Banach J. Math. Anal. 5 (2011), no. 1, 38–50

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

QUASI-CONTRACTIONS ON SYMMETRIC AND CONE SYMMETRIC SPACES

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Communicated by P. E. T. Jorgensen

Abstract. The purpose of this paper is to introduce the concept of a cone symmetric space and to investigate relationship between (cone) metric spaces and (cone) symmetric spaces. Among other things, we shall also extend some fixed point results from metric spaces to cone metric spaces (Theorem 3.3), and to symmetric spaces (Theorems 3.2 and 3.5) under some new contraction conditions.

1. Introduction and preliminaries

Ordered normed spaces and cones have applications in applied mathematics, for instance, in using Newton's approximation method [12, 19, 21] and in optimization theory [5]. K-metric and K-normed spaces were introduced in the mid-20th century ([12], see also [13, 14, 16, 18, 19, 21]) by using an ordered Banach space instead of the set of real numbers, as the codomain for a metric. Huang and Zhang [7] re-introduced such spaces under the name of cone metric spaces, but went further, defining convergent and Cauchy sequences in the terms of interior points of the underlying cone. Recently, in [1, 8, 9, 17], some common fixed point theorems have been proved for maps on cone metric spaces. However, in [1, 7, 8] the authors usually obtain their results for normal cones. In this paper we obtain some of our results without using the normality condition for cones (see also [11]).

Date: Received: 31 January 2010; Accepted: 13 May 2010.

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2010 *Mathematics Subject Classification.* Primary 47H10; Secondary 54H25.

Key words and phrases. Normal and non-normal cone, cone metric space, symmetric space, fixed point, quasi-contraction.

The authors are thankful to the Ministry of Science and Technology of Serbia.

Recall that in [20] the concept of a "symmetric" on a set X was introduced, as a function $d : X \times X \to \mathbb{R}$ possessing all the properties of a metric except the triangle inequality. This concept was used in some recent papers (see, e.g., [2, 3, 6, 10, 22]) to obtain certain fixed point results.

The purpose of this paper is to introduce the concept of a cone symmetric space and to investigate relationship between (cone) metric spaces and (cone) symmetric spaces. We shall also extend some fixed point results from metric spaces to cone metric spaces (Theorem 3.3), and to symmetric spaces (Theorems 3.2 and 3.5) under some new contraction conditions. For instance, our Theorem 3.5 is a proper generalization of a result from metric spaces [4] to symmetric spaces.

We need the following definitions and results, consistent with [5] and [7], in the sequel.

Let E be a real Banach space. A subset P of E is called a *cone* if:

(i) P is closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$;
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$ (interior of P).

There exist two kinds of cones: normal and nonnormal ones. A cone $P \subset E$ is *normal* if there is a number $K > 0$ such that for all $x, y \in P$,

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq K \|y\|, \tag{1.1}$$

or, equivalently, if $x_n \leq y_n \leq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x.$$

The least positive number K satisfying (1.1) is called the normal constant of P. It is clear that $K \geq 1$. Most of ordered Banach spaces used in applications posses a normal cone with the normal constant $K = 1$. For details see [5].

Example 1.1. [5] Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{\, x \in E : x(t) \geq 0 \text{ on } [0,1] \,\}$. This cone is not normal. Consider, for example, $x_n(t) = \frac{1-\sin nt}{n+2}$ and $y_n(t) = \frac{1+\sin nt}{n+2}$. Then $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| = \frac{2}{n+2} \to 0$.

Definition 1.2. [5, 7] Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X and (X, d) is called a *cone metric space*.

In the case of a normal cone, the concept of a cone metric space is more general than that of a metric space. Indeed, each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty[$ (see [7, Example 1], [17, Examples 1.2 and 2.2]). If the cone P is nonnormal, then this is not true.

Remark 1.3. (1) If $u \leq v$ and $v \ll w$, then $u \ll w$.
(2) If $u \ll v$ and $v \leq w$, then $u \ll w$.
(3) If $u \ll v$ and $v \ll w$, then $u \ll w$.
(4) If $0 \leq u \ll c$ for each $c \in \operatorname{int} P$ then $u = 0$.
(5) If $a \leq b + c$ for each $c \in \operatorname{int} P$ then $a \leq b$.
(6) If $a \leq \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.

