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MULTIPLE HILBERT'S TYPE INEQUALITIES WITH A HOMOGENEOUS KERNEL

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ABSTRACT. The main objective of this paper is a study of some new generalizations of Hilbert's and Hardy–Hilbert's type inequalities. We apply our general results to homogeneous functions. Also, we obtain the best possible constants when the parameters satisfy appropriate conditions.

1. INTRODUCTION

Hilbert's and Hardy–Hilbert's type inequalities (see [1]) are very significant weight inequalities which play an important role in many fields of mathematics. Similar inequalities, in operator form, appear in harmonic analysis where one investigates properties of boundedness of such operators. This is the reason why Hilbert's inequality is so popular and is of great interest to numerous mathematicians, since Hilbert till nowadays.

During the past century Hilbert-type inequalities were generalized in many different directions and also the numerous mathematicians reproved them using various technics. Some possibilities of generalizing such inequalities are, for example, various choices of non-negative measures, kernels, sets of integration, extension to multi-dimensional case etc. Several generalizations involve very important notions such as Hilbert's transform, Laplace transform, singular integrals, Weyl operators.

In this paper we refer to a paper of Bicheng Yang, [7], where Hilbert-type

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inequality was obtained for conjugate parameters and kernel $K(x_1, \ldots, x_n) = (x_1 + \ldots + x_n)^{-s}$, $s > 0$. See also [6, 8] as illustrative papers for recent progress in studying Hilbert and Hardy–Hilbert type inequalities. However, we shall keep an attention on the result with a special homogeneous function of degree $-s$. Yang's result is contained in:

Theorem 1.1. *If $n \in \mathbb{N} \setminus \{1\}$, $p_i > 1$, $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $s > 0$, $f_i : (0, \infty) \to \mathbb{R}$, $i = 1, \ldots, n$, non-negative functions which satisfy*

$$0 < \int_0^\infty x^{p_i - s - 1} f_i^{p_i}(x) dx < \infty \quad (i = 1, 2, \ldots, n), \tag{1.1}$$

then

$$\int_{(0,\infty)^n} \frac{\prod_{i=1}^{n} f_i(x_i)}{(\sum_{j=1}^{n} x_j)^s} dx_1 \ldots dx_n < \frac{1}{\Gamma(s)} \prod_{i=1}^{n} \Gamma\left(\frac{s}{p_i}\right) \left(\int_0^\infty x^{p_i - s - 1} f_i^{p_i}(x) dx\right)^{\frac{1}{p_i}},$$

where the constant factor $\frac{1}{\Gamma(s)} \prod_{i=1}^{n} \Gamma\left(\frac{s}{p_i}\right)$ is the best possible.

Our main objective is to emphasize the previous theorem. Our generalization will include a general homogeneous kernel which satisfies some special conditions. Also, applying our main results we shall obtain generalization of the following theorem (see [9]):

Theorem 1.2. *Let $a \geq 0$, $b > 0$, and $T : L^2(0, \infty) \to L^2(0, \infty)$ be defined as follows:*

$$(Tf)(y) := \int_0^\infty \frac{f(x)}{a \min\{x, y\} + b \max\{x, y\}} dx \quad (y \in (0, \infty)).$$

Then $\|T\| = D(a, b)$, where

$$D(a, b) = \int_0^\infty \frac{\left(\frac{x}{y}\right)^{\frac{1}{2}}}{a \min\{x, y\} + b \max\{x, y\}} dy, \quad x \in (0, \infty),$$

and for any f, $g \in L^2(0, \infty)$, such that $0 < \int_0^\infty f^2(x) dx < \infty$, $0 < \int_0^\infty g^2(x) dx < \infty$, we have $(Tf, g) < D(a, b) \|f\| \, \|g\|$,
i. e.

$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{a \min\{x, y\} + b \max\{x, y\}} dx dy < D(a, b) \left(\int_0^\infty f^2(x) dx\right)^{\frac{1}{2}} \left(\int_0^\infty g^2(x) dx\right)^{\frac{1}{2}}$$

where the constant factor $D(a, b)$ is the best possible.

2. Some lemmas

To prove our main results we need some lemmas.

