



STABILIZING ISOMORPHISMS FROM $\ell_p(\ell_2)$ INTO $L_p[0, 1]$

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ABSTRACT. Let $1 < p \neq 2 < \infty$, $\varepsilon > 0$ and let T be an isomorphism from $\ell_p(\ell_2)$ into $L_p[0, 1]$. Then there is a subspace $Y \subset \ell_p(\ell_2)$, $(1 + \varepsilon)$ -isomorphic to $\ell_p(\ell_2)$ such that $T|_Y$ is an $(1 + \varepsilon)$ -isomorphism and $T(Y)$ is K_p -complemented in $L_p[0, 1]$, with K_p depending only on p . Moreover, $K_p \leq (1 + \varepsilon)\gamma_p$ if $p > 2$ and $K_p \leq (1 + \varepsilon)\gamma_{p/(p-1)}$ if $1 < p < 2$, where γ_r is the L_r norm of a standard Gaussian variable.

1. INTRODUCTION

Let B be one of the 5 “classical” subspaces of $L_p = L_p[0, 1]$; by these we mean ℓ_p , ℓ_2 , $\ell_p \oplus_p \ell_2$, $\ell_p(\ell_2)$ and L_p itself. Here $\ell_p(\ell_2)$ is the space of (say, real) matrices $A = (a_{i,j})_{i,j=1}^\infty$ with norm $\|A\|_{\ell_p(\ell_2)} = (\sum_{j=1}^\infty (\sum_{i=1}^\infty a_{i,j}^2)^{p/2})^{1/p}$. It is well known that these five spaces isometrically embed in L_p , $1 \leq p < \infty$, and, for $1 < p < \infty$, have embeddings which are complemented. (For $p = 1$ this last statement holds only for L_1 and ℓ_1 .)

It was known for some time that if X is any subspace of L_p , $1 < p < \infty$, isomorphic to one of these B then there is a subspace of X isomorphic to B and complemented in L_p . For ℓ_p see [10], for ℓ_2 , [14]. The case $\ell_p \oplus_p \ell_2$ follows from these two results. The quite complicated case of L_p was proved in [8] (and for $p = 1$ perviously in [5]). The case of $\ell_p(\ell_2)$ can be proved using a variation of the method of [8] (and there is also a much simpler proof for $p > 2$) and was known to the second named author for a long time but not published (the simpler proof for $p > 2$ is included in [7]).

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Recently, there were three paper which address this property for some of the spaces B above and related questions again. This was done mostly because some strengthening of this property was needed for other purposes. Firstly, Haydon, Odell and Schlumprecht proved in [7, Theorem 6.8] that, for $p > 2$, any subspace of L_p isomorphic to $\ell_p(\ell_2)$ contains a subspace $(1 + \varepsilon)$ -isomorphic to $\ell_p(\ell_2)$ and complemented in L_p by means of a projection of norm at most $(1 + \varepsilon)\gamma_p$, γ_p being the L_p norm of a standard Gaussian variable. Their proof uses the fact that a similar theorem holds for the space ℓ_2 in L_p ; i.e., any subspace of L_p , $p > 2$, isomorphic to ℓ_2 contains a subspace $(1 + \varepsilon)$ -isomorphic to ℓ_2 and complemented in L_p by means of a projection of norm at most $(1 + \varepsilon)\gamma_p$. This later deep fact, hidden already in Aldous' [1], was recently given a simpler proof by Alspach [2]. The third paper is by Dosev, Johnson and the second named author [4, Theorem 3.4] from which the following follows:

For each $1 < p < \infty$ there is a constant K_p , depending only on p such that if $T : L_p \rightarrow L_p$ is an isomorphism (into) then there is a subspace X of L_p K_p -isomorphic to L_p such that $T|_X$ is a K_p -isomorphism and $T(X)$ is K_p -complemented in L_p . We remark that a similar theorem for $p = 1$ (with the constant K_1 arbitrarily close to 1) is due to Enflo and Starbird [5]; see also [15] for a somewhat simpler exposition.

The main purpose of the current paper is to prove a similar theorem for $\ell_p(\ell_2)$.

Theorem 1.1. *Let $1 < p \neq 2 < \infty$, $\varepsilon > 0$ and let $T : \ell_p(\ell_2) \xrightarrow{\text{onto}} X$ be an isomorphism where $X \subset L_p[0, 1]$. Then there is a subspace $Y \subset \ell_p(\ell_2)$ $(1 + \varepsilon)$ -isomorphic to $\ell_p(\ell_2)$ such that $T|_Y$ is an $(1 + \varepsilon)$ -isomorphism and $T(Y)$ is complemented in $L_p[0, 1]$ by means of a projection of norm at most $(1 + \varepsilon)\gamma_p$ if $p > 2$ and $(1 + \varepsilon)\gamma_{p/(p-1)}$ if $1 < p < 2$.*

For $p > 2$ this theorem follows easily from the result of [7] mentioned above so the main innovation here is the case $1 < p < 2$. However, since the addition needed to present a uniform proof for $1 < p < 2$ and $p > 2$ is minimal, we shall prove both cases. The proof is very much in the spirit of [4] but we wrote it in such a manner that one does not need to refer to that paper.

2. PRELIMINARIES

The proofs below will assume familiarity with basic techniques of Banach space theory. In particular techniques related to bases. We shall use freely notions like unconditional bases, block bases, small perturbations of bases, gliding hump arguments and similar notions. They can all be found in the first chapter of [11].

