



## ON A HILBERT-TYPE INTEGRAL INEQUALITY IN THE SUBINTERVAL AND ITS OPERATOR EXPRESSION

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**ABSTRACT.** In this paper, by using the methods of real analysis and functional analysis, a Hilbert-type integral inequality in the subinterval  $(a, \infty)$  ( $a > 0$ ) with the homogeneous kernel of  $-\lambda$ -degree and a best constant factor and its operator expression are given. As applications, a few improved results, the equivalent forms and some new inequalities with the particular kernels are obtained.

### 1. INTRODUCTION

If  $f, g \geq 0$ ,  $f, g \in L^2(0, \infty)$ ,  $\|f\| = \{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}}$  and  $\|g\| = \{\int_0^\infty g^2(x)dx\}^{\frac{1}{2}}$ , then we have the following Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \pi \|f\| \cdot \|g\|, \quad (1.1)$$

where the constant factor  $\pi$  is the best possible. Inequality (1.1) is important in analysis and its applications (cf. [2, 5]). Define an integral operator  $T : L^2(0, \infty) \rightarrow L^2(0, \infty)$  as: for  $f(\geq 0) \in L^2(0, \infty)$ ,

$$T(f)(y) := \int_0^\infty \frac{f(x)}{x+y} dx (y \in (0, \infty)). \quad (1.2)$$

Then inequality (1.1) is rewritten as:  $(Tf, g) \leq \pi \|f\| \cdot \|g\|$ , where  $(Tf, g) := \int_0^\infty (\int_0^\infty \frac{f(x)}{x+y} dx) g(y) dy$  is the inner product of  $Tf$  and  $g$ . We named of  $T$  Hilbert

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integral operator. By (1.1), we can prove the equivalent form that  $\|Tf\| \leq \pi\|f\|$ , and conclude that  $\|T\| = \pi$ ; [1].

If we replace  $\frac{1}{x+y}$  by a bilinear function  $k(x, y)(\geq 0)$  in (1.1), then the problem is how to make sure the conditions of  $k(x, y)$  for giving an integral operator  $T$  as (1.2) and the inequality with a best constant factor as (1.1). In recent years, Yang [7, 8] considered the case of  $k(x, y)$  being continuous and symmetric in the function space  $L^p(0, \infty)$ , Yang [9, 10, 11] considered the same case of  $k(x, y)$  in the disperse space  $l^p$ , and Zhong et al. [18] considered the case of  $k(x, y)$  in  $L^p(R_+^n)$ . But their given conditions are not quite simple.

In 1998, by introducing  $\lambda \in (0, 1]$  and the Beta function  $B(u, v)$  as[6]:

$$B(u, v) := \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{-u+1} dt (u, v > 0), \tag{1.3}$$

Yang [12] gave an extension of (1.1) in the subinterval  $(a, \infty)(a > 0)$  as:

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)dxdy}{(x+y)^\lambda} \leq k_\lambda \left\{ \int_a^\infty \sigma(x)x^{1-\lambda} f^2(x)dx \int_a^\infty \sigma(x)x^{1-\lambda} g^2(x)dx \right\}^{\frac{1}{2}} \tag{1.4}$$

where  $k_\lambda = B(\frac{\lambda}{2}, \frac{\lambda}{2})$  and  $\sigma(x) = 1 - \frac{1}{2}(\frac{a}{x})^{\frac{\lambda}{2}}$ . When  $\lambda = 1, a \rightarrow 0^+$ , inequality (1.4) deduces to (1.1). In recent years, a number of papers studied some improvements and extensions of (1.4) (cf. [13, 14, 15, 17]).

In this paper, a simple condition of the homogeneous kernel with  $-\lambda$ -degree ( $\lambda > 0$ ) is considered. By using the methods of real analysis and functional analysis, a Hilbert-type integral inequality in the subinterval  $(a, \infty)$  with the homogeneous kernel and a best constant factor and its operator expression are given. As applications, a few improved results, the equivalent forms and some new inequalities with the particular kernels are obtained.

## 2. LEMMAS AND MAIN RESULTS

If  $\lambda > 0$ , the function  $k_\lambda(x, y)$  is non-negative measurable in  $(0, \infty) \times (0, \infty)$ , satisfying  $k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y)$  for any  $u, x, y > 0$ , then we call  $k_\lambda(x, y)$  the homogeneous function of  $-\lambda$ -degree; if for any  $x, y > 0, k_\lambda(x, y) = k_\lambda(y, x)$ , then we call the homogeneous function  $k_\lambda(x, y)$  is symmetric. Assume that  $r > 1, \frac{1}{r} + \frac{1}{s} = 1$ . Setting  $k_\lambda(r)$  and  $\tilde{k}_\lambda(s)$  as

$$k_\lambda(r) := \int_0^\infty k_\lambda(u, 1)u^{\frac{\lambda}{r}-1} du, \tilde{k}_\lambda(s) := \int_0^\infty k_\lambda(1, u)u^{\frac{\lambda}{s}-1} du,$$

then it follows  $k_\lambda(r) = \tilde{k}_\lambda(s)$ . In fact, setting  $v = \frac{1}{u}$ , we obtain

$$\tilde{k}_\lambda(s) = \int_0^\infty k_\lambda(1, \frac{1}{v})v^{\frac{-\lambda}{s}+1} \frac{dv}{v^2} = \int_0^\infty k_\lambda(v, 1)v^{\frac{\lambda}{r}-1} dv = k_\lambda(r).$$

