



THE GELFAND–PHILLIPS PROPERTY IN CLOSED SUBSPACES OF SOME OPERATOR SPACES

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Communicated by D. E. Alspach

ABSTRACT. By introducing the concept of limited completely continuous operators between two arbitrary Banach spaces X and Y , we give some properties of this concept related to some well known classes of operators and specially, related to the Gelfand–Phillips property of the space X or Y . Then some necessary and sufficient conditions for the Gelfand–Phillips property of closed subspace M of some operator spaces, with respect to limited complete continuity of some operators on M , so-called, evaluation operators, are verified.

1. INTRODUCTION AND PRELIMINARIES

A subset A of a Banach space X is called limited (resp. Dunford–Pettis (DP)), if every weak* null (resp. weak null) sequence (x_n^*) in X^* converges uniformly on A , that is,

$$\lim_{n \rightarrow \infty} \sup_{a \in A} |\langle a, x_n^* \rangle| = 0.$$

Also if $A \subseteq X^*$ and every weak null sequence (x_n) in X converges uniformly on A , we say that A is an L-set.

We know that every relatively compact subset of X is limited and clearly every limited set is DP and every DP subset of a dual Banach space is an L-set, but the converse of these assertions, in general, are false. If every limited subset of a Banach space X is relatively compact, then X has the Gelfand–Phillips (GP)

Date: Received: 13 January 2011; Accepted: 8 April 2011.

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2010 *Mathematics Subject Classification.* Primary 47L05; Secondary 47L20, 46B28.

Key words and phrases. Gelfand–Phillips property, Schur property, limited set, evaluation operator, operator ideal.

property and X has the relatively compact Dunford–Pettis property (DPrcP), if every DP set in X is relatively compact. It is clear that DPrcP imply the GP property, but if X is a Grothendieck space (i.e., weak and weak* convergence of sequences in X^* are coincide), then these properties are the same on X . For example, the classical Banach spaces c_0 and ℓ_1 have the GP property and every separable Banach space, every Schur space (i.e., weak and norm convergence of sequences in X are coincide), and spaces containing no copy of ℓ_1 , such as reflexive spaces, have the same property [2].

Also the Banach space X has the GP property if and only if every limited and weakly null sequence (x_n) in X is norm null [5]. The reader can be find some useful and additional properties of limited and DP sets and Banach spaces with the GP property in [2, 5, 8, 10, 17].

Here, by introducing the concept of limited completely continuous operators between Banach spaces, we obtain some characterizations of it and then the relation between the Gelfand–Phillips property of X and limited complete continuity of operators on X is treated. Finally, we shall obtain some necessary and/or sufficient conditions for the GP property of a closed subspace of some operator spaces, relative to the limited complete continuity of special operators, so-called, evaluation operators.

The notations and terminologies are standard. We use the symbols X , Y and Z for arbitrary Banach spaces. We denoted the closed unit ball of X by B_X , the dual of X by X^* and T^* refers to the adjoint of the operator T . Also we use $\langle x, x^* \rangle$ for the duality between $x \in X$ and $x^* \in X^*$. We refer the reader for undefined terminologies, to the classical references [6, 12].

2. LIMITED COMPLETELY CONTINUOUS OPERATORS

Let X and Y be arbitrary Banach spaces and $T : X \rightarrow Y$ be a bounded linear operator. We remember that:

- (a) T is (weakly) compact, if $T(B_X)$ is a relatively (weakly) compact set in Y .
- (b) T is limited, if $T(B_X)$ is limited in Y .
- (c) T is completely continuous, if it carries weakly null sequences in X to norm null sequences in Y .

Since the limited and weakly convergent sequences, play an important role in the study of Gelfand–Phillips property of Banach spaces, we define the following class of operators:

- (d) T is called limited completely continuous (abb. lcc), if T carries limited and weakly null sequences in X to norm null ones.