Lemma 2.1. *(see [5]) If* $n \in \mathbb{N}$, $r_i > 0$, $i = 1, \dots, n$, *then*

$$\int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{r_i - 1}}{\left(1 + \sum_{i=1}^{n-1} u_i\right)^{\sum_{i=1}^{n} r_i}} du_1 \dots du_{n-1} = \frac{\prod_{i=1}^{n} \Gamma(r_i)}{\Gamma\left(\sum_{i=1}^{n} r_i\right)}.$$

By using Lemma 2.1 we have

Lemma 2.2. *If* $n \in \mathbb{N}$, s, $\lambda > 0$, $\beta_i > -1$, $i = 1, \dots, n-1$, *and* $\sum_{i=1}^{n-1} \beta_i < \lambda s - n + 1$, *then*

$$\begin{aligned}&\int_{(0,\infty)^{n-1}} \frac{\prod_{i=1}^{n-1} t_i^{\beta_i}}{\left(1 + \sum_{i=1}^{n-1} t_i^{\lambda}\right)^{s}} dt_1 \dots dt_{n-1} \\ &= \frac{1}{\Gamma(s)\lambda^{n-1}} \left(\prod_{i=1}^{n-1} \Gamma\left(\frac{\beta_i + 1}{\lambda}\right)\right) \Gamma\left(s - \frac{1}{\lambda}\sum_{i=1}^{n-1}(\beta_i + 1)\right).\end{aligned}$$

Proof. The proof follows directly from Lemma 2.1 using the substitution $u_i = t_i^{\lambda}$, $i = 1, \dots, n-1$. □

Lemma 2.3. *Let* $s > 0$, $a \geq 0$ *and* $b > 0$. *Let* α_1, $\alpha_2 > -1$, $\alpha_1 + \alpha_2 < s - 2$ *and*

$$k(\alpha_1, \alpha_2) := \int_0^{\infty} \int_0^{\infty} \frac{t_1^{\alpha_1} t_2^{\alpha_2}}{(a \min\{1, t_1, t_2\} + b \max\{1, t_1, t_2\})^s} dt_1 dt_2.$$

Then

$$\begin{aligned} k(\alpha_1, \alpha_2) =& \frac{b^{-s}}{(\alpha_1 + 1)(\alpha_2 + 1)} \sum_{i=1}^{2} F\left(s, \alpha_i + 1; \alpha_i + 2; -\frac{a}{b}\right) \\ &- \frac{b^{-s}}{(\alpha_1 + 1)(\alpha_2 + 1)} F\left(s, \alpha_1 + \alpha_2 + 2; \alpha_1 + \alpha_2 + 3; -\frac{a}{b}\right) \\ &+ b^{-s} \sum_{i=1}^{2} \left(\int_0^1 t_i^{\alpha_i} F\left(s, s - \alpha_{i+1} - 1; s - \alpha_{i+1}; -\frac{a}{b} t_i\right) dt_i\right) \\ &+ b^{-s}(\alpha_1 + \alpha_2 + 2)[(\alpha_1 + 1)(\alpha_2 + 1)(s - \alpha_1 - \alpha_2 - 1)]^{-1} \\ &\times F\left(s, s - \alpha_1 - \alpha_2 - 2; s - \alpha_1 - \alpha_2 - 1; -\frac{a}{b}\right) \\ &- b^{-s} \sum_{i=1}^{2} \frac{1}{(\alpha_i + 1)(s - \alpha_{i+1} - 1)} F\left(s, s - \alpha_{i+1} - 1; s - \alpha_{i+1}; -\frac{a}{b}\right) \end{aligned} \tag{2.1}$$

where $F(\alpha, \beta; \gamma; z)$ *denotes the hypergeometric function defined by*

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 t^{\beta - 1}(1 - t)^{\gamma - \beta - 1}(1 - zt)^{-\alpha} dt, \ \gamma > \beta > 0, z < 1$$

and the indices are taken modulo 2.