Suppose that  $k_\lambda(r)$  is a positive number. For  $a > 0, x, y \in (a, \infty)$ , define the weight functions  $\omega_\lambda(r, y, a)$  and  $\varpi_\lambda(s, x, a)$  as:

$$\omega_\lambda(r, y, a) := \int_a^\infty k_\lambda(x, y) \frac{y^{\frac{\lambda}{s}}}{x^{1-\frac{\lambda}{r}}} dx, \varpi_\lambda(s, x, a) := \int_a^\infty k_\lambda(x, y) \frac{x^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{s}}} dy. \tag{2.1}$$

Setting  $u = \frac{y}{x}$  in the integral  $\omega_\lambda(r, y, a)$ , for any  $y \in (a, \infty)$ , we find

$$\omega_\lambda(r, y, a) = \int_0^{\frac{y}{a}} k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du \leq \int_0^\infty k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du = \tilde{k}_\lambda(s).$$

Similarly,  $\varpi_\lambda(s, x, a) \leq k_\lambda(r)$  ( $x \in (a, \infty)$ ). Setting  $\theta_\lambda(r)$  and  $\tilde{\theta}_\lambda(s)$  as

$$\theta_\lambda(r) := \int_0^1 k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du, \tilde{\theta}_\lambda(s) := \int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du,$$

if  $\theta_\lambda(r), \tilde{\theta}_\lambda(s) > 0$ , then for any  $y > a$ , we find

$$\omega_\lambda(r, y, a) = \int_0^{\frac{y}{a}} k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du \geq \int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s}-1} du = \tilde{\theta}_\lambda(s) > 0.$$

Similarly,  $\varpi_\lambda(s, x, a) \geq \theta_\lambda(r) > 0$  ( $x > a$ ). Hence by (2.1), for fixed  $y > a$ ,  $k_\lambda(x, y) > 0$  a.e. in  $(a, \infty)$ , and for fixed  $x > a$ ,  $k_\lambda(x, y) > 0$  a.e. in  $(a, \infty)$ .

**Lemma 2.1.** *If both  $k_\lambda(1, u), k_\lambda(u, 1) \geq l_\lambda > 0, u \in (0, 1]$ , then we have*

$$\omega_\lambda(r, y, a) \leq k_\lambda(r) \left[ 1 - \frac{rl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right] \quad (y \in (a, \infty)); \quad (2.2)$$

$$\varpi_\lambda(s, x, a) \leq k_\lambda(r) \left[ 1 - \frac{sl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \quad (x \in (a, \infty)). \quad (2.3)$$

*Proof.* Setting  $u = \frac{x}{y}$ , we find

$$\begin{aligned} \omega_\lambda(r, y, a) &= \int_{\frac{a}{y}}^\infty k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du = k_\lambda(r) - \int_0^{\frac{a}{y}} k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du \\ &\leq k_\lambda(r) - l_\lambda \int_0^{\frac{a}{y}} u^{\frac{\lambda}{r}-1} du = k_\lambda(r) - \frac{rl_\lambda}{\lambda} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \quad (y \in (a, \infty)). \end{aligned}$$

Hence we have (2.2). Similarly we have (2.3). The lemma is proved.  $\square$

**Lemma 2.2.** *If both  $k_\lambda(1, u)$  and  $k_\lambda(u, 1)$  are derivable decreasing function in  $(0, 1]$ , then we have*

$$\omega_\lambda(r, y, a) \leq k_\lambda(r) \left[ 1 - \frac{\theta_\lambda(r)}{k_\lambda(r)} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right] \quad (y \in (a, \infty)); \quad (2.4)$$

$$\varpi_\lambda(s, x, a) \leq k_\lambda(r) \left[ 1 - \frac{\tilde{\theta}_\lambda(s)}{k_\lambda(r)} \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \quad (x \in (a, \infty)). \quad (2.5)$$

*In particular, if  $k_\lambda(x, y)$  is symmetric, setting  $k_\lambda := k_\lambda(2)$ , then*

$$\omega_\lambda(2, y, a) \leq k_\lambda \left[ 1 - \frac{1}{2} \left( \frac{a}{y} \right)^{\frac{\lambda}{2}} \right]; \varpi_\lambda(2, x, a) \leq k_\lambda \left[ 1 - \frac{1}{2} \left( \frac{a}{x} \right)^{\frac{\lambda}{2}} \right] \quad (x, y > a). \quad (2.6)$$

*Proof.* Since  $k'_\lambda(u, 1) \leq 0, u \in (0, 1)$ , for  $y \in (0, 1)$ , we obtain

$$\begin{aligned} \frac{d}{dy} \left[ y^{-\frac{\lambda}{r}} \int_0^y k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du \right] &= \frac{-\lambda}{r} y^{-\frac{\lambda}{r}} \int_0^y k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du + k_\lambda(y, 1) y^{-1} \\ &= -y^{-\frac{\lambda}{r}} \int_0^y k_\lambda(u, 1) du^{\frac{\lambda}{r}} + k_\lambda(y, 1) y^{-1} = y^{-\frac{\lambda}{r}} \int_0^y k'_\lambda(u, 1) u^{\frac{\lambda}{r}} du \leq 0, \end{aligned}$$

and  $y^{-\frac{\lambda}{r}} \int_0^y k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du \geq \int_0^1 k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du = \theta_\lambda(r)$ . Hence we find

$$\begin{aligned} \omega_\lambda(r, y, a) &= k_\lambda(r) - \left[ \left( \frac{a}{y} \right)^{-\frac{\lambda}{r}} \int_0^{\frac{a}{y}} k_\lambda(u, 1) u^{\frac{\lambda}{r}-1} du \right] \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \\ &\leq k_\lambda(r) - \theta_\lambda(r) \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \quad (y \in (a, \infty)). \end{aligned}$$