Recall that the Haar system is the following sequence of functions on $[0, 1]$: $h_{0,0}(t) \equiv 1$ and, for $n = 0, 1, \dots$ and $i = 1, 2, \dots, 2^n$,

$$h_{n,i}(t) = \begin{cases} 1 & \text{if } t \in ((2i - 2)2^{-(n+1)}, (2i - 1)2^{-(n+1)}) \\ -1 & \text{if } t \in ((2i - 1)2^{-(n+1)}, 2i2^{-(n+1)}) \\ 0 & \text{otherwise} \end{cases}$$

This system forms an unconditional basis for $L_p = L_p[0, 1]$ for each $1 < p < \infty$ (but not in L_1 in which it is, in its natural order, a non-unconditional Schauder

basis) see e.g. [12]. We denote by H_p its unconditional constant; i.e.,

$$\left\| \sum a_{n,i} h_{n,i} \right\|_p \leq H_p \left\| \sum \varepsilon_{n,i} a_{n,i} h_{n,i} \right\|_p$$

for any sequence of coefficients $\{a_{n,i}\}$ and any sequence of signs $\{\varepsilon_{n,i}\}$. (We shall use the real field although all the arguments easily carry over to the complex field with minimal changes, one of which should be in this definition). The best constant H_p is known and is of order $\max\{p, p/(p-1)\}$.

Recall Khinchine's inequality; For $1 \leq p < \infty$,

$$A_p \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \leq (\text{Ave}_{\pm} \left| \sum_{i=1}^n \pm a_i \right|^p)^{1/p} \leq B_p \left(\sum_{i=1}^n a_i^2 \right)^{1/2}$$

for all n and all coefficients $\{a_i\}_{i=1}^n$. The best constants A_p and B_p are known and in particular A_p is between $2^{-1/2}$ and 1 for $1 \leq p \leq 2$ and 1 for $p > 1$ and B_p is 1 for $1 \leq p \leq 2$ and of order $p^{1/2}$ for $p > 2$.

We shall make an intensive use of the square function with respect to the Haar system. For $f \in L_p$, $1 < p < \infty$, with expansion $f = \sum a_{n,i} h_{n,i}$ we denote its square function with respect to the Haar system by

$$S(f) = \left(\sum a_{n,i}^2 h_{n,i}^2 \right)^{1/2}.$$

The unconditionality of the Haar system and Khinchine's inequality easily imply that

$$H_p^{-1} A_p \|S(f)\|_p \leq \|f\|_p \leq H_p B_p \|S(f)\|_p, \quad 1 < p < \infty.$$

Equi-integrability of some sets of functions will play an important role in the sequel. Recall that a set F of Lebesgue integrable functions on $[0, 1]$ is said to be *equi-integrable* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for every subset A of $[0, 1]$ of measure at most δ , $\int_A |f| dt < \varepsilon$ for all $f \in F$. Equivalently, if for all $\varepsilon > 0$ there is a positive R such that $\int |f| \mathbf{1}_{|f| > R} dt < \varepsilon$ for all $f \in F$.

For $0 < r < \infty$ the set F is *r-equi-integrable* if $\{|f|^r; f \in F\}$ is equi-integrable.

Finally, by a *K-isomorphism* we mean a linear map T from one normed space X into another Y such that $A^{-1}\|x\| \leq \|Tx\| \leq B\|x\|$ for all $x \in X$ with $AB \leq K$. In particular, for small ε , a $(1 + \varepsilon)$ -isomorphism T does not necessarily almost preserve the norm of each x (as is sometimes assumed in other places) but of course some multiple of T does.

3. STABILIZING EMBEDDINGS OF ℓ_2 INTO L_p

In this section we consider the analogue of Theorem 1.1 for the space ℓ_2 instead of $\ell_p(\ell_2)$. As we indicated above this theorem is known although not simple, especially if one wants to achieve the best constants. (For a somewhat weaker Theorem, in terms of the constants achieved, see Theorem 3.1 in [14].) The purpose of this section is to survey its proof and point the reader to the relevant references.

Theorem 3.1. *Let $\varepsilon > 0$ and let $T : \ell_2 \rightarrow L_p[0, 1]$, $1 \leq p < \infty$, be an isomorphism. Then there is an infinite dimensional subspace $X \subseteq \ell_2$ such that $T|_X$ is an $(1 + \varepsilon)$ -isomorphism and, for $1 < p < \infty$, TX is $(1 + \varepsilon)\gamma_p$ -complemented in*

L_p . Here, for $p > 2$, γ_p is the L_p norm of a standard Gaussian variable and, for $1 < p < 2$, the $L_{\frac{p}{p-1}}$ norm of such a variable.

Note first that in order to prove the first part of the theorem, the existence of a $(1 + \varepsilon)$ -isomorphism on $X \subseteq \ell_2$, it is enough to prove that $T\ell_2$ contains a Y which is $(1 + \varepsilon)$ -isomorphic to ℓ_2 . Indeed if this is the case let $\{f_i\}_{i=1}^\infty \subseteq Y$ be a basis $(1 + \varepsilon)$ -equivalent to an orthonormal basis of ℓ_2 and put $e_i = T^{-1}f_i$, $i = 1, 2, \dots$. Since $\{e_i\}_{i=1}^\infty$ is a weakly null sequence in ℓ_2 for which $\{\|e_i\|\}_{i=1}^\infty$ is bounded and bounded away from zero, we may find a subsequence $\{e_{i_j}\}_{j=1}^\infty$ such that $\{e_{i_j}\}_{j=1}^\infty$ is as small perturbation as we wish of an orthogonal sequence and $\{\|e_{i_j}\|\}_{j=1}^\infty$ is as close as we want to a constant sequence. $X = \text{span} \{e_{i_j}\}_{j=1}^\infty$ is then the subspace we are after. That TX contains a subspace $(1 + \varepsilon)$ -isomorphic to ℓ_2 is by now well known and follows from the stability of L_p (see [9]).