We denote the class of all weakly compact and compact operators from X to Y by $W(X, Y)$ and $K(X, Y)$, respectively. Also, we shall use the symbols $Li(X, Y)$, $Cc(X, Y)$ and $Lcc(X, Y)$, for the class of all limited, completely continuous and limited completely continuous operators from X to Y , respectively.

It is clear that the class $Lcc(X, Y)$ is a closed linear subspace of $L(X, Y)$, consisting of all bounded linear operators from X to Y ; which has the ideal property, that is, for each $T \in Lcc(X, Y)$ and each two bounded linear operators R and S , which can be composed with T , one has RTS is a limited completely continuous operator.

Now we will establish some additional properties of lcc operators.

Theorem 2.1. *A bounded operator T on X is lcc if and only if for each limited set $A \subseteq X$, the set $T(A)$ is relatively compact.*

Proof. suppose that $T : X \rightarrow Y$ is lcc and $A \subseteq X$ is limited. Since every limited set is weakly conditionally compact [6], every sequence (x_n) in A has a subsequence, denoted again by (x_n) , that is weakly Cauchy. On the other hands, the difference set $A - A$ is limited [2]. Thus the sequence $(x_n - x_m)$ is limited and weakly null, and then the sequence (Tx_n) is Cauchy and so is norm convergent in the Banach space Y . Thus $T(A)$ is relatively compact.

Conversely, if (x_n) is a limited weakly null sequence in X ; it is enough to show that every subsequence of (x_n) , has a subsequence (x_{n_k}) such that (Tx_{n_k}) is norm null.

For each such subsequence of (x_n) , which denoted again by (x_n) , by hypothesis, the sequence (Tx_n) is relatively compact and so has a convergent subsequence (Tx_{n_k}) . Since T carries weakly null sequences to weakly null ones, the sequence (Tx_{n_k}) is weakly null and so $\|Tx_{n_k}\| \rightarrow 0$ as $k \rightarrow \infty$. This shows that the sequence (Tx_n) is norm null and then T is lcc. \square

In the following theorem, we give a characterization of GP spaces, with respect to lcc operators.

Theorem 2.2. *For a Banach space X , the following are equivalent:*

- (a) X has the GP property,
- (b) For each Banach space Y , $Lcc(X, Y) = L(X, Y)$,
- (c) For each Banach space Y , $Lcc(Y, X) = L(Y, X)$.

Proof. (a) \Rightarrow (b). By [5], we know that the Banach space X has the GP property if and only if every limited weakly null sequence in X is norm convergent. So if X has the GP property and $(x_n) \subset X$ is limited and weakly null, then (x_n) is norm null. Now for each bounded operator T on X , $\|Tx_n\| \rightarrow 0$; that is $Lcc(X, Y) = L(X, Y)$.

(b) \Rightarrow (a). If $Y = X$, then (b) implies that the identity operator on X is lcc. In other wise, X has the GP property. The proof of equivalence of (a) and (c) is similar. \square

For the following corollary, we need a theorem from [2]:

Theorem 2.3. *A subset $A \subseteq X$ is limited if and only if for each bounded operator $S : X \rightarrow c_0$, the subset $S(A)$ of c_0 is relatively compact.*

Corollary 2.4. *For an operator $T : X \rightarrow Y$, the following are equivalent:*

- (a) T is limited,

- (b) For each Banach space Z and each lcc operator $S : Y \rightarrow Z$, the composition operator ST is compact,
(c) For each operator $S : Y \rightarrow c_0$, the operator ST is compact.

Proof. (a) \Rightarrow (b). If T is limited and $S : Y \rightarrow Z$ is lcc, then $T(B_X)$ is limited in Y and by Theorem 2.1, $S(T(B_X))$ is relatively compact. So the operator ST is compact.

(b) \Rightarrow (c). Since c_0 has the GP property, by Theorem 2.2, every operator $S : Y \rightarrow c_0$ is lcc. Now apply the statement (b).