Proof. We start with the identity $k(\alpha_1,\alpha_2) = I_1 + I_2$, where

$$I_1 = \int_0^1 t_1^{\alpha_1} \left(\int_0^\infty \frac{t_2^{\alpha_2}}{(a\min\{t_1,t_2\} + b\max\{1,t_2\})^s} dt_2 \right) dt_1$$

and

$$I_2 = \int_1^\infty t_1^{\alpha_1} \left(\int_0^\infty \frac{t_2^{\alpha_2}}{(a\min\{1,t_2\} + b\max\{t_1,t_2\})^s} dt_2 \right) dt_1.$$

Here we used Fubini's theorem. In what follows we shall express the integral I_1 by a hypergeometric function. It is easy to see that holds the equality $I_1 = I_{11} + I_{12}$, where

$$I_{11} = \int_0^1 t_1^{\alpha_1} \left(\int_0^1 \frac{t_2^{\alpha_2}}{(a\min\{t_1,t_2\} + b)^s} dt_2 \right) dt_1$$

and

$$I_{12} = \int_0^1 t_1^{\alpha_1} \left(\int_1^\infty \frac{t_2^{\alpha_2}}{(at_1 + bt_2)^s} dt_2 \right) dt_1.$$

The integral I_{11} can be easily transformed in the following way

$$I_{11} = \int_0^1 t_1^{\alpha_1} \left(\int_0^{t_1} \frac{t_2^{\alpha_2}}{(at_2 + b)^s} dt_2 \right) dt_1 + \int_0^1 t_1^{\alpha_1} \left(\int_{t_1}^1 \frac{t_2^{\alpha_2}}{(at_1 + b)^s} dt_2 \right) dt_1. \tag{2.2}$$

By applying classical real analysis we have

$$\begin{aligned}
&\int_0^1 t_1^{\alpha_1} \left(\int_0^{t_1} \frac{t_2^{\alpha_2}}{(at_2 + b)^s} dt_2 \right) dt_1 \qquad (2.3)\\
&= \int_0^1 t_2^{\alpha_2} \left(\int_{t_2}^1 \frac{t_1^{\alpha_1}}{(at_2 + b)^s} dt_1 \right) dt_2 \\
&= \int_0^1 t_2^{\alpha_2} (at_2 + b)^{-s} \left(\int_{t_2}^1 t_1^{\alpha_1} dt_1 \right) dt_2 \\
&= \frac{b^{-s}}{\alpha_1 + 1} \int_0^1 t_2^{\alpha_2} (1 - t_2^{\alpha_1 + 1}) \left(1 + \frac{a}{b} t_2 \right)^{-s} dt_2 \\
&= \frac{b^{-s}}{\alpha_1 + 1} \left[\frac{1}{\alpha_2 + 1} F\left(s, \alpha_2 + 1; \alpha_2 + 2; -\frac{a}{b} \right) \right. \\
&\qquad \left. - \frac{1}{\alpha_1 + \alpha_2 + 2} F\left(s, \alpha_1 + \alpha_2 + 2; \alpha_1 + \alpha_2 + 3; -\frac{a}{b} \right) \right],
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_0^1 t_1^{\alpha_1} \left(\int_{t_1}^1 \frac{t_2^{\alpha_2}}{(at_1 + b)^s} dt_2 \right) dt_1 \qquad (2.4)\\
&= \frac{b^{-s}}{\alpha_2 + 1} \left[\frac{1}{\alpha_1 + 1} F\left(s, \alpha_1 + 1; \alpha_1 + 2; -\frac{a}{b} \right) \right. \\
&\qquad \left. - \frac{1}{\alpha_1 + \alpha_2 + 2} F\left(s, \alpha_1 + \alpha_2 + 2; \alpha_1 + \alpha_2 + 3; -\frac{a}{b} \right) \right].
\end{aligned}$$

Now, we substitute (2.3), (2.4) in (2.2), and we have

$$\begin{aligned} I_{11} =& \frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)} \sum_{i=1}^{2} F\left(s, \alpha_i+1; \alpha_i+2; -\frac{a}{b}\right) \\ & - \frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)} F\left(s, \alpha_1+\alpha_2+2; \alpha_1+\alpha_2+3; -\frac{a}{b}\right). \end{aligned} \tag{2.5}$$