We are left with the problem of complementation, and especially the norm of the best projection. For $p > 2$ the theorem is specifically stated and proved in [7]. There is a simpler proof in [2]. For $1 \leq p < 2$ this follows, as we shall indicate momentarily from [1]. This is not at all an easy paper to follow and it would be nice if somebody finds an easier proof maybe a-la-[2]. We shall only sketch how to get the result from [1] and then give a much simpler argument which however gives a somewhat weaker estimate.

Let $Y \subseteq L_p[0, 1]$, $1 < p < 2$, be isomorphic to ℓ_2 . We would like to find a subspace $Y_0 \subseteq Y$ $(1 + \varepsilon)$ -isomorphic to ℓ_2 and $(1 + \varepsilon)\gamma_{\frac{p}{p-1}}$ -complemented in $L_p[0, 1]$.

By [10] the unit ball of Y is p -equi-integrable and the L_1 and L_p norms are equivalent on Y . From here on we shall use the notations of [1]. By the combination of Proposition 3.9 and Theorem 3.10 of [1], there is a uniformly integrable sequence X_n in Y such that $i(X_n) \xrightarrow{wm} \sigma(q, \alpha)$ for some $1 < q \leq 2$. Recall that $i(X_n)$ is the random measure δ_{X_n} and that for random measures $\xi_n, \xi, \xi_n \xrightarrow{wm} \xi$ denotes $\langle f, \xi_n \rangle \xrightarrow{w} \langle f, \xi \rangle$ for all $f \in C(\mathbb{R})$ (and \xrightarrow{w} denotes weak convergence in L_1). Finally, for a function $\alpha \geq 0$, $\sigma(q, \alpha)$ is the random measure whose characteristic function is $e^{-\alpha^\alpha |t|^q}$; i.e., it is a mixture of symmetric q -stable random variables.

In our case, by [1, Proposition 3.11] the only possible value for q is $q = 2$. So we get a sequence X_n which tends in some sense to a mixture of Gaussian variables. Proposition 3.11 and its proof then say that some subsequence of X_n is $(1 + \varepsilon)$ equivalent to the unit vector basis of ℓ_2 . The proof really gives more: some subsequence of X_n is arbitrarily close, in L_p norm to a sequence of the form αZ_n where, given α , Z_n are i.i.d $N(0, 1)$. This means that after a change of density we may assume that $\{X_n\}$ is a small perturbation in the L_p norm of a sequence of i.i.d $N(0, 1)$ variables Z_n . By a change of density we mean an operator of the form $T_\varphi : L_p \rightarrow L_p([0, 1], \varphi dt)$, $T_\varphi f = \frac{f}{\varphi^{1/p}}$ where φ is a density which is strictly positive on the union of the supports of the X_n -s. It is thus enough to show that the span of such a sequence is $\gamma_{\frac{p}{p-1}}$ complemented in L_p . Since $\{Z_n\}_{n=1}^\infty \subseteq L_{\frac{p}{p-1}}$ it is clearly enough to show that the orthogonal projection $P : L_2 \xrightarrow{\text{onto}} [Z_n]$ has norm $\gamma_{\frac{p}{p-1}}$

when considered as an operator on $L_{\frac{p}{p-1}}$. Since for all $f \in L_{\frac{p}{p-1}}$ (so also $f \in L_2$) Pf is a Gaussian variable $\|Pf\|_{\frac{p}{p-1}} = \gamma_{\frac{p}{p-1}} \|Pf\|_2 \leq \gamma_{\frac{p}{p-1}} \|f\|_2 \leq \gamma_{\frac{p}{p-1}} \|f\|_{\frac{p}{p-1}}$. This concludes the (admittedly very rough) sketch of the proof of Theorem 3.1

We now present a sketch of a **proof of the complementation** part for $1 < p < 2$ which does not use [1], but gives a somewhat worse constant than $(1 + \varepsilon) \gamma_{\frac{p}{p-1}}$. Let $\{f_n\}_{n=1}^\infty$ be a normalized sequence in TX which is $(1 + \varepsilon)$ equivalent to the unit vector basis of ℓ_2 . Let $g_n = |f_n|^{p-1} \text{sign} f_n$ so that $\|g_n\|_{\frac{p}{p-1}} = 1$ and $\langle g_n, f_n \rangle = 1$. Passing to a subsequence we may assume $g_n \xrightarrow{w} g$. Since $f_n \xrightarrow{w} 0$, passing to another subsequence we may assume that $\{f_n\}_{n=1}^\infty$ and $\{g_n - g\}_{n=1}^\infty$ are arbitrarily close to a biorthogonal sequence; i.e.