Especially properties (1), (4) and (6) of this remark are often used (particularly when dealing with nonnormal cones), so we give their proofs.

Proof. (1) We have to prove that $w - u \in \operatorname{int} P$ if $v - u \in P$ and $w - v \in \operatorname{int} P$. There exists a neighborhood V of 0 in E such that $w - v + V \subset P$. Then, from $v - u \in P$ it follows that

$$w - u + V = (w - v) + V + (v - u) \subset P + P \subset P,$$

since P is convex.

(4) Since $c - u \in \operatorname{int} P$ for each $c \in \operatorname{int} P$, it follows that $\frac{1}{n}c - u \in \operatorname{int} P$ for each $n \in \mathbb{N}$. Thus,

$$\lim_{n\to\infty} \left(\frac{1}{n}c - u\right) = 0 - u \in \overline{P} = P.$$

Hence $u \in -P \cap P = \{0\}$, i.e., $u = 0$.

(6) The condition $a \leq \lambda a$ means that $\lambda a - a \in P$, i.e., $-(1-\lambda)a \in P$. Since $a \in P$ and $1 - \lambda > 0$, we have also $(1-\lambda)a \in P$. Thus we have $(1-\lambda)a \in P \cap (-P) = \{0\}$, and $a = 0$. □

Let (X, d) be a cone metric space and $\{x_n\}$ a sequence in X. Then it is said [7] that $\{x_n\}$ is:

(e) a *Cauchy sequence* if for every c in E with $0 \ll c$, there is a positive integer N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;
(f) a *convergent sequence* if for every c in E with $0 \ll c$, there is a positive integer N such that for all $n > N$, $d(x_n, x) \ll c$ for some fixed x in X.

A cone metric space X is said to be *complete* if every Cauchy sequence in X is convergent in X.

In the case of a normal cone it is known [7] that $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$, and $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Remark 1.4. (1) If $c \in \operatorname{int} P$, $0 \leq a_n$ and $a_n \to 0$, then there exists a positive integer n_0 such that $a_n \ll c$ for all $n > n_0$.
(2) If $0 \leq d(x_n, x) \leq b_n$ and $b_n \to 0$, then $d(x_n, x) \ll c$ where x_n and x are a sequence and a given point in X, respectively.
(3) If $0 \leq a_n \leq b_n$ and $a_n \to a$, $b_n \to b$, then $a \leq b$ for an arbitrary cone P.

Proof. (1) Let $0 \ll c$ be given. Choose a symmetric neighborhood V such that $c + V \subset P$. Since $a_n \to 0$ there is an n_0 such that $a_n \in V = -V$ for $n > n_0$. This means that $c \pm a_n \in c + V \subset P$ for $n > n_0$; that is $a_n \ll c$. □

It follows that a sequence $\{x_n\}$ converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$. For a nonnormal cone only one part of Lemmas 1 and 4 from [7] are valid. Also, the fact that $d(x_n, y_n) \to d(x, y)$ if $x_n \to x$ and $y_n \to y$ cannot be applied.

2. Cone symmetric spaces

2.1. **Definition and properties of cone symmetrics.** In the sequel we assume only that E is a Banach space and that P is a cone in E with $\operatorname{int} P \neq \emptyset$ (this assumption is necessary in order to obtain reasonable results connected with convergence and continuity). In some situations (but not always) normality of the cone will be also assumed. The partial ordering induced by the cone P will be denoted by $\leq$.

We shall define now a new concept.

Definition 2.1. Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies:

(s1) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(s2) $d(x, y) = d(y, x)$, for all $x, y \in X$.

Then d is called a *cone symmetric* on X, and (X, d) is called a *cone symmetric space*.

Example 2.2. Let $X = \mathbb{R}$, $E = \mathbb{R}^2$, $P = \{(a, b) : a \geq 0, b \geq 0\}$ and let $d : X \times X \to E$ be defined by

$$d(x, y) = (e^{|x-y|} - 1, \alpha(e^{|x-y|} - 1)), \quad \alpha \geq 0.$$

Then (X, d) is a cone symmetric space which is not a cone metric space.