By using the substitution $u = 1/t_2$, we find

$$\begin{aligned} I_{12} &= \int_0^1 t_1^{\alpha_1} \left(\int_1^\infty \frac{t_2^{\alpha_2}}{(at_1+bt_2)^s} dt_2 \right) dt_1 \\ &= b^{-s} \int_0^1 t_1^{\alpha_1} \left(\int_0^1 u^{s-\alpha_2-2} \left(1+\frac{a}{b}t_1 u\right)^{-s} du \right) dt_1 \\ &= b^{-s} \int_0^1 t_1^{\alpha_1} F\left(s, s-\alpha_2-1; s-\alpha_2; -\frac{a}{b}t_1\right) dt_1. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6) we get

$$\begin{aligned} I_1 &= I_{11} + I_{12} \\ &= \frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)} \sum_{i=1}^{2} F\left(s, \alpha_i+1; \alpha_i+2; -\frac{a}{b}\right) \\ & \quad -\frac{b^{-s}}{(\alpha_1+1)(\alpha_2+1)} F\left(s, \alpha_1+\alpha_2+2; \alpha_1+\alpha_2+3; -\frac{a}{b}\right) \\ & \quad +b^{-s} \int_0^1 t_1^{\alpha_1} F\left(s, s-\alpha_2-1; s-\alpha_2; -\frac{a}{b}t_1\right) dt_1. \end{aligned}$$

If we continue with the described procedure for the integrals

$$I_{21} = \int_1^\infty t_1^{\alpha_1} \left(\int_0^1 \frac{t_2^{\alpha_2}}{(at_2+bt_1)^s} dt_2 \right) dt_1$$

and

$$I_{22} = \int_1^\infty t_1^{\alpha_1} \left(\int_1^\infty \frac{t_2^{\alpha_2}}{(a+b\max\{t_1,t_2\})^s} dt_2 \right) dt_1,$$

such that $I_{21} + I_{22} = I_2$, we obtain the equality (2.1) and the proof is completed. □

3. Main results

In what follows we suppose that $K(x_1, \ldots, x_n)$ is nonnegative measurable homogeneous function of degree $-s$, $s > 0$. To obtain the main results we define the function $k(\alpha_1, \ldots, \alpha_{n-1})$ by:

$$k(\alpha_1, \ldots, \alpha_{n-1}) := \int_{(0,\infty)^{n-1}} K(1, t_1 \ldots, t_{n-1}) t_1^{\alpha_1} \cdots t_{n-1}^{\alpha_{n-1}} dt_1 \cdots dt_{n-1}, \tag{3.1}$$

where we suppose that $k(\alpha_1, \ldots, \alpha_{n-1}) < \infty$ for $\alpha_1, \ldots, \alpha_{n-1} > -1$ and $\alpha_1 + \cdots + \alpha_{n-1} + n < s+1$. Note that the function $k(\alpha_1, \alpha_2)$ from Lemma 2.3 satisfies these conditions.

Due to technical reasons, we introduce real parameters A_{ij}, $i,j = 1,2,\dots,k$ satisfying

$$\sum_{i=1}^{n} A_{ij} = 0, \quad j = 1,2,\dots,n. \tag{3.2}$$

We also define

$$\alpha_i = \sum_{j=1}^{n} A_{ij}, \quad i = 1,2,\dots,n. \tag{3.3}$$

Main results will be based on the following equivalent inequalities proven in [4] with the real parameters A_{ij}, $i,j = 1,\dots,n$, and α_i, $i = 1,\dots,n$, satisfying (3.2) and (3.3). Let $p_1,\dots,p_n$ be conjugate parameters such that $p_i > 1$, $i = 1,\dots,n$, and $f_i : (0,\infty) \to \mathbb{R}$ non-negative functions such and $0 < \int_0^\infty x^{n-s-1+p_i\alpha_i} f_i^{p_i}(x)dx < \infty$, $i = 1,\dots,n$. Then the following inequalities hold and are equivalent

$$\int_{(0,\infty)^n} K(x_1,\dots,x_n) \prod_{i=1}^{n} f_i(x_i) dx_1 \cdots dx_n < L \prod_{i=1}^{n} \left(\int_0^\infty x_i^{n-s-1+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{1}{p_i}} \tag{3.4}$$