$$\sum_{n,m} |\langle f_n, g_m - g \rangle - \delta_{nm}| < \varepsilon$$

Note that $1 - \varepsilon \leq \|g_n - g\| \leq 2$ for all n (the lower bound follows from $\langle f_n, g_n - g \rangle \geq 1 - \varepsilon$). We may also assume, by passing to a further subsequence, that $g_n - g$ is a martingale difference sequence (or a block basis of the Haar system). Consequently, for all coefficients $\{a_n\}_{n=1}^\infty$

$$\begin{aligned} \left\| \sum a_n (g_n - g) \right\|_{\frac{p}{p-1}} &\leq H_p \mathbb{E} \left\| \sum \pm a_n (g_n - g) \right\|_{\frac{p}{p-1}} \\ &\leq 2H_p B_p (\sum a_n^2)^{1/2} \end{aligned}$$

where H_p is the unconditionality constant of the Haar basis and $B_p = \gamma_{\frac{p}{p-1}}$ is the type 2 constant of $L_{\frac{p}{p-1}}$. Define now $P : L_p \rightarrow [f_n]$ by $Pf = \sum_{n=1}^\infty \langle f, g_n - g \rangle f_n$ then it is easily seen that P is a projection of norm $\leq 2K_p B_p (1 + \varepsilon)$.

4. PROOF OF THE MAIN RESULT

Here we shall prove Theorem 1.1. We shall denote by $\{e_{i,j}\}_{i,j=1}^\infty$ the canonical basis of $\ell_p(\ell_2)$ i.e.,

$$\left\| \sum_{i,j=1}^\infty a_{i,j} e_{i,j} \right\| = \left(\sum_{j=1}^\infty \left(\sum_{i=1}^\infty a_{i,j}^2 \right)^{p/2} \right)^{1/p}$$

for all $\{a_{i,j}\}_{i,j=1}^\infty \subseteq \mathbb{R}$. By passing to a subsequence in each column of $\{e_{i,j}\}_{i,j=1}^\infty$, a gliding hump argument (applied in the order (11), (12), (21), (13), (22), (31), ... of the indices) and a simple perturbation argument, we can assume that for some infinite subsequences $N_j \subseteq \mathbb{N}$, $\{Te_{i,j}\}_{j=1, i \in N_j}^\infty$ is a block basis of the Haar system in L_p . By that we mean that if the perturbed operator satisfies the conclusion of the theorem so does the original one. Also, since $\{e_{i,j}\}_{j=1, i \in N_j}^\infty$ spans an isometric $\ell_p(\ell_2)$, and we are anyhow interested only in a subspace of $\ell_p(\ell_2)$ isometric to $\ell_p(\ell_2)$, we may also assume that $N_j = \mathbb{N}$ for all j ; i.e., $\{Te_{i,j}\}_{i,j=1}^\infty$ is a block basis of the Haar system. We shall assume that from now on.

Given a finite $E \subseteq \mathbb{N}$ and $i \in \mathbb{N}$ set $v_i(E) = S \left(\sum_{j \in E} Te_{i,j} \right)$. The next two lemmas are basically taken from [4]. We repeat the proofs for completeness.

Lemma 4.1. *For all finite $E \subseteq \mathbb{N}$ and $1 < p < 2$ the convex hull of $\{v_i^2(E)\}_{i \in \mathbb{N}}$ is $p/2$ -equi-integrable; i.e., the set of $p/2$ powers of functions in the convex hull of $\{v_i^2(E)\}_{i \in \mathbb{N}}$ is equi-integrable.*

Proof. Fix a finite $E \subseteq \mathbb{N}$ and write $v_i = v_i(E)$. Assume that the convex hull of $\{v_i^2\}_{i \in \mathbb{N}}$ is not $p/2$ -equi-integrable. Then, there exists $\varepsilon_0 > 0$, a sequence $\{u_j^2\}_{j \in \mathbb{N}}$ of disjoint convex blocks of $\{v_i^2\}_{i \in \mathbb{N}}$ (i.e. $u_j^2 = \sum_{i \in \sigma_j} \alpha_{i,j}^2 v_i^2$ where σ_j are disjoint subsets of \mathbb{N} and $\sum_{i \in \sigma_j} \alpha_{i,j}^2 \leq 1$) and disjoint subsets B_j such that $\int_{B_j} |u_j|^p > \varepsilon_0$ for all $j \in \mathbb{N}$. The fact that we can choose the sequence $\{u_j^2\}$ to be disjointly supported with respect to $\{v_i^2\}$ follows from the easy fact that the convex hull of a finite subset of $\{v_i^2\}_{i=1}^\infty$ is $p/2$ -equi-integrable.

Now, For all $\{a_m\}_{m=1}^\infty \in \ell_2$, we obtain the following inequality, contradicting $p < 2$.

$$\begin{aligned}
\left(\sum_{m=1}^\infty a_m^2\right)^{1/2} &= |E|^{-1/p} \left(\sum_{j \in E} \left(\sum_{m=1}^\infty a_m^2\right)^{p/2}\right)^{1/p} \\
&= |E|^{-1/p} \left\| \sum_{m=1}^\infty a_m \sum_{i \in \sigma_m} \alpha_{i,m} \sum_{j \in E} e_{i,j} \right\|_{\ell_p(\ell_2)} \\
&\geq |E|^{-1/p} \|T\|^{-1} H_p^{-1} A_p \left\| S \left(\sum_{m=1}^\infty a_m \sum_{i \in \sigma_m} \alpha_{i,m} \sum_{j \in E} T e_{i,j} \right) \right\|_p \\
&= |E|^{-1/p} \|T\|^{-1} H_p^{-1} A_p \left(\int \left(\sum_{m=1}^\infty a_m^2 u_m^2(E) \right)^{p/2} d\mu \right)^{1/p} \\
&\geq |E|^{-1/p} \|T\|^{-1} H_p^{-1} A_p \left(\int \left(\sum_{m=1}^\infty a_m^2 u_m^2(E) \chi_{B_m} \right)^{p/2} d\mu \right)^{1/p} \\
&= |E|^{-1/p} \|T\|^{-1} H_p^{-1} A_p \left(\sum_{m=1}^\infty |a_m|^p \int_{B_m} u_m^p(E) d\mu \right)^{1/p} \\
&\geq |E|^{-1/p} \|T\|^{-1} H_p^{-1} A_p \varepsilon_0 \left(\sum_{m=1}^\infty |a_m|^p \right)^{1/p}.
\end{aligned}$$