Hence, we have the following diagram

$$\begin{array}{ccc} \text{cone metric spaces} & \longrightarrow & \text{cone symmetric spaces} \\ \uparrow & & \uparrow \\ \text{metric spaces} & \longrightarrow & \text{symmetric spaces} \end{array}$$

where arrows stand for inclusions. The inverse inclusions do not hold. It is clear that there exists a symmetric (resp. cone symmetric) space which is not a metric (resp. cone metric) space. For vertical arrows see [7, Example 1] (resp. our Example 2.2).

Definitions of convergent and Cauchy sequences are the same as for cone metric spaces.

Similarly as for symmetric spaces ([3, 22]) we have the following possible axioms for cone symmetric spaces:

(W_3) For each sequence $\{x_n\}$ in X, and $x, y \in X$, $x_n \to x$ and $x_n \to y$ (as $n \to \infty$) imply $x = y$ or, equivalently, if for each c, $0 \ll c$, there exists n_0 such that $d(x_n, x) \ll c$ and $d(x_n, y) \ll c$ for all $n > n_0$, then $x = y$.

(W_4) For sequences $\{x_n\}$, $\{y_n\}$ in X and $x \in X$, $d(x_n, x) \ll c$ and $d(x_n, y_n) \ll c$ imply $d(y_n, x) \ll c$, as $n \to \infty$.

(H.E) For sequences $\{x_n\}$, $\{y_n\}$ in X and $x \in X$, $d(x_n, x) \ll c$ and $d(y_n, x) \ll c$ imply $d(x_n, y_n) \ll c$, as $n \to \infty$.

(H_E) For sequences $\{x_n\}$, $\{y_n\}$ in X and $x \in X$, $d(x_n, x) \ll c$ and $d(y_n, y) \ll c$ imply $d(x_n, y_n) \ll c$, as $n \to \infty$.

(C.C) For a sequence $\{x_n\}$ in X and $x, y \in X$, $x_n \to x$ implies $d(x_n, y) \to d(x, y)$, as $n \to \infty$.

Remark 2.3. We shall show that every cone metric space satisfies (W_3), (W_4) and (H.E) if $\operatorname{int} P \neq \emptyset$ (hence, possibly without normality property).

(W_3): Let $0 \ll c$ be given. According to the triangle inequality it follows

$$d(x, y) \leq d(x, x_n) + d(x_n, y) \ll \frac{c}{2} + \frac{c}{2} = c,$$

i.e., $d(x, y) \ll c$. Hence, by Remark 1.3.(4), $d(x, y) = 0$, that is $x = y$.

(W_4): Similarly, $d(y_n, x) \leq d(y_n, x_n) + d(x_n, x) \ll \frac{c}{2} + \frac{c}{2} = c$.

(H.E): Again, $d(x_n, y_n) \leq d(x_n, x) + d(x, y_n) \ll \frac{c}{2} + \frac{c}{2} = c$.

If (X, d) is a cone metric space with a normal cone P then it also satisfies (H_E) and (C.C), [7, Lemma 5]. Evidently, (C.C) is the special case of (H_E) where $y_n = y$ for each $n \in \mathbb{N}$.

2.2. **Topologies $\mathbf{t}_d$ and $\mathbf{t}_D$.** Let d be a cone symmetric on a nonempty set X. For $x \in X$ and $c \in P$, $0 \ll c$, let $K_c(x) = \{y \in X : d(x, y) \ll c\}$. The topology t_d on X is defined as follows: $U \in t_d$ if and only if for each $x \in U$, there exists $c \in P$, $0 \ll c$, such that $K_c(x) \subset U$. A subset S of X is a neighborhood of $x \in X$ if there exists $U \in t_d$ such that $x \in U \subset S$.

A cone symmetric d is a *cone semi-metric* if for each $x \in X$ and each $c \in P$, $0 \ll c$, $K_c(x)$ is a t_d-neighborhood of x.

For the given cone symmetric space (X, d) one can construct a symmetric space (X, D) where "symmetric" $D : X \times X \to \mathbb{R}$ is given by $D(x, y) = \|d(x, y)\|$.

Definition 2.4. The space (X, D) is called *the symmetric space associated with the cone symmetric space* (X, d).

In the case of cone metric spaces with normal cones, the triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$ for each $x, y, z \in X$, implies that the symmetric D satisfies the condition

$$D(x, y) = \|d(x, y)\| \leq K\|d(x, z) + d(z, y)\| \leq K(D(x, z) + D(z, y)),$$

where $K \geq 1$ is the normal constant of the cone P. So, the symmetric D satisfies

(s3) for each $x, y, z \in X$

$$D(x, y) \leq K(D(x, z) + D(z, y)),$$

Hence, in this case the symmetric space (X, D) is "almost" a metric space.