(c) \Rightarrow (a). For every operator $S : Y \rightarrow c_0$, the operator ST is compact. That is, $S(T(B_X))$ is relatively compact and so by Theorem 2.3, $T(B_X)$ is limited in Y . \square

Corollary 2.5. *Every weakly compact operator is lcc.*

Proof. Let $T : X \rightarrow Y$ be a weakly compact operator between Banach spaces X and Y . Then by Davis-Figiel–Johnson–Pelczynski’s Theorem [3], T can be factored through a reflexive Banach space Z ; that is, there are operators $S : X \rightarrow Z$ and $R : Z \rightarrow Y$ such that $T = RS$. Since Z has the GP property, by Theorem 2.2, R and so T is lcc. \square

Corollary 2.6. *Let Y be a Banach space containing no copy of ℓ_1 . Then for every Banach space X , every limited operator $T : X \rightarrow Y$ is lcc.*

Proof. By [6], if Y contains no copy of ℓ_1 , every limited subset of Y is relatively weakly compact. So by hypothesis, $T(B_X)$ is relatively weakly compact and T is a weakly compact operator. \square

The following theorem proves that the validity of the statement (b) of Theorem 2.2 by ℓ_∞ instead of all Banach space Y , is a sufficient condition for the GP property of X .

Theorem 2.7. *A Banach space X has the GP property if and only if $Lcc(X, \ell_\infty) = L(X, \ell_\infty)$.*

Proof. If X does not have the GP property, then by [5], there exists a limited and weakly null sequence (x_n) in X such that $\|x_n\| = 1$, for all n . Choose a normalized sequence (x_n^*) in X^* such that $|\langle x_n, x_n^* \rangle| = 1$, for all n , and define the operator $T : X \rightarrow \ell_\infty$ by

$$Tx = (\langle x, x_n^* \rangle), \quad x \in X.$$

But T is not lcc, since the sequence (x_n) is limited and weakly null and $\|Tx_n\| \geq 1$, for all n . \square

It is clear that every completely continuous operator is lcc, but the converse, in general, is false. In the following, we give a characterization of this converse assertion, with respect to the DP* property of Banach spaces. Recall that a Banach space X has the DP* property, if every relatively weakly compact subset of X is limited, or equivalently, every weakly null sequence in X , is limited.

Theorem 2.8. *A Banach space X has the DP* property if and only if for each Banach space Y , $Cc(X, Y) = Lcc(X, Y)$.*

Proof. If X has the DP^* property and $T : X \rightarrow Y$ is lcc, then for every weakly null sequence (x_n) in X , it is limited and so by hypothesis on T , the sequence (Tx_n) is norm null. Hence T is completely continuous. Conversely, if every lcc operator on X is completely continuous, then X has the DP^* property. In fact, if (x_n) is a non limited and weakly null sequence in X , then by passing to a subsequence, there exist a weak* null sequence (x_n^*) in X^* such that

$$|\langle x_n, x_n^* \rangle| > \epsilon,$$

for all integer n and some positive ϵ , [5]. Now the bounded operator $T : X \rightarrow c_0$ defined by $Tx = (\langle x, x_n^* \rangle)$ is lcc, thanks to Theorem 2.2; while it is not completely continuous, since (x_n) is weakly null and $\|Tx_n\| > \epsilon$ for all n . This is a contradiction. \square

We conclude this section by proving that the operator ideal Lcc of all lcc operators between Banach spaces, by meaning of [4], is injective but it is not surjective. Recall that an operator ideal \mathcal{U} is injective, if for each Banach spaces X, Y and Z and each isometric embedding $J : Y \rightarrow Z$, the operator $T \in L(X, Y)$ belongs to \mathcal{U} if $JT \in \mathcal{U}$. Also \mathcal{U} is surjective, if for each Banach spaces X, Y and Z and each surjection $Q : Z \rightarrow X$, the operator $T \in L(X, Y)$ belongs to \mathcal{U} if $TQ \in \mathcal{U}$.