and

$$\int_0^\infty x_n^{(1-q)(n-1-s)-q\alpha_n} \left(\int_{(0,\infty)^{n-1}} K(x_1,\dots,x_n) \times \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1} \right)^q dx_n < L^q \prod_{i=1}^{n-1} \left(\int_0^\infty x_i^{n-1-s+p_i\alpha_i} f_i^{p_i}(x_i) dx_i \right)^{\frac{q}{p_i}}, \tag{3.5}$$

where $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$ and

$$L = k(p_1A_{12},\dots,p_1A_{1n})^{\frac{1}{p_1}} \cdot k(s-n-p_2(\alpha_2 - A_{22}), p_2A_{23},\dots,p_2A_{2n})^{\frac{1}{p_2}} \cdots k(p_nA_{n2},\dots,p_nA_{n,n-1}, s-n-p_n(\alpha_n - A_{nn}))^{\frac{1}{p_n}},$$

and $p_iA_{ij} > -1$, $i \neq j$, $p_i(A_{ii} - \alpha_i) > n-s-1$.

Besides the approach to the study of Hardy–Hilbert's inequalities presented in this paper, for which inequalities (3.4) and (3.5) are typical results, it should be mentioned that there are numerous papers which study Hardy-type operators with general kernel on weighted Lebesgue spaces. Even more, necessary and sufficient conditions are given such that inequality analogous to (3.5) holds. An interested reader should consult books [3] and [2]. Our approach is to study kernels for which we can find the best possible constant as in the case of the classical Hilbert's inequality.

In this section, we consider the homogeneous kernels $K:(0,\infty)^n\to\mathbb{R}$ of degree $-s$, which satisfy the following condition for every $i=2,\dots,n$:

$$\begin{aligned} K(1,t_2,\dots,t_i,\dots,t_n) \leq & CK(1,t_2,\dots,0,\dots,t_n), \\ & 0\leq t_i\leq 1,\ t_j\geq 0,\ j\neq i, \end{aligned} \tag{3.6}$$

for some $C>0$. Also, to obtain a case of the best possible inequality in (3.4) and (3.5) it is natural to impose the following conditions on the parameters A_{ij}:

$$\begin{aligned} p_jA_{ji} &= s-n-p_i(\alpha_i-A_{ii}),\ i,j=1,2,\dots,n,\ i\neq j, \\ p_iA_{ik} &= p_jA_{jk},\ k\neq i,j. \end{aligned} \tag{3.7}$$

It is easy to see that the parameters A_{ij}, $i,j=1,\dots,n$, defined by

$$A_{ij}=\frac{s-p_j}{p_ip_j},\quad i\neq j,\quad \text{and}\quad A_{ii}=\frac{(s-p_i)(1-p_i)}{p_i^2}, \tag{3.8}$$

satisfy the conditions (3.7).

We proved the following result (see [4]): if the parameters A_{ij}, $i,j=1,\dots,n$, satisfy condition (3.7) and the kernels $K(x_1,\dots,x_n)$ satisfy condition (3.6), then the constants $L=k(p_1A_{12},\dots,p_1A_{1n})$ and L^q are the best possible in the inequalities (3.4) and (3.5).

Now, set the parameters A_{ij} defined by (3.8) in the inequalities (3.4) and (3.5). By using (3.8) we have $n-1-s+p_i\alpha_i=p_i-s-1$ and $(1-q)(n-1-s)-q\alpha_n=\frac{s}{p_n-1}-1$. In this way we proved the following theorem:

Theorem 3.1. *Let $n\geq 2$ be an integer and let $p_1,\dots,p_n$ be conjugate parameters such that $p_i>1$, $i=1,\dots,n$ and let $\frac{1}{q}=\sum_{i=1}^{n-1}\frac{1}{p_i}$. Let $K:(0,\infty)^n\to\mathbb{R}$ be nonnegative measurable homogeneous function of degree $-s$, $s>0$, which satisfies condition (3.6). If $f_i:(0,\infty)\to\mathbb{R}$, $i=1,\dots,n$ are nonnegative measurable functions satisfying (1.1), then the following inequalities hold and are equivalent*