If the cone P is nonnormal then the symmetric D satisfies only (s1) and (s2).

Remark 2.5. If (X, d) is a cone metric space with a normal cone P, then (X, D) is a symmetric space which satisfies all the axioms from [3], that is (W_3)–(C.C), in the setting of symmetric spaces.

Proof. (W_3): According to (s3) we have

$$D(x, y) \leq K(D(x, x_n) + D(x_n, y)) \to K \cdot (0 + 0) = 0,$$

and thus $x = y$.

(W_4): Again using (s3) we obtain

$$D(y_n, x) \leq K(D(y_n, x_n) + D(x_n, x)) \to K \cdot (0 + 0) = 0,$$

wherefrom it follows that $D(y_n, x) \to 0$ as $n \to \infty$.

(H.E): In this case

$$D(x_n, y_n) \leq K(D(x_n, x) + D(x, y_n)) \to K \cdot (0 + 0) = 0,$$

which means that this axiom is satisfied, as well.

(H_E) and C.C are proved in the similar way. □

It follows that the space (X, D) satisfies the conditions of [10, Theorems 2.1–2.3], [2, Theorem 1, Corollary], as well as conditions of [3, Theorems 3.1–3.5].

Now, for $x \in X$ and $\varepsilon > 0$ let $K_\varepsilon(x) = \{y \in X : D(y, x) < \varepsilon\}$. Let t_D be the topology on X generated by the balls of the form $K_\varepsilon(x)$, $x \in X$, $\varepsilon > 0$.

Example 2.6. Let (X, d_1) be a symmetric space which is not semi-metric (see, e.g., [10, p. 353]). Then there exists a cone symmetric on X, $d : X \times X \to E$, $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\}$ such that $d(x, y) = (d_1(x, y), \alpha d_1(x, y))$, $\alpha \geq 0$. It is easy to check that (X, d) is a cone symmetric space which is not a cone semi-metric, so that the associated symmetric space $(X, D) = (X, \|d(x, y)\|)$ is not a semi-metric symmetric space.

For cone metric spaces we have

Theorem 2.7. *Let (X, d) be a cone metric space with a normal cone P and let D be the associated symmetric. Then $t_d = t_D$; moreover, d is a cone semi-metric and D is a semi-metric.*

Proof. Let $K_c(x) = \{y \in X : d(y, x) \ll c\}$ and $K_\varepsilon(x) = \{y \in X : D(y, x) < \varepsilon\}$ be, respectively, a d- and a D-ball with the center x. Since for every $\varepsilon > 0$ there exists $c \in \operatorname{int} P$ such that $K\|c\| < \varepsilon$, it is $K_c(x) \subset K_\delta(x)$ for $\delta = K\|c\|$, i.e., $t_D \leq t_d$.

Conversely, suppose that the ball $K_c(x)$, for fixed $x \in X$ and $c \in \operatorname{int} P$, contains no ball of the form $K_\varepsilon(x)$, i.e., that

$$K_\varepsilon(x) \subsetneq K_c(x);$$

in particular that $K_{\frac{1}{n}}(x) \subsetneq K_c(x)$ for each positive integer n. Then, there exists a sequence of points $x_n \in K_{\frac{1}{n}}(x)$ such that $x_n \notin K_c(x)$. Thus, $d(x_n, x) < \frac{1}{n}$ and $d(x_n, x) \not\ll c$ which is a contradiction because $d(x_n, x) \to 0$ implies $d(x_n, x) \ll c$.

To prove that d is a cone semi-metric, we have to prove that for each $c \in \operatorname{int} P$ and each $x \in X$, the d-ball $K_c(x) = \{y \in X : d(y, x) \ll c\}$ is a t_d-neighborhood of the point x. Take an arbitrary point $z \in K_c(x)$, $z \neq x$; we shall prove that $K_{c-d(z,x)}(z) \subset K_c(x)$. Let $y \in K_{c-d(z,x)}(z)$. Then $d(y, z) \ll c - d(z, x)$, wherefrom, using the triangle inequality, it follows that $d(y, x) \leq d(y, z) + d(z, x) \ll c - d(z, x) + d(z, x) = c$, i.e., $y \in K_c(x)$. This proves that the cone metric d is also a cone semi-metric; moreover, each d-ball is t_d-open. All these facts have been proved without using the normality property of the cone.

