y_2 \in \bar{\mathcal{B}}x_2$. A monotone operator $\bar{\mathcal{B}}$ is said to be maximal if it's graph $G(\bar{\mathcal{B}}) = \{(x, y) : y \in \bar{\mathcal{B}}x\}$ is not properly contained in the graph of any other monotone operator.

It is well known that, if $\bar{\mathcal{B}}$ is a maximal monotone operator, then $\bar{\mathcal{B}}^{-1}0$ is closed and convex.

The following result is also well known.

Lemma 4.1. [26] *Let E be a reflexive, strictly convex and smooth Banach space and $\bar{\mathcal{B}}$ be a monotone operator from E to E^*. Then $\bar{\mathcal{B}}$ is maximal if and only if $R(J + r\bar{\mathcal{B}}) = E^*$ for all $r > 0$.*

Let E be a reflexive, strictly convex and smooth Banach space and $\bar{\mathcal{B}}$ be a maximal monotone operator from E to E^*. Using Lemma 4.1 and the strict convexity of E, it follows that, for all $r > 0$ and $x \in E$, there exists a unique $x_r \in D(\bar{\mathcal{B}})$ such that

$$Jx \in Jx_r + r\bar{\mathcal{B}}x_r.$$

If $J_r x = x_r$, then we can define a single valued mapping $J_r : E \to D(\bar{\mathcal{B}})$ by $J_r = (J + r\bar{\mathcal{B}})^{-1}J$ and such a J_r is called the resolvent of $\bar{\mathcal{B}}$. We know that $\bar{\mathcal{B}}^{-1}0 = F(J_r)$ for all $r > 0$ (see [30, 31] for more details).

The following lemma plays an important role in our next theorem:

Lemma 4.2. [29] *Let E be a uniformly convex and uniformly smooth Banach space, $\bar{\mathcal{B}}$ be a maximal monotone operator from E to E^* and J_r be a resolvent of $\bar{\mathcal{B}}$. Then J_r is closed hemi-relatively nonexpansive mapping.*

We consider the problem of strong convergence concerning maximal monotone operators in a Banach space. Such a problem has been also studied in [15, 13, 24, 25, 27]. Using Theorem 3.1, we obtain the following result:

Theorem 4.3. *Let E be a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying the conditions (B_1)-(B_4). Assume that A is a continuous operator of C into E^* satisfying the conditions (1.2) and (1.3), $\bar{\mathcal{B}}_1, \bar{\mathcal{B}}_2 : C \to C$ are two maximal monotone operator from E to E^*, $J_r^{\bar{\mathcal{B}}_1}$ and $J_r^{\bar{\mathcal{B}}_2}$ are two resolvents of $\bar{\mathcal{B}}_1$ and $\bar{\mathcal{B}}_2$ with $F := \bar{\mathcal{B}}_1^{-1}0 \cap \bar{\mathcal{B}}_2^{-1}0 \cap VI(A, C) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\alpha_n Jx_0 + \beta_n Jx_n + \gamma_n JJ_r^{\bar{\mathcal{B}}_1} x_n + \delta_n JJ_r^{\bar{\mathcal{B}}_2} x_n), \\ y_n = J^{-1}(\lambda_n Jx_n + (1-\lambda_n) J\Pi_C(Jz_n - \beta Az_n)), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq (1-\lambda_n)\alpha_n \phi(z, x_0) \\ \qquad\qquad +[1 - (1-\lambda_n)\alpha_n]\phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$*,* $\{\delta_n\}$ *and* $\{\lambda_n\}$ *are the sequences in* $[0,1]$ *with the following restrictions:*

(a) $\alpha_n+\beta_n+\gamma_n+\delta_n=1$;
(b) $0\le\lambda_n<1$ *and* $\limsup_{n\to\infty}\lambda_n<1$;
(c) $\{r_n\}\subset[a,\infty)$ *for some* $a>0$;
(d) $\lim_{n\to\infty}\alpha_n=0$*,* $\liminf_{n\to\infty}\beta_n\gamma_n>0$ *and* $\liminf_{n\to\infty}\beta_n\delta_n>0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

Proof. From Lemma 4.2, we know that $J_r^{\bar{\mathcal{B}}_1}$ and $J_r^{\bar{\mathcal{B}}_1}$ are two closed hemi-relatively nonexpansive mappings. Furthermore, applying Theorem 3.1, we can obtain that the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$. □

Considering $\lambda_n\equiv 0$ in (4.1), we can directly obtain the following corollary by applying Theorem 4.3:

Corollary 4.4. *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed convex subset of* E*. Let* f *be a bifunction from* $C\times C$ *to* $\mathbb{R}$ *satisfying the conditions* (B_1)*-*(B_4)*. Assume that* A *is a continuous operator of* C *into* E^* *satisfying the conditions* (1.2) *and* (1.3)*,* $\bar{\mathcal{B}}_1,\bar{\mathcal{B}}_2:C\to C$ *are two maximal monotone operator from* E *to* E^**,* $J_r^{\bar{\mathcal{B}}_1}$ *and* $J_r^{\bar{\mathcal{B}}_2}$ *are two resolvents of* $\bar{\mathcal{B}}_1$ *and* $\bar{\mathcal{B}}_2$ *with* $F:=\bar{\mathcal{B}}_1^{-1}0\cap\bar{\mathcal{B}}_2^{-1}0\cap VI(A,C)\cap EP(f)\neq\emptyset$*. Let* $\{x_n\}$ *be the sequence generated by the following iterative scheme:*

$$\begin{cases} x_0\in C \text{ chosen arbitrarily},\\ z_n=\Pi_C(\alpha_nJx_0+\beta_nJx_n+\gamma_nJJ_r^{\bar{\mathcal{B}}_1}x_n+\delta_nJJ_r^{\bar{\mathcal{B}}_2}x_n),\\ y_n=\Pi_C(Jz_n-\beta Az_n)),\\ u_n\in C \quad \text{such that } f(u_n,y)+\frac{1}{r_n}\langle y-u_n,Ju_n-Jy_n\rangle\ge 0,\quad \forall y\in C,\\ C_{n+1}=\{z\in C_n:\phi(z,u_n)\le\alpha_n\phi(z,x_0)+(1-\alpha_n)\phi(z,x_n)\},\\ C_0=C,\\ x_{n+1}=\Pi_{C_{n+1}}Jx_0,\quad \forall n\ge 1,\end{cases}$$

where $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *and* $\{\delta_n\}$ *are the sequences in* $[0,1]$ *with the following restrictions:*

(a) $\alpha_n+\beta_n+\gamma_n+\delta_n=1$;
(b) $\{r_n\}\subset[a,\infty)$ *for some* $a>0$;
(c) $\lim_{n\to\infty}\alpha_n=0$*,* $\liminf_{n\to\infty}\beta_n\gamma_n>0$ *and* $\liminf_{n\to\infty}\beta_n\delta_n>0$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $\Pi_F Jx_0$*, where* Π_F *is the generalized projection from* C *onto* F*.*

When $\{\alpha_n\}\equiv 0$ in 4.2, we can obtain the new modified Mann iteration for the variational inequality (1.1), the equilibrium problem (1.4) and zeros of maximal monotone operators as follows:

Corollary 4.5. *Let* E *be a uniformly convex and uniformly smooth Banach space and* C *be a nonempty closed convex subset of* E*. Let* f *be a bifunction from* $C\times C$ *to* $\mathbb{R}$ *satisfying the conditions* (B_1)*-*(B_4)*. Assume that* A *is a continuous operator* *of C into E^* satisfying the conditions (1.2) and (1.3), $\bar{\mathcal{B}}_1, \bar{\mathcal{B}}_2 : C \to C$ are two maximal monotone operator from E to E^*, $J_r^{\bar{\mathcal{B}}_1}$ and $J_r^{\bar{\mathcal{B}}_2}$ are two resolvents of $\bar{\mathcal{B}}_1$ and $\bar{\mathcal{B}}_2$ with $F := \bar{\mathcal{B}}_1^{-1}0 \cap \bar{\mathcal{B}}_2^{-1}0 \cap VI(A, C) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by the following iterative scheme:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \Pi_C(\beta_n Jx_n + \gamma_n J J_r^{\bar{\mathcal{B}}_1} x_n + \delta_n J J_r^{\bar{\mathcal{B}}_2} x_n), \\ y_n = \Pi_C(Jz_n - \beta A z_n)), \\ u_n \in C \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ C_0 = C, \\ x_{n+1} = \Pi_{C_{n+1}} Jx_0, \quad \forall n \geq 1, \end{cases}$$

where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are the sequences in $[0, 1]$ with the following restrictions:

(a) $\beta_n + \gamma_n + \delta_n = 1$;
(b) $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$;
(c) $\liminf_{n\to\infty} \beta_n\gamma_n > 0$ *and* $\liminf_{n\to\infty} \beta_n\delta_n > 0$.

Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_F Jx_0$, where Π_F is the generalized projection from C onto F.

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