Then we obtain (2.4). Similarly, we obtain (2.5). If  $k_\lambda(x, y)$  is symmetric, then we find  $\theta_\lambda(2) = \tilde{\theta}_\lambda(2)$  and

$$k_\lambda = \theta_\lambda(2) + \int_1^\infty k_\lambda(1, u) u^{\frac{\lambda}{2}-1} du = \theta_\lambda(2) + \int_0^1 k_\lambda(v, 1) v^{\frac{\lambda}{2}-1} dv = 2\theta_\lambda(2).$$

Then by (2.4) and (2.5), we have (2.6). The lemma is proved.  $\square$

For the measurable function  $\varphi(x) > 0$ , set the function spaces as:

$$L_\varphi^\rho(a, \infty) := \{h \geq 0; \|h\|_{\rho, \varphi} := \left\{ \int_a^\infty \varphi(x) h^\rho(x) dx \right\}^{\frac{1}{\rho}} < \infty\} (\rho = p, q).$$

**Theorem 2.3.** Assume that  $p, r > 1, \frac{1}{p} + \frac{1}{q} = 1, \frac{1}{r} + \frac{1}{s} = 1, \lambda > 0, k_\lambda(x, y)$  is a homogeneous function of  $-\lambda$ -degree in  $(0, \infty) \times (0, \infty)$ , satisfying  $k_\lambda(r), \theta_\lambda(r)$  and  $\tilde{\theta}_\lambda(s)$  are positive numbers. For  $a > 0$ , there exist measurable functions  $\kappa(y)$  and  $\tilde{\mu}(x)$ , such that  $0 < \kappa(y), \tilde{\mu}(x) \leq 1$  and

$$\omega_\lambda(r, y, a) \leq k_\lambda(r) \kappa(y), \varpi_\lambda(s, x, a) \leq k_\lambda(r) \tilde{\mu}(x) \quad (x, y \in (a, \infty)). \quad (2.7)$$

If  $\phi_r(x) := x^{p(1-\frac{\lambda}{r})-1}, \psi_s(x) := x^{q(1-\frac{\lambda}{s})-1} (x \in (a, \infty)), f \in L_{\phi_r}^p(a, \infty), g \in L_{\psi_s}^q(a, \infty), \|f\|_{p, \phi_r}, \|g\|_{q, \psi_s} > 0$ , then we have the equivalent inequalities as

$$I_\lambda(a) := \int_a^\infty \int_a^\infty k_\lambda(x, y) f(x) g(y) dx dy < k_\lambda(r) \|f\|_{p, \tilde{\mu} \cdot \phi_r} \|g\|_{q, \kappa \cdot \psi_s}; \quad (2.8)$$

$$J_\lambda(a) := \int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)} \left( \int_a^\infty k_\lambda(x, y) f(x) dx \right)^p dy < k_\lambda^p(r) \|f\|_{p, \tilde{\mu} \cdot \phi_r}^p, \quad (2.9)$$

where the constant factors  $k_\lambda(r)$  and  $k_\lambda^p(r)$  are the best possible.

In particular, for  $\kappa(y) = \tilde{\mu}(x) = 1$ , we have the equivalent inequalities as:

$$I_\lambda(a) < k_\lambda(r) \left\{ \int_a^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y) dy \right\}^{\frac{1}{q}}; \quad (2.10)$$

$$\int_a^\infty y^{\frac{p\lambda}{s}-1} \left( \int_a^\infty k_\lambda(x, y) f(x) dx \right)^p dy < k_\lambda^p(r) \int_a^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx. \quad (2.11)$$

*Proof.* Since  $0 < \theta_\lambda(r)/k_\lambda(r) \leq \kappa(y) \leq k_\lambda(r), 0 < \tilde{\theta}_\lambda(s)/k_\lambda(r) \leq \tilde{\mu}(x) \leq k_\lambda(r)$  ( $x, y \in (a, \infty)$ ), it is obvious that the condition  $0 < \|f\|_{p, \phi_r}, \|g\|_{q, \psi_s} < \infty$  is equivalent to the condition  $0 < \|f\|_{p, \tilde{\mu} \cdot \phi_r}, \|g\|_{q, \kappa \cdot \psi_s} < \infty$ .

By Hölder's inequality [3], in view of (2.1) and Fubini's theorem [4], we find

$$I_\lambda(a) = \int_a^\infty \int_a^\infty k_\lambda(x, y) \left[ \frac{x^{(1-\frac{\lambda}{r})/q}}{y^{(1-\frac{\lambda}{s})/p}} f(x) \right] \left[ \frac{y^{(1-\frac{\lambda}{s})/p}}{x^{(1-\frac{\lambda}{r})/q}} g(y) \right] dx dy$$

$$\begin{aligned}
&\leq \left\{ \int_a^\infty \int_a^\infty k_\lambda(x, y) \frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}} f^p(x) dx dy \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \int_a^\infty \int_a^\infty k_\lambda(x, y) \frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}} g^q(y) dx dy \right\}^{\frac{1}{q}} \\
&= \left\{ \int_a^\infty \varpi_\lambda(s, x, a) \phi_r(x) f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^\infty \omega_\lambda(r, y, a) \psi_s(y) g^q(y) dy \right\}^{\frac{1}{q}}. \quad (2.12)
\end{aligned}$$