$$\begin{aligned} &\int_{(0,\infty)^n} K(x_1,\dots,x_n)\prod_{i=1}^n f_i(x_i)dx_1\cdots dx_n \\ &\qquad < L\prod_{i=1}^n\left(\int_0^\infty x_i^{p_i-s-1}f_i^{p_i}(x_i)dx_i\right)^{\frac{1}{p_i}} \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\int_0^\infty x_n^{\frac{s}{p_n-1}-1}\left(\int_{(0,\infty)^{n-1}} K(x_1,\dots,x_n)\prod_{i=1}^{n-1} f_i(x_i)dx_1\cdots dx_{n-1}\right)^q dx_n \\ &\qquad < L^q\prod_{i=1}^{n-1}\left(\int_0^\infty x_i^{p_i-s-1}f_i^{p_i}(x_i)dx_i\right)^{\frac{q}{p_i}}, \end{aligned} \tag{3.10}$$

where the constants $L=k\left(\frac{s-p_2}{p_2},\dots,\frac{s-p_n}{p_n}\right)$ and L^q are the best possible in the inequalities (3.9) and (3.10).

Remark 3.2. Note that the kernel $K(x_1, \dots, x_n) = (x_1 + \dots + x_n)^{-s}$ is homogeneous function of degree $-s$ which satisfies condition (3.6). In this case using Lemma 2.1 we have $L = k\left(\frac{s-p_2}{p_2}, \dots, \frac{s-p_n}{p_n}\right) = \frac{1}{\Gamma(s)} \prod_{i=1}^n \Gamma\left(\frac{s}{p_i}\right)$. Hence, Theorem 3.1 can be seen as generalization of Theorem A.

Remark 3.3. Setting $n = 2$, $K(x, y) = (a \min\{x, y\} + b \max\{x, y\})^{-1}$, $p = q = 2$ in Theorem 3.1, we obtain Theorem B with the best possible constant $D(a, b) = k(-1/2) = 4/bF(1, 1/2; 3/2; -a/b)$. For each choice of the parameters a, b we compute the best possible constant $D(a, b)$:

(i) $a = b = 1$, $D(1, 1) = \pi$, as in [9],
(ii) $a = 0$, $b = 1$, $D(0, 1) = 4$, as in [9],
(iii) $a = 1$, $b = 2$, $D(1, 2) = 2\sqrt{2} \arctan \frac{1}{\sqrt{2}}$,
(iv) $a = 1$, $b = 3$, $D(1, 3) = \frac{2\pi}{3\sqrt{3}}$,
(v) $a = 2$, $b = 1$, $D(2, 1) = 2\sqrt{2} \arctan \sqrt{2}$.

We proceed with some special homogeneous functions. Since the function $K(x_1, \dots, x_n) = \left(\sum_{i=1}^n x_i^{s(\lambda-1)}\right) / \left(\sum_{i=1}^n x_i^\lambda\right)^s$, $\lambda > 1$, is homogeneous of degree $-s$, satisfying the condition (3.6), by using Theorem 3.1 we obtain:

Corollary 3.4. *Let $n \geq 2$ be an integer and let $p_1, \dots, p_n$ be conjugate parameters such that $p_i > 1$, $i = 1, \dots, n$ and let $\frac{1}{q} = \sum_{i=1}^{n-1} \frac{1}{p_i}$. If $f_i : (0, \infty) \to \mathbb{R}$, $i = 1, \dots, n$ are nonnegative measurable functions satisfying (1.1), then the following inequalities hold and are equivalent*

$$\int_{(0,\infty)^n} \frac{\sum_{i=1}^n x_i^{s(\lambda-1)}}{\left(\sum_{i=1}^n x_i^\lambda\right)^s} \prod_{i=1}^n f_i(x_i) dx_1 \dots dx_n$$
$$< L_1 \prod_{i=1}^n \left(\int_0^\infty x_i^{p_i-s-1} f_i^{p_i}(x_i) dx_i\right)^{\frac{1}{p_i}} \tag{3.11}$$

and

$$\left[\int_0^\infty x_n^{\frac{s}{p_n-1}-1} \left(\int_{(0,\infty)^{n-1}} \frac{\sum_{i=1}^n x_i^{s(\lambda-1)}}{\left(\sum_{i=1}^n x_i^\lambda\right)^s} \times \prod_{i=1}^{n-1} f_i(x_i) dx_1 \cdots dx_{n-1}\right)^q dx_n\right]^{\frac{1}{q}}$$
$$< L_1 \prod_{i=1}^{n-1} \left(\int_0^\infty x_i^{p_i-s-1} f_i^{p_i}(x_i) dx_i\right)^{\frac{1}{p_i}}, \tag{3.12}$$