□

Lemma 4.2. *There are successive convex combinations $\nu_k(\cdot)$ of $\{v_i^2(\cdot)\}$ such that for all finite $E \subseteq \mathbb{N}$ $\nu_k(E) \xrightarrow[k \rightarrow \infty]{} \Lambda(E)$ in $L_{p/2}$. $\Lambda(E)$ is a $L_{p/2}^+$ additive valued measure, Λ , on the finite subsets on \mathbb{N} .*

Proof. Case 1 ($1 < p < 2$): Let $V = \{(\sum_{n=1}^\infty \alpha_n^2 v_n^2) : \sum \alpha_n^2 \leq 1\}$. Since V is bounded in $L_{p/2}$, by a result of Nikishin [13] for each $\varepsilon > 0$ there is a set $D = D_\varepsilon \subset [0, 1]$ of measure larger than $1 - \varepsilon$ such that $\sup_{v \in V} \int_D v d\mu < \infty$. As in the proof of [8, Lemma 6.4] (or see [4, Proposition 5.2] for more details), we can find successive convex combinations $\nu_k(\cdot)$ of the $v_n^2(\cdot)$ such that $\nu_k(E) \mathbf{1}_D$ converges pointwise and in L_1 to $\Lambda(E) \mathbf{1}_D$ for every finite $E \subset \mathbb{N}$, where $\Lambda(E)$ is L_0^+ -valued (and $\Lambda \mathbf{1}_D$ is L_1^+ -valued). By passing to a subsequence of the ν_k and a simple diagonal argument we can find, for every $\varepsilon_n \rightarrow 0$ a sequence of sets D_n with $\mu(D_n) > 1 - \varepsilon_n$ and such that $\nu_k(E) \mathbf{1}_{D_n}$ converges, as $k \rightarrow \infty$, pointwise and in L_1 to $\Lambda(E) \mathbf{1}_{D_n}$ for every finite $E \subset \mathbb{N}$ and every n . In particular, $\nu_k(E)$ converges pointwise to $\Lambda(E)$ for every finite $E \subset \mathbb{N}$. It remains to show that the convergence is also in $L_{p/2}$ (on the whole interval). Since for each E , $\{\nu_k(E)\}_{k \in \mathbb{N}}$ is $p/2$ -equi-integrable, it follows that, given any $\delta > 0$, if n is large enough $\int_{D_n^c} \nu_k(E)^{p/2} d\mu < \delta$ for all k . Consequently, also $\int_{D_n^c} \Lambda(E)^{p/2} d\mu \leq \delta$ and

$$\begin{aligned}
\limsup_{k \rightarrow \infty} \int |\nu_k(E) - \Lambda(E)|^{p/2} d\mu &\leq \limsup_{k \rightarrow \infty} \int_{D_n} |\nu_k(E) - \Lambda(E)|^{p/2} d\mu + 2\delta \\
&\leq \limsup_{k \rightarrow \infty} \|(\nu_k(E) - \Lambda(E)) \mathbf{1}_{D_n}\|_1^{p/2} + 2\delta \\
&= 2\delta.
\end{aligned}$$

Case 2 ($2 < p < \infty$):

$$\|v_i^2(E)\|_{p/2}^{1/2} = \|v_i(E)\|_p \leq H_p \left\| \sum_{j \in E} T e_{i,j} \right\|_p \leq H_p \|T\| \left\| \sum_{j \in E} e_{i,j} \right\|_{\ell_p(\ell_2)} = H_p \|T\| |E|^{1/p}.$$

Thus, there exists a subsequence of $\{v_i^2\}_{i=1}^\infty$ that converges weakly in $L_{p/2}[0, 1]$ for every $E \in 2^\mathbb{N}$. Denote the limit by $\Lambda(E)$. By the reflexivity of $L_{p/2}[0, 1]$ and another diagonal argument, there exists a sequence $\{\sigma_k\}_{k=1}^\infty$ of disjoint finite subsets of the integers and non-negative numbers $\{\alpha_i\}_{i=1}^\infty$ such that $\sum_{i \in \sigma_k} \alpha_i^2 = 1$ and $\sum_{i \in \sigma_k} \alpha_i^2 v_i^2(E) \rightarrow \Lambda(E)$ as $k \rightarrow \infty$ for all finite E where the convergence is in the $L_{p/2}[0, 1]$ norm. Set $\nu_k(\cdot) = \sum_{i \in \sigma_k} \alpha_i^2 v_i^2(\cdot)$. ν_k is clearly additive on the finite subsets of \mathbb{N} . \square

It is clear that the sequence $\{\Lambda(\{j\})\}_{j=1}^\infty$ is positively equivalent to the unit vector basis of $\ell_{p/2}$; i.e., putting $\Lambda_j = \Lambda(\{j\})$, $\|\sum a_j^2 \Lambda_j\|_{p/2} \approx (\sum |a_j|^p)^{2/p}$ for all sequences of coefficients $\{a_j\}_{j=1}^\infty$. Next we would like to improve the equivalence constant to be arbitrarily close to 1 by blocking the Λ_j -s.