If inequality (2.12) keeps the form of equality, then [3] there exist constants  $A$  and  $B$ , such that they are not all zero and

$$A \frac{x^{(1-\frac{\lambda}{r})(p-1)}}{y^{1-\frac{\lambda}{s}}} f^p(x) = B \frac{y^{(1-\frac{\lambda}{s})(q-1)}}{x^{1-\frac{\lambda}{r}}} g^q(y) \quad a.e. \text{ in } (a, \infty) \times (a, \infty).$$

It follows  $Ax^{p(1-\frac{\lambda}{r})} f^p(x) = By^{q(1-\frac{\lambda}{s})} g^q(y)$  a.e. in  $(a, \infty) \times (a, \infty)$ . Assuming that  $A \neq 0$ , there exists  $y > a$ ,  $x^{p(1-\frac{\lambda}{r})-1} f^p(x) = [By^{q(1-\frac{\lambda}{s})} g^q(y)] \frac{1}{Ax}$  a.e. in  $x \in (a, \infty)$ . This contradicts the fact that  $0 < \|f\|_{p, \phi_r} < \infty$ . Then inequality (2.12) keeps the strict form and inequality (2.12) is valid by using (2.7).

For  $x \in (a, \infty)$ , setting a bounded measurable function  $[f(x)]_n$  as

$$[f(x)]_n := \min\{f(x), n\} = \begin{cases} f(x), & \text{for } f(x) \leq n \\ n, & \text{for } f(x) > n, \end{cases}$$

since  $\|f\|_{p, \phi_r} > 0$ , there exists  $n_0 \in N$ , such that  $\int_a^n \phi_r(x) [f(x)]_n^p dx > 0$  ( $n \geq n_0$ ), and then  $\int_a^n \tilde{\mu}(x) \phi_r(x) [f(x)]_n^p dx > 0$ . Setting  $\tilde{g}_n(y)$  as

$$\tilde{g}_n(y) := \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)} \left( \int_a^n k_\lambda(x, y) [f(x)]_n dx \right)^{p-1} \quad (y \in (a, n); n \geq n_0),$$

then by (2.8), we have

$$\begin{aligned}
0 &< \int_a^n \kappa(y) \psi_s(y) \tilde{g}_n^q(y) dy = \int_a^n \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)} \left( \int_a^n k_\lambda(x, y) [f(x)]_n dx \right)^p dy \\
&= \int_a^n \int_a^n k_\lambda(x, y) [f(x)]_n \tilde{g}_n(y) dx dy \\
&< k_\lambda(r) \left\{ \int_a^n \tilde{\mu}(x) \phi_r(x) [f(x)]_n^p dx \right\}^{\frac{1}{p}} \left\{ \int_a^n \kappa(y) \psi_s(y) \tilde{g}_n^q(y) dy \right\}^{\frac{1}{q}} < \infty; \quad (2.13)
\end{aligned}$$

$$0 < \int_a^n \kappa(y) \psi_s(y) \tilde{g}_n^q(y) dy < k_\lambda^p(r) \int_a^\infty \tilde{\mu}(x) \phi_r(x) f^p(x) dx < \infty. \quad (2.14)$$

It follows  $0 < \|g\|_{q, \kappa \cdot \psi_s} < \infty$  and  $0 < \|g\|_{q, \psi_s} < \infty$ . For  $n \rightarrow \infty$ , by (2.8), both (2.13) and (2.14) still keep the forms of strict inequality. Hence we have (2.9).

On the other-hand, suppose that (2.9) is valid. By Hölder's inequality,

$$\begin{aligned}
J_\lambda(a) &= \int_a^\infty [y^{\frac{-1}{p} + \frac{\lambda}{s}} \kappa^{\frac{-1}{q}}(y)] \int_a^\infty k_\lambda(x, y) f(x) dx [\kappa^{\frac{1}{q}}(y) y^{\frac{1}{p} - \frac{\lambda}{s}} g(y)] dy \\
&\leq \left\{ \int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{\kappa^{p-1}(y)} \left( \int_a^\infty k_\lambda(x, y) f(x) dx \right)^p dy \right\}^{\frac{1}{p}} \left\{ \int_a^\infty \kappa(y) \psi_s(y) g^q(y) dy \right\}^{\frac{1}{q}}. \quad (2.15)
\end{aligned}$$

In view of (2.9), we have (2.8). Hence (2.8) is equivalent to (2.9).

For  $n \in N, n > \max\{\frac{\lambda}{r}, \frac{\lambda}{s}\}$ , setting  $f_n, g_n$  as:  $f_n(x) = x^{\frac{\lambda}{r} - \frac{1}{np} - 1}, g_n(x) = x^{\frac{\lambda}{s} - \frac{1}{nq} - 1}$ , for  $x \in (a, \infty)$ , if there exists  $0 < K \leq k_\lambda(r) = \tilde{k}_\lambda(s)$ , such that (2.8) is still valid if we replace  $k_\lambda(r)$  by  $K$ , then we have

$$I_\lambda^{(n)}(a) := \int_a^\infty \int_a^\infty k_\lambda(x, y) f_n(x) g_n(y) dx dy < K \|f_n\|_{p, \phi_r} \|g_n\|_{q, \psi_s} = \frac{nK}{a^{\frac{1}{n}}}; \quad (2.16)$$