Now, using the normality of the cone, we have a previously proved inclusion $K_c(x) \subset K_\varepsilon(x)$, $\varepsilon = K \cdot \|c\|$ and so $K_c(x)$ is a $t_d = t_D$-neighborhood of each of its points. Hence, the symmetric D is a semi-metric (under the assumption of the normality of the cone). □

Fixed point problems on symmetric spaces which need not be metric, have been investigated intensively in the last few years. To compensate the lack of the triangle inequality, symmetrics which are semi-metrics have often been used. The previous theorem shows that the frame of cone metric spaces, introduced in [7], is a good resource for obtaining such symmetrics.

3. Fixed point theorems for cone symmetric spaces

3.1. Fixed points of contractions and D-contractions.

Definition 3.1. Let (X, d) be a cone metric space and (X, D) the associated symmetric space. The self-map $f : X \to X$ is called a *contraction* [7] if for some $\lambda \in (0, 1)$ and for all $x, y \in X$

$$d(fx, fy) \leq \lambda d(x, y).$$

holds true. The map f is called a *quasicontraction* [8] if for some $\lambda \in (0, 1)$ and for all $x, y \in X$ there exists

$$u(x, y) \in \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$$

such that

$$d(fx, fy) \leq \lambda \cdot u(x, y).$$

We call f a *D-contraction* if for some $\lambda \in (0, 1)$ and for $x, y \in X$

$$D(fx, fy) \leq \lambda D(x, y). \tag{3.1}$$

f is a *D-quasicontraction* if for some $\lambda \in (0, 1)$ and for all $x, y \in X$

$$D(fx, fy) \leq \lambda \cdot \max\{D(x, y), D(x, fx), D(y, fy), D(x, fy), D(y, fx)\}.$$

It follows from [7] and [9] that each contraction (quasicontraction) in a complete cone metric space has a unique fixed point if the cone is normal.

In the proof of the following fixed point theorem for D-contractions we have to use the normality of the cone, opposite to the situation with contractions where it is sufficient to assume that the cone has nonempty interior, see Theorem 3.3 below.

Theorem 3.2. *Let f be a D-contraction with the coefficient λ, on a complete cone metric space (X, d). If $\lambda < 1/K$, where K is the normal constant of the cone, then f has a unique fixed point, e.g., p. Moreover, for each $x \in X$ the sequence $\{f^n x\}$ of Picard iterations converges to p and the estimate*

$$D(f^n x, p) \leq \frac{K\lambda^n}{1 - \lambda K} D(x, fx)$$

holds.

Proof. Using condition (3.1) for the D-contraction and induction, we obtain that $D(f^m x_1, f^m x_2) \leq \lambda^m D(x_1, x_2)$ for the m-th iteration f^m of the map f. Using the relation

$$d(x_1, x_2) \leq d(x_1, fx_1) + d(fx_1, fx_2) + d(fx_2, x_2),$$

the fact that the cone is normal and (3.1) we obtain

$$D(x_1, x_2) \leq \frac{K}{1 - \lambda K}(D(x_1, fx_1) + D(x_2, fx_2)). \tag{3.2}$$

In terms of [15] this is the "fundamental contraction inequality" for the D-contraction f on the symmetric space (X, D) associated with the cone metric space (X, d). This implies the following: if x_1 and x_2 are two fixed points of a D-contraction, then they coincide. In other words, similarly as in [15], a D-contraction can have at most one fixed point.

Let now $x \in X$ be arbitrary. Replacing x_1 and x_2 in (3.2) with $f^n x$ and $f^m x$, and using (3.1), we obtain

$$\begin{aligned} D(f^n x, f^m x) &\leq \frac{K}{1 - \lambda K}(D(f^n x, f^n(fx)) + D(f^m x, f^m(fx))) \\ &\leq \frac{K}{1 - \lambda K}(\lambda^n + \lambda^m) D(x, fx) \to 0, \end{aligned} \tag{3.3}$$

when $m, n \to \infty$. Hence, the sequence $\{f^n x\}$ is a D-Cauchy sequence for fixed $x \in X$. Since D- and d-Cauchy sequences are the same, and since the given cone metric space is complete, it follows that there exists $p \in X$ such that $f^n x \to p$ when $n \to \infty$. Obviously, p is a fixed point of the D-contraction f, since it is D-continuous and so d-continuous (which is equivalent with $t_D = t_d$-continuous).