Lemma 4.3. *Let $1 < p < \infty$, $\varepsilon > 0$ and $\varepsilon_k \searrow 0$. There are successive disjoint $\sigma_k \subseteq \mathbb{N}$ $k = 1, 2, \dots$ and coefficients $\{\alpha_j\}_{j=1}^\infty$ such that putting $\phi_k = \sum_{j \in \sigma_k} \alpha_j^2 \Lambda_j$, for all $\{a_k\}_{k=1}^\infty$,*

$$\left(\sum_{k=1}^\infty |a_k|^p \right)^{2/p} \leq \left\| \sum_{k=1}^\infty a_k^2 \phi_k \right\|_{p/2} \leq (1 + \varepsilon) \left(\sum_{k=1}^\infty |a_k|^p \right)^{2/p}.$$

In addition, there are disjoint sets $\{B_k\}_{k=1}^\infty$ such that $\|\phi_k \mathbf{1}_{B_k^c}\| \leq \varepsilon_k$.

Proof. By the subsequence splitting lemma (see e.g. [3, Lemma 5.2.8]) we may assume, passing to a subsequence, that there are disjoint sets $\{A_j\}_{j=1}^\infty$ such that $\{\Lambda_j^{p/2} \mathbf{1}_{A_j^c}\}_{j=1}^\infty$ is equi-integrable. For $p > 2$ this already implies that $\int \Lambda_j^{p/2} \mathbf{1}_{A_j^c} \rightarrow 0$. Otherwise, there is an $R > 0$ and $\delta > 0$ such that $\int \Lambda_j^{p/2} \mathbf{1}_{\Lambda_j \leq R} \geq \delta$ for $j \in J$ for some infinite $J \subseteq \mathbb{N}$ and it would follow that for $j \in J$ $\int \Lambda_j \geq \delta R^{1-p/2}$ and for all coefficients $\{a_i\}$, $\int (\sum a_j^2 \Lambda_j)^{p/2} \geq (\int \sum a_j^2 \Lambda_j)^{p/2} \geq \delta^{p/2} R^{\frac{p}{2}(1-\frac{p}{2})} (\sum a_j^2)^{p/2}$, contradicting the positive equivalence to the unit vector basis of $\ell_{p/2}$. We can now take the σ_k -s to be singletons and B_k -s to be some subsequence of the A_j -s.

For $1 < p < 2$, the proof is a bit more complicated. Set $h_j = \Lambda_j \mathbf{1}_{A_j^c}$ and let $\delta_k \searrow 0$ be a sequence to be determined later. By the equi-integrability, there

exists an $R > 0$ such that

$$\int \left| \frac{1}{n^{2/p}} \sum_{j=1}^n h_j \mathbf{1}_{\{h_j \geq R\}} \right|^{p/2} d\mu \leq \frac{1}{n} \sum_{i=1}^n \int h_i^{p/2} \mathbf{1}_{\{h_i \geq R\}} d\mu \leq \frac{\delta_1^{p/2}}{2}.$$

We also have,

$$\int \left| \frac{1}{n^{2/p}} \sum_{j=1}^n h_j \mathbf{1}_{\{h_j < R\}} \right|^{p/2} d\mu < \int \left| \frac{1}{n^{2/p}} \sum_{j=1}^n R \right|^{p/2} d\mu = R^{p/2} \frac{1}{n} n^{p/2} \leq \frac{\delta_1^{p/2}}{2}$$

for n sufficiently large. Set $m_1 = \frac{1}{n^{2/p}} \sum_{j=1}^n \Lambda_j$ and $B_1 = \cup_{j=1}^n A_j$. Then

$$\begin{aligned} \int |m_1 \mathbf{1}_{B_1^c}|^{p/2} d\mu &= \int \left| \frac{1}{n^{2/p}} \sum_{j=1}^n \Lambda_j \mathbf{1}_{B_1^c} \right|^{p/2} d\mu \\ &\leq \int \left| \frac{1}{n^{2/p}} \sum_{j=1}^n \Lambda_j \mathbf{1}_{A_j^c} \right|^{p/2} d\mu \leq \delta_1^{p/2}. \end{aligned}$$

Similarly, we find $m_2 = \frac{1}{(n_2 - n_1)^{2/p}} \sum_{j=n_1+1}^{n_2} \Lambda_j$ and $B_2 = \cup_{j=n_1+1}^{n_2} A_j$ that satisfy

$$\int |m_2 \mathbf{1}_{B_2^c}|^{p/2} d\mu \leq \delta_2^{p/2}.$$

Continuing in this manner we find $m_i = \frac{1}{(n_i - n_{i-1})^{2/p}} \sum_{j=n_{i-1}+1}^{n_i} \Lambda_j$ and disjoint $\{B_j\}$ satisfying $\int |m_i \mathbf{1}_{B_i^c}|^{p/2} d\mu \leq \delta_i^{p/2}$. The sequence $\{\|m_k\|_{p/2}\}$ is bounded and bounded away from zero. Consequently, putting $\Phi_k = \frac{m_k}{\|m_k\|_{p/2}}$ we get that for an appropriate choice of the δ_k -s, $\{\Phi_k, B_k\}$ satisfy the conclusion of the lemma. \square