Theorem 2.9. *The operator ideal Lcc is injective but not surjective.*

Proof. Suppose that $T \in L(X, Y)$ and $J : Y \rightarrow Z$ is an isometric embedding, such that JT is lcc. If (x_n) is limited and weakly null in X , then $\|JT x_n\| \rightarrow 0$ as $n \rightarrow \infty$. By hypothesis on J , $\|Tx_n\| \rightarrow 0$ and so T is lcc too. Hence Lcc is injective. Now, for the proof of non surjectivity of Lcc , suppose that X is a Banach space without the GP property. Then the identity operator $i : X \rightarrow X$ is not lcc. On the other hands, if one define $Q_X : \ell_1(B_X) \rightarrow X$ via

$$Q_X(\phi) = \sum_{x \in B_X} \phi(x)x, \quad \phi \in \ell_1(B_X),$$

then by [11], Q_X is a surjective operator. Thus the Schur property and so GP property of $\ell_1(B_X)$ implies that the operator $Q_X = iQ_X$ is lcc, while the identity operator i is not. \square

3. EVALUATION OPERATORS AND GP PROPERTY

For each two Banach spaces X and Y , by meaning of [4] or [11], let $\mathcal{U}(X, Y)$ be the component of operator ideal \mathcal{U} of all operators from X to Y that belongs to \mathcal{U} . If M is a closed subspace of $\mathcal{U}(X, Y)$, for each arbitrary elements $x \in X$ and $y^* \in Y^*$, the evaluation operators $\phi_x : M \rightarrow Y$ and $\psi_{y^*} : M \rightarrow X^*$ on M are defined by

$$\phi_x(T) = Tx, \quad \psi_{y^*}(T) = T^*y^*, \quad T \in M.$$

Also, the point evaluation sets related to $x \in X$ and $y^* \in Y^*$ are the images of the closed unit ball B_M of M , under the evaluation operators ϕ_x and ψ_{y^*} and are denoted by $M_1(x)$ and $\widetilde{M}_1(y^*)$ respectively.

In the following, among other things, we give some necessary and sufficient conditions of GP property of some closed subspace M of operator ideals with respect

to limited complete continuity of all evaluation operators. At the first, we give some generalizations of [14, Theorem 2.5], [16, Theorem 4] and [18, Theorem 1]. Recall that a Banach space X has the RDP property [9], if every DP subset of X is relatively weakly compact.

Theorem 3.1. *Let X and Y be two Banach spaces and M be a closed subspace of operator ideal $\mathcal{U}(X, Y)$, such that M^* has the Schur property.*

- (a) *If X^* and Y are Grothendieck spaces, then all of the point evaluation sets $M_1(x)$ and $\widetilde{M}_1(y^*)$ are limited.*
- (b) *If X^* and Y have RDP property, then all evaluation sets are relatively weakly compact.*
- (c) *If X^* and Y have the Grothendieck and GP properties, then all evaluation sets are relatively compact.*
- (d) *If X and Y^* contain no copy of ℓ_1 , then all evaluation sets are relatively compact.*

Proof. If M^* has the Schur property, then for every weak null sequence (Λ_n) in M^* , it is norm null and so is uniformly convergent on the unit ball B_M of M . This shows that B_M is a DP set. Since the DP sets preserve under each bounded linear operator, we see that all evaluation sets are DP too.

For completeness of the proof of (a), it is enough to note that every DP subset of a Grothendieck space is limited. The investigation of the rest assertions (b)-(d) are similar. For the proof of (d), one need combine the above technique with a theorem from [7] that states, every DP set in the dual of Banach spaces containing no copy of ℓ_1 , is relatively compact. \square

The following theorem shows that the limited complete continuity of all evaluation operators is a necessary condition for the GP property of closed subspace $M \subseteq \mathcal{U}(X, Y)$:

Theorem 3.2. *For each two Banach spaces X and Y , if the closed subspace M of arbitrary operator ideal $\mathcal{U}(X, Y)$ has the GP property, then all of the evaluation operators ϕ_x and ψ_{y^*} are lcc.*

Proof. Since all $\phi_x : M \rightarrow Y$ and $\psi_{y^*} : M \rightarrow X^*$ are bounded linear operators, it is an easy consequence of Theorem 2.2. \square

By a similar method, we have the following necessary condition for the GP property of the dual of closed subspace $M \subseteq \mathcal{U}(X, Y)$.