$$\begin{aligned} I_\lambda^{(n)}(a) &= \int_a^\infty \left[ \int_a^\infty k_\lambda(x, y) x^{\frac{\lambda}{r} - \frac{1}{np} - 1} y^{\frac{\lambda}{s} - \frac{1}{nq} - 1} dx \right] dy \\ &\stackrel{u=y/x}{=} \int_a^\infty y^{-1 - \frac{1}{n}} \left[ \int_0^{\frac{y}{a}} k_\lambda(1, u) u^{\frac{\lambda}{s} + \frac{1}{np} - 1} du \right] dy \\ &= \frac{n}{a^{\frac{1}{n}}} \int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s} + \frac{1}{np} - 1} du + \int_a^\infty y^{-1 - \frac{1}{n}} \int_1^{\frac{y}{a}} k_\lambda(1, u) u^{\frac{\lambda}{s} + \frac{1}{np} - 1} du dy \\ &= \frac{n}{a^{\frac{1}{n}}} \int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s} + \frac{1}{np} - 1} du + \int_1^\infty \left( \int_{au}^\infty y^{-1 - \frac{1}{n}} dy \right) k_\lambda(1, u) u^{\frac{\lambda}{s} + \frac{1}{np} - 1} du \\ &= \frac{n}{a^{\frac{1}{n}}} \left[ \int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s} + \frac{1}{np} - 1} du + \int_1^\infty k_\lambda(1, u) u^{\frac{\lambda}{s} - \frac{1}{nq} - 1} du \right]. \end{aligned} \quad (2.17)$$

Hence by (2.16) and (2.17), we have

$$\int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s} + \frac{1}{np} - 1} du + \int_1^\infty k_\lambda(1, u) u^{\frac{\lambda}{s} - \frac{1}{nq} - 1} du < K,$$

and by Fatou's lemma [4], it follows

$$\begin{aligned} \tilde{k}_\lambda(s) &= \int_0^1 \lim_{n \rightarrow \infty} k_\lambda(1, u) u^{\frac{\lambda}{s} + \frac{1}{np} - 1} du + \int_1^\infty \lim_{n \rightarrow \infty} k_\lambda(1, u) u^{\frac{\lambda}{s} - \frac{1}{nq} - 1} du \\ &\leq \frac{\lim_{n \rightarrow \infty}}{n} \left[ \int_0^1 k_\lambda(1, u) u^{\frac{\lambda}{s} + \frac{1}{np} - 1} du + \int_1^\infty k_\lambda(1, u) u^{\frac{\lambda}{s} - \frac{1}{nq} - 1} du \right] \leq K. \end{aligned}$$

Therefore  $K = k_\lambda(r)$  is the best constant factor of (2.8). If the constant factor in (2.9) is not the best possible, then by (2.15), we can get a contradiction that the constant factor in (2.8) is not the best possible. The theorem is proved.  $\square$

Define an operator  $T_a : L_{\phi_r}^p(a, \infty) \rightarrow L_{\psi_s^{1-p}}^p(a, \infty)$  as: for  $f \in L_{\phi_r}^p(a, \infty)$ ,

$$(T_a f)(y) := \int_a^\infty k_\lambda(x, y) f(x) dx \quad (y \in (a, \infty)).$$

In view of (2.11), it follows  $T_a f \in L_{\psi_s^{1-p}}^p(a, \infty)$ . For  $g \in L_{\psi_s^q}^q(a, \infty)$ , define the formal inner of  $T_a f$  and  $g$  as:

$$(T_a f, g) := \int_a^\infty \int_a^\infty k_\lambda(x, y) f(x) g(y) dx dy.$$

Hence the equivalent inequalities (2.10) and (2.11) may be rewritten as

$$(T_a f, g) < k_\lambda(r) \|f\|_{p, \phi_r} \|g\|_{q, \psi_s}; \|T_a f\|_{p, \psi_s^{1-p}} < k_\lambda(r) \|f\|_{p, \phi_r},$$

where the constant factor is the best possible,  $T_a$  is obviously bounded and  $\|T_a\| = k_\lambda(r)$ . We call  $T_a$  Hilbert-type integral operator with the homogeneous kernel of  $-\lambda$ -degree in the subinterval  $(a, \infty)$ .

**Corollary 2.4.** *Let the assumptions of Theorem 2.3 be fulfilled and additionally both  $k_\lambda(1, u), k_\lambda(u, 1) \geq l_\lambda > 0$ ,  $u \in (0, 1]$ . Then we have the following equivalent inequalities:*

$$(T_a f, g) < k_\lambda(r) \left\{ \int_a^\infty \left[ 1 - \frac{sl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \phi_r(x) f^p(x) dx \right\}^{\frac{1}{p}} \\ \times \left\{ \int_a^\infty \left[ 1 - \frac{rl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right] \psi_s(y) g^q(y) dy \right\}^{\frac{1}{q}}; \quad (2.18)$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{\left[ 1 - \frac{rl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right]^{p-1}} \left( \int_a^\infty k_\lambda(x, y) f(x) dx \right)^p dy \\ < k_\lambda^p(r) \int_a^\infty \left[ 1 - \frac{sl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \phi_r(x) f^p(x) dx, \quad (2.19)$$

where the constant factors  $k_\lambda(r)$  and  $k_\lambda^p(r)$  are the best possible. We still have the following two pairs of equivalent inequalities:

$$(T_a f, g) < k_\lambda(r) \left\{ \int_a^\infty \left[ 1 - \frac{sl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \phi_r(x) f^p(x) dx \right\}^{\frac{1}{p}} \|g\|_{q, \psi_s}, \quad (2.20)$$

$$\|T_a f\|_{p, \psi_s^{1-p}}^p < k_\lambda^p(r) \int_a^\infty \left[ 1 - \frac{sl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \phi_r(x) f^p(x) dx; \quad (2.21)$$

$$(T_a f, g) < k_\lambda(r) \|f\|_{p, \phi_r} \left\{ \int_a^\infty \left[ 1 - \frac{rl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right] \psi_s(y) g^q(y) dy \right\}^{\frac{1}{q}}, \quad (2.22)$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{\left[ 1 - \frac{rl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right]^{p-1}} \left( \int_a^\infty k_\lambda(x, y) f(x) dx \right)^p dy < k_\lambda^p(r) \|f\|_{p, \phi_r}^p. \quad (2.23)$$

*Proof.* By Lemma 2.1, setting  $\kappa(y) = 1 - \frac{rl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}}$  and  $\tilde{\mu}(x) = 1 - \frac{sl_\lambda}{\lambda k_\lambda(r)} \left( \frac{a}{x} \right)^{\frac{\lambda}{s}}$  in (2.8) and (2.9), we have (2.18) and (2.19). Since  $\kappa(y) \leq 1$ , by (2.18) and (2.19), we have (2.20) and (2.21). Similarly, since  $\tilde{\mu}(x) \leq 1$ , we have (2.22) and (2.23). The corollary is proved.  $\square$

**Corollary 2.5.** *Let the assumptions of Theorem 2.3 be fulfilled and additionally both  $k_\lambda(1, u)$  and  $k_\lambda(u, 1)$  are derivable decreasing function in  $(0, 1]$ . Then we have the following equivalent inequalities with the best constant factors:*

$$(T_a f, g) < k_\lambda(r) \left\{ \int_a^\infty \left[ 1 - \frac{\tilde{\theta}_\lambda(s)}{k_\lambda(r)} \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \phi_r(x) f^p(x) dx \right\}^{\frac{1}{p}} \\ \times \left\{ \int_a^\infty \left[ 1 - \frac{\theta_\lambda(r)}{k_\lambda(r)} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right] \psi_s(y) g^q(y) dy \right\}^{\frac{1}{q}}; \quad (2.24)$$

$$\begin{aligned} & \int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{\left[1 - \frac{\theta_\lambda(r)}{k_\lambda(r)}\left(\frac{a}{y}\right)^{\frac{\lambda}{r}}\right]^{p-1}} \left(\int_a^\infty k_\lambda(x, y)f(x)dx\right)^p dy \\ & < k_\lambda^p(r) \int_a^\infty \left[1 - \frac{\tilde{\theta}_\lambda(s)}{k_\lambda(r)}\left(\frac{a}{x}\right)^{\frac{\lambda}{s}}\right] \phi_r(x) f^p(x) dx. \end{aligned} \quad (2.25)$$

If  $k_\lambda(x, y)$  is symmetric and  $\sigma(x) = 1 - \frac{1}{2}\left(\frac{a}{x}\right)^{\frac{\lambda}{2}}$  as (1.4), then we have the equivalent inequalities as:

$$I_\lambda(a) < k_\lambda \left\{ \int_a^\infty \sigma(x) \phi_2(x) f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^\infty \sigma(y) \psi_2(y) g^q(y) dy \right\}^{\frac{1}{q}}; \quad (2.26)$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{2}-1}}{[\sigma(y)]^{p-1}} \left(\int_a^\infty k_\lambda(x, y) f(x) dx\right)^p dy < k_\lambda^p \int_a^\infty \sigma(x) \phi_2(x) f^p(x) dx. \quad (2.27)$$

*Proof.* By Lemma 2.2, setting  $\kappa(y) = 1 - \frac{\theta_\lambda(r)}{k_\lambda(r)}\left(\frac{a}{y}\right)^{\frac{\lambda}{r}}$  and  $\tilde{\mu}(x) = 1 - \frac{\tilde{\theta}_\lambda(s)}{k_\lambda(r)}\left(\frac{a}{x}\right)^{\frac{\lambda}{s}}$  in (2.8) and (2.9), we have (2.24) and (2.25). For  $r = s = 2$ , by (2.6), we have (2.26) and (2.27). The corollary is proved.  $\square$

### 3. APPLICATIONS TO SOME PARTICULAR KERNELS

In the following, we assume that  $a, \lambda > 0, p, r > 1, \frac{1}{p} + \frac{1}{q} = 1, \frac{1}{r} + \frac{1}{s} = 1, \phi_r(x) := x^{p(1-\frac{\lambda}{r})-1}, \psi_s(x) := x^{q(1-\frac{\lambda}{s})-1}, \sigma(x) = 1 - \frac{1}{2}\left(\frac{a}{x}\right)^{\frac{\lambda}{2}}, f, g \geq 0, 0 < \|f\|_{p, \phi_r}, \|g\|_{q, \psi_s} < \infty$ . Some words that the constants are the best possible are omitted.