Proof of Theorem 1.1 In the following let σ_k and α_j be as in the statement of Lemma 4.3. Let $\varepsilon_{l,k} > 0$ and for each k let $\{\sigma_{l,k}\}_{l=1}^\infty$ be successive finite subsets of \mathbb{N} and $\{\beta_{i,k}\}_{i \in \sigma_{l,k}}$ coefficients such that $\sum_{i \in \sigma_{l,k}} \beta_{i,k}^2 = 1$ and $\left\| \sum_{i \in \sigma_{l,k}} \beta_{i,k}^2 \nu_i^2(\sigma_k) - \Lambda(\sigma_k) \right\|_{p/2} < \varepsilon_{l,k}$. Then

$$\left\| \sum_{j \in \sigma_k} \alpha_j^2 \sum_{i \in \sigma_{l,k}} \beta_{i,k}^2 \nu_i^2(j) - \sum_{j \in \sigma_k} \alpha_j^2 \Lambda_j \right\|_{p/2} \leq \left(\max_{j \in \sigma_k} |\alpha_j|^p \right)^{2/p} \varepsilon_{l,k}.$$

Put $f_{l,k} = \sum_{j \in \sigma_k} \alpha_j \sum_{i \in \sigma_{l,k}} \beta_{i,k} e_{i,j}$. Then, if the $\varepsilon_{l,k}$ -s are small enough,

$$\begin{aligned} \left\| \sum_{l,k} a_{l,k} f_{l,k} \right\|_{\ell_p(\ell_2)} &\approx \left\| \sum_{l,k} a_{l,k}^2 S^2(T(f_{l,k})) \right\|_{p/2}^{1/2} \\ &= \left\| \sum_{l,k} a_{l,k}^2 \sum_{j \in \sigma_k} \alpha_j^2 \sum_{i \in \sigma_{l,k}} \beta_{i,k}^2 \nu_i^2(j) \right\|_{p/2}^{1/2} \\ &\stackrel{1+\varepsilon}{\approx} \left\| \sum_{l,k} a_{l,k}^2 \sum_{j \in \sigma_k} \alpha_j^2 \Lambda_j \right\|_{p/2}^{1/2} \\ &\stackrel{1+\varepsilon}{\approx} \left(\sum_{k=1}^\infty \left(\sum_{l=1}^\infty a_{l,k}^2 \right)^{p/2} \right)^{1/p}. \end{aligned}$$

This shows that $\{f_{l,k}\}_{l=1, k=1}^{\infty, \infty}$ is equivalent to the unit vector basis of $\ell_p(\ell_2)$. The constant of the equivalence depends however on $\|T\|, \|T^{-1}\|$ and p . We next correct this: we can find, for each k , a block basis $g_{u,k} = \sum_{l \in \tau_{u,k}} \gamma_{l,k} f_{l,k}$, $u =$

1, 2, ..., such that $\|\sum_{u=1}^{\infty} a_u g_{u,k}\|_{\ell_p(\ell_2)} \stackrel{1+\varepsilon}{\approx} (\sum a_u^2)^{1/2}$.

Since for each k $g_{u,k}$ is supported only on columns in σ_k , we get from that that $\{g_{u,k}\}_{u,k}$ is $(1 + \varepsilon)$ -equivalent to the unit vector basis of $\ell_p(\ell_2)$.

Next we would like to apply a similar stabilization procedure to $\{T(g_{u,k})\}_{u,k}$. This of course is more complicated since this sequence does not lie in $\ell_p(\ell_2)$ any more.

Note that, by passing to further subsequences we may assume that for some positive constant b (depending on T), and for all k and $\{a_n\}$ with $\sum_{n=1}^{\infty} a_n^2 = 1$

$$\left\| S^2 \left(\sum_{n=1}^{\infty} a_n T g_{n,k} \right) - b \phi_k \right\|_{p/2} < \varepsilon_k$$

where ε_k is the preassigned sequence appearing in Lemma 4.3. In particular,

$$\begin{aligned} & \left\| S^2 \left(\sum_{n=1}^{\infty} a_n T g_{n,k} \right) \mathbf{1}_{B_k^c} \right\|_{p/2}^{p/2} \\ & \leq \left\| \left(S^2 \left(\sum_{n=1}^{\infty} a_n T g_{n,k} \right) - b \phi_k \right) \mathbf{1}_{B_k^c} \right\|_{p/2}^{p/2} + b^{p/2} \left\| \phi_k \mathbf{1}_{B_k^c} \right\|_{p/2}^{p/2} \\ & \leq \varepsilon_k^{p/2} + b^{p/2} \varepsilon_k^{p/2}. \end{aligned}$$

This shows that any sequence of the form $\{S(\sum_{n=1}^{\infty} a_{n,k} T g_{n,k})\}_{k=1}^{\infty}$ with, say, $\sum_{n=1}^{\infty} a_{n,k}^2 = 1$ for all k is essentially (with respect to the L_p norm) disjointly supported. We would like to have a similar statement with the S removed. This is where the Haar functions (rather than some other unconditional basis of L_p) play a role. First we may assume that the sets B_k are each a finite union of dyadic intervals. Next notice that for each k if l is large enough then $T f_{l,k}$ is a dyadic simple function such that each Haar function appearing in its expansion (with non zero coefficient) has support which is either contained or disjoint of B_k . Consequently,

$$S(T f_{l,k} \mathbf{1}_{B_k}) = S(T f_{l,k}) \mathbf{1}_{B_k} \quad \text{and} \quad S(T f_{l,k} \mathbf{1}_{B_k^c}) = S(T f_{l,k}) \mathbf{1}_{B_k^c}. \quad (4.1)$$