Letting $m \to \infty$ with n fixed in (3.3), we obtain the estimate $D(f^n x, p) \leq \frac{\lambda^n K}{1-\lambda K} D(x, fx)$, which is consistent with the respective estimate in metric spaces. The theorem is proved. □

We can conclude that the elegant Palais' proof [15] of the Banach Contraction Principle can be applied to D-contractions, but in the case of the symmetric space (X, D) associated with the given cone metric space (X, d). In the case of arbitrary symmetric spaces additional assumptions are needed [2, 3, 10, 22].

The Banach Contraction Principle was in the case of cone metric spaces proved in [7] using normality of the cone, and then in [17] without this assumption. The only assumption in the next theorem will be that $\operatorname{int} P \neq \emptyset$. Besides, we shall obtain an estimate that, to the best of our knowledge, appears for the first time in this context.

Theorem 3.3. *Each contraction f on a complete cone metric space (X, d) with a cone having the nonempty interior, has a unique fixed point, e.g., p. Moreover, for each $x \in X$, the sequence $\{f^n x\}$ of Picard iterations converges to p and the following estimate is valid*

$$d(f^n x, p) \leq \frac{\lambda^n}{1 - \lambda} d(x, fx). \tag{3.4}$$

Proof. If $x \in X$ is an arbitrary point, then, similarly as in the proof of Theorem 3.2, we have

$$d(f^n x, f^m x) \leq \frac{\lambda^n + \lambda^m}{1-\lambda} d(x, fx). \tag{3.5}$$

Since $\frac{\lambda^n+\lambda^m}{1-\lambda} d(x, fx) \to 0$, $m, n \to \infty$, in the Banach norm, then by Remark 1.4 $\frac{\lambda^n+\lambda^m}{1-\lambda} d(x, fx) \ll c$ for each $c \in \operatorname{int} P$. Then, by Remark 1.3.(1), $d(f^n x, f^m x) \ll c$, which means that $\{f^n x\}$ is a Cauchy sequence. Thus, it converges to a unique point p. Since f is a continuous mapping (the proof of that fact can be deduced without using normality of the cone), p is also the fixed point of f.

If n is a fixed positive integer and p the fixed point of f, i.e., the limit of the sequence $\{f^m x\}$ (for some given x), then we have

$$d(f^n x, p) \leq d(f^n x, f^m x) + d(f^m x, p),$$

for each $m \in \mathbb{N}$. Applying (3.5) we obtain

$$\begin{aligned} d(f^n x, p) &\leq \frac{\lambda^n + \lambda^m}{1-\lambda} d(x, fx) + d(f^m x, p) \\ &= \frac{\lambda^n}{1-\lambda} d(x, fx) + \frac{\lambda^m}{1-\lambda} d(x, fx) + d(f^m x, p) \\ &= \frac{\lambda^n}{1-\lambda} d(x, fx) + v_m, \end{aligned}$$

where $v_m = \frac{\lambda^m}{1-\lambda} d(x, fx) + d(f^m x, p)$. It follows that $v_m \ll c$ for each interior point c. Indeed, the first summand tends to zero in the Banach space E, so by Remark 1.4.(1) it is $\frac{\lambda^m}{1-\lambda} d(x, fx) \ll \frac{c}{2}$ for m sufficiently large and $d(f^m x, p) \ll \frac{c}{2}$ follows by definition since $f^m x \to p$. Since n, x, p, λ are fixed and $v_m \ll c$ for each interior point c of the cone P, Remark 1.3.(5) implies estimate (3.4) and the theorem is proved. □

Remark 3.4. Taking $E = \mathbb{R}$, $P = [0, +\infty[$, $\|\cdot\| = |\cdot|$ in Theorem 3.3 we obtain a proper generalization of the Banach Contraction Principle (including the estimate, as a crucial part), and also a supplement of the Palais' proof of the same principle in the setting of cone metric spaces.