**Example 3.1.** Let  $k_\lambda(x, y) = \frac{1}{(x^\alpha + y^\alpha)^{\lambda/\alpha}}$  ( $\alpha > 0$ ), which is symmetric. Since both  $k_\lambda(1, u)$  and  $k_\lambda(u, 1)$  are derivable decreasing in  $(0, 1]$ , and  $k_\lambda(u, 1) = \frac{1}{(u^\alpha + 1)^{\lambda/\alpha}} \geq l_\lambda = \frac{1}{2^{\lambda/\alpha}}$  ( $u \in (0, 1]$ ), setting  $v = u^\alpha$ , by (1.3), we have

$$\tilde{k}_\lambda(r) := \int_0^\infty \frac{u^{\frac{\lambda}{r}-1} du}{(u^\alpha + 1)^{\frac{\lambda}{\alpha}}} = \frac{1}{\alpha} \int_0^\infty \frac{v^{\frac{\lambda}{\alpha r}-1} dv}{(v + 1)^{\frac{\lambda}{\alpha}}} = \frac{1}{\alpha} B\left(\frac{\lambda}{\alpha r}, \frac{\lambda}{\alpha s}\right).$$

By (2.18), (2.19) and (2.26), (2.27), we have two pairs of equivalent inequalities as:

$$\begin{aligned} H(a) & : = \int_a^\infty \int_a^\infty \frac{f(x)g(y)dx dy}{(x^\alpha + y^\alpha)^{\frac{\lambda}{\alpha}}} \\ & < \tilde{k}_\lambda(r) \left\{ \int_a^\infty \left[1 - \frac{s}{2^{\frac{\lambda}{\alpha}} \lambda \tilde{k}_\lambda(r)} \left(\frac{a}{x}\right)^{\frac{\lambda}{s}}\right] \phi_r(x) f^p(x) dx \right\}^{\frac{1}{p}} \\ & \quad \times \left\{ \int_a^\infty \left[1 - \frac{r}{2^{\frac{\lambda}{\alpha}} \lambda \tilde{k}_\lambda(r)} \left(\frac{a}{y}\right)^{\frac{\lambda}{r}}\right] \psi_s(y) g^q(y) dy \right\}^{\frac{1}{q}}, \\ & \int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{\left[1 - \frac{r}{2^{\frac{\lambda}{\alpha}} \lambda \tilde{k}_\lambda(r)} \left(\frac{a}{y}\right)^{\frac{\lambda}{r}}\right]^{p-1}} \left[\int_a^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^{\frac{\lambda}{\alpha}}} dx\right]^p dy \\ & < \tilde{k}_\lambda^p(r) \int_a^\infty \left[1 - \frac{s}{2^{\frac{\lambda}{\alpha}} \lambda \tilde{k}_\lambda(r)} \left(\frac{a}{x}\right)^{\frac{\lambda}{s}}\right] \phi_r(x) f^p(x) dx; \end{aligned}$$



$$H(a) < \tilde{k}_\lambda(2) \left\{ \int_a^\infty \sigma(x) \phi_2(x) f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^\infty \sigma(y) \psi_2(y) g^q(y) dy \right\}^{\frac{1}{q}}, \quad (3.1)$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{2}-1}}{\sigma^{p-1}(y)} \left[ \int_a^\infty \frac{f(x) dx}{(x^\alpha + y^\alpha)^{\frac{\lambda}{\alpha}}} \right]^p dy < \tilde{k}_\lambda^p(2) \int_a^\infty \sigma(x) \phi_2(x) f^p(x) dx.$$

**Example 3.2.** Let  $k_\lambda(x, y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$ , which is symmetric. We find that both  $k_\lambda(1, u)$  and  $k_\lambda(u, 1)$  are derivable decreasing in  $(0, 1]$ , [16] and  $k_\lambda(u, 1) = \frac{\ln u}{u^\lambda - 1} \geq l_\lambda = \frac{1}{\lambda}$  ( $u \in (0, 1]$ ). Setting  $v = u^\lambda$ , we obtain [2]

$$k_\lambda(r) = \int_0^\infty \frac{(\ln u) u^{\frac{\lambda}{r}-1}}{u^\lambda - 1} du = \int_0^\infty \frac{(\ln v) v^{\frac{1}{r}-1}}{\lambda^2(v-1)} dv = \left[ \frac{\pi}{\lambda \sin(\frac{\pi}{r})} \right]^2.$$

By (2.18), (2.19) and (2.26), (2.27), we have two pairs of equivalent inequalities as:

$$\begin{aligned} H'(a) &:= \int_a^\infty \int_a^\infty \frac{\ln(\frac{x}{y}) f(x) g(y)}{x^\lambda - y^\lambda} dx dy \\ &< \left[ \frac{\pi}{\lambda \sin(\frac{\pi}{r})} \right]^2 \left\{ \int_a^\infty \left[ 1 - s \left[ \frac{\sin(\frac{\pi}{r})}{\pi} \right]^2 \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \phi_r(x) f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_a^\infty \left[ 1 - r \left[ \frac{\sin(\frac{\pi}{r})}{\pi} \right]^2 \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right] \psi_s(y) g^q(y) dy \right\}^{\frac{1}{q}}, \\ &\quad \int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{\left[ 1 - r \left[ \frac{\sin(\frac{\pi}{r})}{\pi} \right]^2 \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right]^{p-1}} \left( \int_a^\infty \frac{\ln(\frac{x}{y}) f(x)}{x^\lambda - y^\lambda} dx \right)^p dy \\ &< \left[ \frac{\pi}{\lambda \sin(\frac{\pi}{r})} \right]^{2p} \int_a^\infty \left[ 1 - s \left[ \frac{\sin(\frac{\pi}{r})}{\pi} \right]^2 \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \phi_r(x) f^p(x) dx; \end{aligned}$$

$$H'(a) < \left( \frac{\pi}{\lambda} \right)^2 \left\{ \int_a^\infty \sigma(x) \phi_2(x) f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_a^\infty \sigma(y) \psi_2(y) g^q(y) dy \right\}^{\frac{1}{q}},$$

$$\int_a^\infty \frac{y^{\frac{p\lambda}{2}-1}}{\sigma^{p-1}(y)} \left[ \int_a^\infty \frac{\ln(\frac{x}{y}) f(x)}{x^\lambda - y^\lambda} dx \right]^p dy < \left( \frac{\pi}{\lambda} \right)^{2p} \int_a^\infty \sigma(x) \phi_2(x) f^p(x) dx.$$