In particular, this implies that for some constant K_p (depending on p only), for all k and all $\{a_n\}_{n=1}^{\infty}$,

$$\begin{aligned} \left\| \left(\sum_{n=1}^{\infty} a_n T g_{n,k} \right) \mathbf{1}_{B_k^c} \right\|_p & \leq K_p \left\| S^2 \left(\sum_{n=1}^{\infty} a_n T g_{n,k} \right) \mathbf{1}_{B_k^c} \right\|_{p/2}^{1/2} \\ & \leq K_p (1 + b^{p/2})^{1/p} \varepsilon_k^{1/2} (\sum_{n=1}^{\infty} a_n^2)^{1/2}. \end{aligned}$$

We thus get that if ε_k are chosen small enough, the two sequences $\{T g_{n,k} \mathbf{1}_{B_k}\}_{n,k=1}^{\infty}$ and $\{T g_{n,k}\}_{n,k=1}^{\infty}$ are small perturbations one of the other. Define $\tilde{T} : \ell_p(\ell_2) \rightarrow [(\sum_{n,k=1}^{\infty} T g_{n,k} \mathbf{1}_{B_k})]$ by $\tilde{T} e_{n,k} = (T g_{n,k}) \mathbf{1}_{B_k}$. The perturbation above is such that if we show that for some subspace $X \subset \ell_p(\ell_2)$, spanned by blocks $\{u_{n,k}\}_{n,k=1}^{\infty}$ of $\{e_{n,k}\}_{n,k=1}^{\infty}$ 1-equivalent to the unit vector basis of $\ell_p(\ell_2)$, and such that $\{\tilde{T} u_{n,k}\}_{n,k=1}^{\infty}$ is $(1 + \varepsilon)$ -equivalent to a multiple of the unit vector basis of $\ell_p(\ell_2)$ and $[\tilde{T} u_{n,k}]_{n,k=1}^{\infty}$ is $(1 + \varepsilon)$ $\gamma_{p/(p-1)}$ -complemented in L_p , then the same (with $1 + 2\varepsilon$ replacing $1 + \varepsilon$) holds for $\{T u_{n,k}\}_{n,k=1}^{\infty}$.

The way we choose $u_{n,k}$ is similar to the way we have chosen $g_{n,k}$: since for each k $\{Tg_{n,k}\}_{k=1}^\infty$ is equivalent to the unit vector basis of ℓ_2 we can find blocks $\{u_{n,k}\}$ of $\{g_{n,k}\}$ such that $\{(Tu_{n,k})\mathbf{1}_{B_k}\}$ (which are blocks of $\{(Tg_{n,k})\mathbf{1}_{B_k}\}_{n,k=1}^\infty$) are $(1 + \varepsilon)$ equivalent to the unit vector basis of ℓ_2 and $(1 + \varepsilon)\gamma_p$ complemented in L_p . Call the projection P_k . Note also that we may assume that the ℓ_2 norm of the coefficients of $\{Tu_{n,k}\}$ relative to $\{Tf_{n,k}\}$ all differ by a multiplicative constant of at most $1 + \varepsilon$. (This is important since we want $T|_X$ to be a $(1 + \varepsilon)$ -isomorphism). Finally, define $P : L_p \rightarrow [(Tu_{n,k})\mathbf{1}_{B_k}]$ by $Pf = \sum P_k(f\mathbf{1}_{B_k})$. \square

5. CONCLUDING REMARKS

1. We first remark that the constants γ_p and $\gamma_{p/(p-1)}$ that appear in the statement of Theorem 1.1 (and also in the theorem in section 3) are best possible. Actually, these are lower bounds on the norm of the best projection onto an (isometric) ℓ_2 subspace of L_p . This was proved in [6].
2. If one wants to avoid the use of the material in section 3 at the price of getting worse constants, still depending only on p , one can use [14, Theorem 3.1] instead.
3. As we already remarked, it would be nice if somebody comes up with a simpler proof of the theorem of section 3. Maybe along the lines of [2].
4. Since ℓ_2 is not isomorphic to a complemented subspace of L_1 there is of course no complete analogue of Theorem 1.1 for L_1 . However, the weaker statement that any isomorphism $T : \ell_1(\ell_2) \rightarrow L_1$ stabilizes; i.e., is a $(1 + \varepsilon)$ -isomorphism when restricted to some subspace of $\ell_1(\ell_2)$ $(1 + \varepsilon)$ -isomorphic to $\ell_1(\ell_2)$, is still possible. Of course the proof as is written above makes an heavy use of the unconditionality of the Haar system and thus cannot be used. However, note that for most of the proof we could replace the Haar system with any unconditional basic sequence containing $T(\ell_1(\ell_2))$, even with constant depending on T . So we could use $Te_{i,j}$ as such a sequence. The problem is in (4.1) where we use the “eventual commutativity” of the square function operation and the restriction to a given dyadic set. This seems like a not very essential use of the Haar system but we couldn’t overcome it easily and we leave it for future research.

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