3.2. **Fixed points of D-quasicontractions.** In this subsection we shall prove a theorem on fixed points of D-quasicontractions defined on a cone metric space with a normal cone.

Theorem 3.5. *Let (X, d) be a complete cone metric space over a normal cone P, with the normal constant $K \geq 1$, and let $f : X \to X$. If for some $\lambda \in (0, 1/K^2)$ and each $x, y \in X$,*

$$D(fx, fy) \leq \lambda \cdot \max\{D(x, y), D(x, fx), D(y, fy), D(x, fy), D(y, fx)\},$$

then f has a unique fixed point, say p. Moreover, for each $x \in X$, the sequence $\{f^n x\}$ of Picard iterations converges to p and for each $n \in \mathbb{N}$ the estimate

$$D(f^n x, p) \leq \frac{\lambda^n K}{1-\lambda K} D(x, fx).$$

holds.

In order to prove the theorem we shall need some lemmas. The orbit of a selfmap $f : X \to X$ at a point $x \in X$ shall be denoted as $O_f(x;\infty) = \{f^n x : n = 0, 1, 2, \dots\}$. $O_f(x;n)$ shall stand for $\{x, fx, f^2x, \dots, f^n x\}$.

Lemma 3.6. *Let f be a D-quasicontraction on a cone metric space (X, d) with a normal cone P having the normal constant $K \geq 1$, and let $\lambda < 1/K$. If $O_f(x;\infty)$ is the orbit of f at a point $x \in X$, then*

$$\operatorname{diam} O_f(x;\infty) \leq \frac{K}{1-\lambda K} D(x, fx). \tag{3.6}$$

Proof. Observe first that for $1 \leq i, j \leq n$,

$$\begin{aligned} D(f^i x, f^j x) &= D(f f^{i-1}x, f f^{j-1}x) \\ &\leq \lambda \cdot \max\{D(f^{i-1}x, f^{j-1}x), D(f^{i-1}x, f^i x), \\ &\quad D(f^{j-1}x, f^j x), D(f^{i-1}x, f^j x), D(f^{j-1}x, f^j x)\}. \end{aligned}$$

Since $f^{i-1}x, f^i x, f^{j-1}x, f^j x \in O_f(x;n)$ we have for $1 \leq i, j \leq n$

$$D(f^i x, f^j x) \leq \lambda \cdot \operatorname{diam} O_f(x;n) < \operatorname{diam} O_f(x;n).$$

Therefore, for some $k \leq n$, $\operatorname{diam} O_f(x;n) = D(x, f^k x)$. From the inequality

$$d(x, f^k x) \leq d(x, fx) + d(fx, f^k x),$$

using normality of the cone, it follows that

$$\|d(x, f^k x)\| \leq K(\|d(x, fx) + d(fx, f^k x)\|),$$

i.e.,

$$\begin{aligned} D(x, f^k x) &\leq KD(x, fx) + KD(fx, f^k x), \quad \text{and so} \\ \operatorname{diam} O_f(x;n) &\leq KD(x, fx) + K\lambda \cdot \operatorname{diam} O_f(x;n). \end{aligned}$$

Hence, $\operatorname{diam} O_f(x;n) \leq \frac{K}{1-\lambda K} D(x, fx)$, wherefrom, passing to the supremum in n, (3.6) follows. ☐

Observe that it follows from the given proof that a D-quasicontraction has a bounded orbit at each point of a cone metric space, which is consistent with the results from [4, 9, 11] for metric and cone metric spaces.

Lemma 3.7. *The sequence $\{f^n x\}$ of Picard iterations of a D-quasicontraction defined on a cone metric space (X, d) with a normal cone (for an arbitrary point x) is a Cauchy sequence in that space, provided that $\lambda < 1/K$.*

Proof. Let $m, n \in \mathbb{N}$, $m > n$. Then

$$\begin{aligned} D(f^n x, f^m x) &= D(f f^{n-1}x, f^{m-n+1} f^{n-1}x) \\ &\leq \lambda \cdot \operatorname{diam}\{f^{n-1}x, f^n x, \dots, f^{m-n+1}x\} \\ &< \operatorname{diam}\{f^{n-1}x, f^n x, \dots, f^{m-n+1}x\} \\ &= D(f^{n-1}x, f^{k_1} f^{n-1}x), \end{aligned}$$

for some $k_1 \le m - n + 1$. Furthermore,

$$\begin{aligned} D(f^{n-1}x, f^{k_1} f^{n-1}x) &= D(f f^{n-2}x, f^{k_1+1} f^{n-2}x) \\ &\le \lambda \cdot \operatorname{diam}\{f^{n-2}x, f^{n-1}x, \dots, f^{(n-2)+k_1+1}x\} \\ &\le \lambda \cdot \operatorname{diam}\{f^{n-2}x, f^{n-1}x, \dots, f^{(n-2)+m-n+2}x\}. \end{aligned}$$