where the constant

$$L_1 = \frac{1}{\Gamma(s)\lambda^{n-1}} \sum_{j=1}^n \left[\left(\prod_{i=1, i\neq j}^n \Gamma\left(\frac{s}{p_i \lambda}\right)\right) \cdot \Gamma\left(\frac{sp_j(\lambda-1)+s}{p_j \lambda}\right)\right] \tag{3.13}$$

is the best possible in the inequalities (3.11) and (3.12).

Proof. It is enough to calculate the constant $L_1 = k\left(\frac{s-p_2}{p_2}, \dots, \frac{s-p_n}{p_n}\right)$. Using the definition of the function $k(\alpha_1, \dots, \alpha_{n-1})$ given by (3.1) we have

$$L_1 = \int_{(0,\infty)^{n-1}} \frac{1 + t_1^{s(\lambda-1)} + \dots + t_{n-1}^{s(\lambda-1)}}{\left(1 + \sum_{i=1}^{n-1} t_i^{\lambda}\right)^s} t_1^{\frac{s}{p_2}-1} \dots t_{n-1}^{\frac{s}{p_n}-1} dt_1 \dots dt_{n-1} = \sum_{k=0}^{n-1} I_k, \tag{3.14}$$

where

$$I_0 = \int_{(0,\infty)^{n-1}} \frac{t_1^{\frac{s}{p_2}-1} \dots t_{n-1}^{\frac{s}{p_n}-1}}{\left(1 + \sum_{i=1}^{n-1} t_i^{\lambda}\right)^s} dt_1 \dots dt_{n-1}$$

and

$$I_k = \int_{(0,\infty)^{n-1}} \frac{t_1^{\frac{s}{p_2}-1} \dots t_k^{s(\lambda-1)+\frac{s}{p_{k+1}}-1} \dots t_{n-1}^{\frac{s}{p_n}-1}}{\left(1 + \sum_{i=1}^{n-1} t_i^{\lambda}\right)^s} dt_1 \dots dt_{n-1}, \quad \text{for } k = 1, \dots, n-1.$$

By using Lemma 2.2 we get

$$I_0 = \frac{1}{\Gamma(s)\lambda^{n-1}} \left(\prod_{i=2}^{n} \Gamma\left(\frac{s}{p_i \lambda}\right)\right) \cdot \Gamma\left(\frac{sp_1(\lambda-1)+s}{p_1\lambda}\right),$$

and similarly

$$I_k = \frac{1}{\Gamma(s)\lambda^{n-1}} \left(\prod_{i=1, i\neq k+1}^{n} \Gamma\left(\frac{s}{p_i \lambda}\right)\right) \cdot \Gamma\left(\frac{sp_{k+1}(\lambda-1)+s}{p_{k+1}\lambda}\right),$$

for $k = 1, \dots, n-1$. Now, from (3.14) we get (3.13). □

Similarly, for the homogeneous function of degree $-s$, $K(x_1, x_2, x_3) = (a \min\{x_1, x_2, x_3\} + b \max\{x_1, x_2, x_3\})^{-s}$, $a \geq 0$, $b > 0$, which satisfies condition (3.6), we have:

Corollary 3.5. *Let $p_1,\ p_2,\ p_3$ be conjugate parameters such that $p_i > 1$, $i = 1, 2, 3$ and let $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$. If $f_i : (0, \infty) \to \mathbb{R}$, $i = 1, 2, 3$ are nonnegative measurable functions satisfying (1.1), then the following inequalities hold and are equivalent*

$$\int_{(0,\infty)^3} \frac{f_1(x_1) f_2(x_2) f_3(x_3)}{(a \min\{x_1, x_2, x_3\} + b \max\{x_1, x_2, x_3\})^s} dx_1 dx_2 dx_3$$
$$< L_2 \prod_{i=1}^{3} \left(\int_0^\infty x_i^{p_i - s - 1} f_i^{p_i}(x_i) dx_i\right)^{\frac{1}{p_i}} \tag{3.15}$$