Theorem 3.3. *Suppose that X^{**} and Y^* have the DP^* property such that the dual M^* of a closed subspace $M \subseteq \mathcal{U}(X, Y)$ has the GP property, Then all of the evaluation sets $M_1(x)$ and $\widetilde{M}_1(y^*)$ are DP sets.*

Proof. Since M^* has the GP property, by Theorem 2.2, the adjoint operators $\phi_x^* : Y^* \rightarrow M^*$ and $\psi_{y^*}^* : X^{**} \rightarrow M^*$ are lcc. So by Theorem 2.8, these operators are completely continuous.

For the proof $M_1(x) \subseteq Y$ is a DP set in Y , suppose that (y_n^*) is a weakly null

sequence in Y^* . Then $\|\phi_x^* y_n^*\| \rightarrow 0$ as $n \rightarrow \infty$, for all $x \in X$. Since

$$\begin{aligned} \|\phi_x^* y_n^*\| &= \sup\{|\langle \phi_x^* y_n^*, T \rangle| : T \in B_M\} \\ &= \sup\{|\langle y_n^*, Tx \rangle| : T \in B_M\}, \end{aligned}$$

the sequence (y_n^*) converges uniformly on $M_1(x)$. This shows that $M_1(x)$ is a DP set in Y , for all $x \in X$. A similar proof shows that all $\widetilde{M}_1(y^*)$ are DP in X^* . \square

Finally, we conclude this section by proving that the lcc of evaluation operators is a sufficient condition for the GP property of the subspace. For the proof of the following two theorems, we remember two theorems from [1, 13, 15]:

Theorem 3.4. ([1, 15]) *Let X and Y be Banach spaces and H be a subset of $L(X, Y)$ such that*

- (a) $H(B_X) := \{Tx : T \in H, x \in B_X\}$ *is relatively compact,*
- (b) $H^* y^* := \{T^* y^* : T \in H\}$ *is relatively compact, for all y^* in Y^* .*

Then H is relatively compact.

Recall that a subset $H \subseteq L(X, Y)$ is uniformly completely continuous, if for every weakly null sequence (x_n) in X ,

$$\lim_{n \rightarrow \infty} \sup_{T \in H} \|Tx_n\| = 0.$$

Theorem 3.5. ([13]) *If X contains no copy of ℓ_1 , then a subset $H \subseteq K(X, Y)$ is relatively compact if and only if H is uniformly completely continuous and for each $x \in X$, the set $\phi_x(H)$ is relatively compact in Y .*

Theorem 3.6. *Suppose that M is a closed linear subspace of $Li(X, Y)$ that the closed linear span of the set $M(X) := \{Tx : T \in M, x \in X\}$ is a GP subspace of Y . If all evaluation operators ψ_{y^*} are lcc, then M has the GP property.*

Proof. Let $H \subseteq M$ be limited subset of M . For the proof of relative compactness of H in M , By Theorem 3.4, it is enough to show that $H(B_X)$ and all $H^*(y^*)$ are relatively compact in Y and X^* respectively.

We know that a bounded operator $T : X \rightarrow Y$ is limited if and only if the adjoint operator T^* is weak*-norm sequentially continuous [2] and also, a subset A of X is limited if and only if every pointwise convergent sequence of bounded linear operators on X , converges uniformly on A [17].

If (y_n^*) is a weak* null sequence in Y^* , then the weak*-norm sequential continuity of the adjoint of each $T \in H$, implies that $\|\psi_{y_n^*}(T)\| = \|T^* y_n^*\| \rightarrow 0$ as $n \rightarrow \infty$.