Now

$$\begin{aligned} D(f^n x, f^m x) &\le \lambda \cdot \operatorname{diam}\{f^{n-1}x, f^n x, \dots, f^{m-n+1}x\} \\ &\le \lambda^2 \cdot \operatorname{diam}\{f^{n-2}x, f^{n-1}x, \dots, f^{(n-2)+m-n+2}x\}. \end{aligned}$$

In a similar way we obtain the inequality

$$\begin{aligned} D(f^n x, f^m x) &\le \lambda \cdot \operatorname{diam}\{f^{n-1}x, f^n x, \dots, f^{m-n+1}x\} \\ &\le \dots \le \lambda^n \operatorname{diam}\{x, fx, \dots, f^m x\}. \end{aligned}$$

Using Lemma 3.6 we obtain that

$$D(f^n x, f^m x) \le \frac{\lambda^n K}{1 - \lambda K} D(x, fx),$$

which means that the sequence $\{f^n x\}$ of Picard iterations is a D-Cauchy sequence, and so also a d-Cauchy sequence by Theorem 2.7. □

Lemma 3.8. *If for some point x of a cone metric space (X, d) with a normal cone the sequence $\{f^n x\}$ of Picard iterations of a D-quasi-contraction $f : X \to X$ converges to a point $p \in X$, then p is its fixed point, provided that $\lambda < 1/K^2$.*

Proof. For the limit point p and its image fp we have

$$\begin{aligned} D(p, fp) &\le K D(p, f^{n+1}x) + K D(f^{n+1}x, fp) \\ &\le K D(p, f^{n+1}x) + \lambda K \max\{D(f^n x, p), D(f^n x, f^{n+1}x), \\ &\quad D(p, fp), D(f^n x, fp), D(p, f^{n+1}x)\}, \end{aligned}$$

Passing to the limit when $n \to \infty$ we obtain that

$$D(p, fp) \le K \cdot 0 + \lambda \cdot K \max\{0, 0, D(p, fp), K \cdot (0 + D(p, fp)), 0\} = \lambda \cdot K^2 D(p, fp).$$

It follows that $fp = p$. □

Proof of Theorem 3.5.

Let $x \in X$ be an arbitrary point. By Lemma 3.7, the sequence $\{f^n x\}$ of Picard iterations is a D-Cauchy sequence, and so also a d-Cauchy sequence. Since (X, d) is a complete cone metric space, this sequence converges to a point $p \in X$. According to Lemma 3.8, the point p is a fixed point of the D-quasi-contraction f. By Lemma 3.6,

$$D(f^n x, f^m x) \le \frac{\lambda^n K}{1 - \lambda K} D(x, fx),$$

wherefrom, passing to the limit when $m \to \infty$ (and using that D is a continuous function in each of its variables, i.e., that it satisfies the axiom (H_E)) we obtain that

$$D(f^n x, p) \le \frac{\lambda^n K}{1 - \lambda K} D(x, fx).$$

This proves the stated estimate.

We conclude with an example of a D-quasicontraction which is not a D-contraction.

Example 3.9. Let $X = \{1, 2, 3\}$, $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\}$ and $f : X \to X$ with $f1 = 2$, $f2 = 2$, $f3 = 1$. Furthermore, let $d : X \times X \to E$ with $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) = (1, 1)$; $d(1, 3) = d(3, 1) = (2, 2)$ and $d(x, x) = 0$ for $x \in X$. (X, d) is a cone metric space. By a careful calculations one can get that f is not a D-contraction. On the other hand, for $\frac{1}{2} \leq \lambda < 1$, f is a D-quasicontraction and all the conditions of Theorem 3.5 (which is a generalization of Theorem 3.2) are fulfilled. The point $x = 2$ is the fixed point of the map f.

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