and

$$\left[\int_0^\infty x_3^{\frac{s}{p_3-1}-1}\left(\int_{(0,\infty)^2} f_1(x_1)f_2(x_2)\right.\right.$$
$$\left.\left.\times\,(a\min\{x_1,x_2,x_3\}+b\max\{x_1,x_2,x_3\})^{-s}dx_1dx_2\right)^q dx_3\right]^{\frac{1}{q}}$$
$$< L_2\prod_{i=1}^{2}\left(\int_0^\infty x_i^{p_i-s-1}f_i^{p_i}(x_i)dx_i\right)^{\frac{1}{p_i}}, \tag{3.16}$$

where the constant

$$\begin{aligned} L_2 =&\frac{p_2p_3}{b^s s^2}\sum_{i=2}^{3}F\left(s,\frac{s}{p_i};\frac{s}{p_i}+1;-\frac{a}{b}\right) \\ &-\frac{p_2p_3}{b^s s^2}F\left(s,\frac{s}{p_2}+\frac{s}{p_3};\frac{s}{p_2}+\frac{s}{p_3}+1;-\frac{a}{b}\right) \\ &+b^{-s}\sum_{i=2}^{3}\left(\int_0^1 t^{\frac{s}{p_i}-1}F\left(s,\frac{s}{p_1}+\frac{s}{p_i};\frac{s}{p_1}+\frac{s}{p_i}+1;-\frac{a}{b}t\right)dt\right) \\ &+\frac{p_1(p_2+p_3)}{b^s s(s+p_1)}F\left(s,\frac{s}{p_1};\frac{s}{p_1}+1;-\frac{a}{b}\right) \\ &-\frac{p_2p_3}{b^s s^2}\sum_{i=2}^{3}\frac{1}{p_i-1}F\left(s,s-\frac{s}{p_i};s-\frac{s}{p_i}+1;-\frac{a}{b}\right) \end{aligned} \tag{3.17}$$

is the best possible in the inequalities (3.15) *and* (3.16)*. In particular, for* $s=1$*,* $p_1=p_2=p_3=3$ *and* $a=b=1$*, we obtain*

$$L_2=\frac{55}{2}\log 2-\frac{3\sqrt{3}}{2}\pi.$$

Proof. The proof follows directly from Theorem 3.1 setting the kernel $K(x_1,x_2,x_3)=(a\min\{x_1,x_2,x_3\}+b\max\{x_1,x_2,x_3\})^{-s}$. Namely, in that case as in Corollary 3.4, it is enough to calculate the constant $L_2=k\left(\frac{s-p_2}{p_2},\frac{s-p_3}{p_3}\right)$. For this purpose, by using Lemma 2.3 we get (3.17). □

Remark 3.6. Setting the kernel $K(x_1,x_2,x_3)=(x_1+x_2+x_3-\min\{x_1,x_2,x_3\})^{-s}$, $s>0$, in Theorem 3.1, in such a way as in Corollary 3.5 we obtain the best possible constant

$$\begin{aligned} L_3 &= k\left(\frac{s-p_2}{p_2},\frac{s-p_3}{p_3}\right) \\ &= \frac{p_2+p_3}{s}F\left(s,\frac{s}{p_2}+\frac{s}{p_3};\frac{s}{p_2}+\frac{s}{p_3}+1;-1\right) \\ &\quad+\frac{1}{s}\sum_{i=2}^{3}p_iF\left(s,\frac{s}{p_1}+\frac{s}{p_i};\frac{s}{p_1}+\frac{s}{p_i}+1;-1\right) \\ &\quad+\frac{p_1(p_2+p_3)}{s^2}F\left(s,\frac{s}{p_1};\frac{s}{p_1}+1;-1\right) \\ &\quad-\frac{p_2p_3}{s^2}\sum_{i=2}^{3}\frac{1}{p_i-1}F\left(s,s-\frac{s}{p_i};s-\frac{s}{p_i}+1;-1\right). \end{aligned}$$

Similarly as in Corollary 3.5, for $s=1$ and $p_1=p_2=p_3=3$ we obtain $L_3=\frac{10\sqrt{3}}{3}\pi+\log 4$.

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