That is the sequence $\psi_{y_n^*}$ of bounded linear operators on M , converges pointwise and so converges uniformly on the limited subset H of M . Hence

$$\begin{aligned} &\sup \{|\langle Tx, y_n^* \rangle| : T \in H, x \in B_X\} \\ &= \sup\{|\langle x, T^* y_n^* \rangle| : T \in H, x \in B_X\} \\ &= \sup\{\|T^* y_n^*\| : T \in H\}, \end{aligned}$$

tends to zero, as $n \rightarrow \infty$. Thus $H(B_X)$ is limited and the hypothesis on the GP property implies that $H(B_X)$ is relatively compact in Y . On the other hands, for each $y^* \in Y^*$, the evaluation operator ψ_{y^*} is lcc and by Theorem 2.1, it carries

the limited set H to relatively compact set $H^*(y^*) = \psi_{y^*}(H)$. Now Theorem 3.4 implies that H is relatively compact in M . \square

Theorem 3.7. *Let X and Y be two Banach spaces such that X contains no copy of ℓ_1 . If M is closed subspace of $K(X, Y)$ such that each evaluation operators ϕ_x is lcc on M , then M has the GP property.*

Proof. Suppose that $H \subseteq M$ is a limited set. If (x_n) is a weakly null sequence in X , then complete continuity of each operator $T \in H$, implies that the sequence (ϕ_{x_n}) is norm null at each point $T \in H$ and then by [17], it is uniformly convergent on the limited set H . This shows that H is uniformly completely continuous. On the other hands, each ϕ_x is lcc and so by Theorem 2.1, $\phi_x(H)$ is relatively compact, for all $x \in X$. Now an appeal to Theorem 3.5 shows that H is relatively compact in $K(X, Y)$ and so M has the GP property. \square

The following theorem extends [8, Theorem 2].

Theorem 3.8. *Let X and Y be two Banach spaces such that Y has the Schur property. If M is a closed subspace of $L(X, Y)$ such that each evaluation operators ψ_{y^*} is lcc on M , then M has the GP property.*

Proof. If M does not have the GP property, then by [5], there exists a limited and weakly null sequence (T_n) in M that is not norm null and by passing to a subsequence, we may assume that $\|T_n\| > \epsilon$, for all integer n and some $\epsilon > 0$. Choose a sequence (x_n) in B_X such that for all n , $\|T_n x_n\| > \epsilon$. On the other hands, for each $y^* \in Y^*$, the evaluation operator $\psi_{y^*} : M \rightarrow X^*$ is lcc, so $\|T_n^* y^*\| = \|\psi_{y^*} T_n\| \rightarrow 0$ and then

$$|\langle T_n x_n, y^* \rangle| \leq \|T_n^* y^*\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

This means that the sequence $(T_n x_n)$ is weakly null and so norm null, thanks to the Schur property of Y . This contradiction shows that M has the GP property. \square

Finally, by a similar technique, we give a sufficient condition for the GP property of closed subspaces of $L_{w^*}(X^*, Y)$, consisting of all bounded weak*-weak continuous operators from X^* to Y , and note that for each operator $T \in L_{w^*}(X^*, Y)$, the adjoint operator T^* tends Y^* into X .

Theorem 3.9. *Let X and Y be two Banach spaces such that X has the Schur property. If M is a closed subspace of $L_{w^*}(X^*, Y)$ such that each evaluation operators ϕ_{x^*} is lcc on M , then M has the GP property.*

Proof. If one can choose a limited and weakly null sequence (T_n) in M such that $\|T_n\| > \epsilon$ for some $\epsilon > 0$ and all integer n , then by hypothesis on evaluation operators, for each $x^* \in X^*$, $\|T_n x^*\| \rightarrow 0$, as $n \rightarrow \infty$. Since $\|T_n^*\| > \epsilon$, there exists a sequence (y_n^*) in B_{Y^*} such that $\|T_n^* y_n^*\| > \epsilon$, for all n . But the Schur property of X shows that the weakly null sequence $(T_n^* y_n^*)$ is norm null, which is a contradiction. \square

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