**Example 3.3.** Let  $k_\lambda(x, y) = \frac{1}{(\max\{x, y\})^\lambda}$ , which is symmetric. Since both  $k_\lambda(1, u)$  and  $k_\lambda(u, 1)$  are derivable decreasing in  $(0, 1]$ , and  $k_\lambda(u, 1) = l_\lambda = 1$  ( $u \in (0, 1]$ ), we have

$$k_\lambda(r) = \int_0^\infty \frac{u^{\frac{\lambda}{r}-1}}{(\max\{u, 1\})^\lambda} du = \int_0^1 u^{\frac{\lambda}{r}-1} du + \int_1^\infty \frac{u^{\frac{\lambda}{r}-1}}{u^\lambda} du = \frac{rs}{\lambda}.$$

Then by (2.18) and (2.19), we have two equivalent inequalities as follows:

$$\begin{aligned} \int_a^\infty \int_a^\infty \frac{f(x) g(y) dx dy}{(\max\{x, y\})^\lambda} &< \frac{rs}{\lambda} \left\{ \int_a^\infty \left[ 1 - \frac{1}{r} \left( \frac{a}{x} \right)^{\frac{\lambda}{s}} \right] \phi_r(x) f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \int_a^\infty \left[ 1 - \frac{1}{s} \left( \frac{a}{y} \right)^{\frac{\lambda}{r}} \right] \psi_s(y) g^q(y) dy \right\}^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} & \int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{[1 - \frac{1}{s}(\frac{a}{y})^\frac{\lambda}{r}]^{p-1}} [\int_a^\infty \frac{f(x)dx}{(\max\{x, y\})^\lambda}]^p dy \\ & < (\frac{rs}{\lambda})^p \int_a^\infty [1 - \frac{1}{r}(\frac{a}{x})^\frac{\lambda}{s}] \phi_r(x) f^p(x) dx; \end{aligned}$$

**Example 3.4.** Let  $k_\lambda(x, y) = \frac{|\ln(x/y)|}{(\max\{x, y\})^\lambda}$ , which is symmetric. We find that

$$\begin{aligned} k_\lambda(r) &= \int_0^\infty \frac{|\ln u| u^{\frac{\lambda}{r}-1} du}{(\max\{u, 1\})^\lambda} \\ &= \int_0^1 (-\ln u) u^{\frac{\lambda}{r}-1} du + \int_1^\infty \frac{(\ln u) u^{\frac{\lambda}{r}-1}}{u^\lambda} du = \frac{r^2 + s^2}{\lambda^2}; \\ \omega_\lambda(r, y, a) &= \int_{\frac{a}{y}}^\infty \frac{|\ln u| u^{\frac{\lambda}{r}-1}}{(\max\{u, 1\})^\lambda} du = \frac{r^2 + s^2}{\lambda^2} - \int_0^{\frac{a}{y}} (-\ln u) u^{\frac{\lambda}{r}-1} du \\ &= \frac{r^2 + s^2}{\lambda^2} - \frac{r}{\lambda} \int_0^{\frac{a}{y}} (-\ln u) du^{\frac{\lambda}{r}} \leq \frac{r^2 + s^2}{\lambda^2} \kappa(y), \\ \kappa(y) &= 1 - \frac{r^2}{r^2 + s^2} (\frac{a}{y})^\frac{\lambda}{r}; \\ \varpi_\lambda(s, x, a) &\leq \frac{r^2 + s^2}{\lambda^2} \tilde{\mu}(x), \tilde{\mu}(x) = 1 - \frac{s^2}{r^2 + s^2} (\frac{a}{x})^\frac{\lambda}{s}. \end{aligned}$$

Then by (2.8) and (2.9), we have two equivalent inequalities as follows:

$$\begin{aligned} & \int_a^\infty \int_a^\infty \frac{|\ln(\frac{x}{y})| f(x) g(y)}{(\max\{x, y\})^\lambda} dx dy \\ & < \frac{r^2 + s^2}{\lambda^2} \{ \int_a^\infty [1 - \frac{s^2}{r^2 + s^2} (\frac{a}{x})^\frac{\lambda}{s}] \phi_r(x) f^p(x) dx \}^\frac{1}{p} \\ & \quad \times \{ \int_a^\infty [1 - \frac{r^2}{r^2 + s^2} (\frac{a}{y})^\frac{\lambda}{r}] \psi_s(y) g^q(y) dy \}^\frac{1}{q}; \\ & \int_a^\infty \frac{y^{\frac{p\lambda}{s}-1}}{[1 - \frac{r^2}{r^2 + s^2} (\frac{a}{y})^\frac{\lambda}{r}]^{p-1}} [\int_a^\infty \frac{|\ln(\frac{x}{y})| f(x) dx}{(\max\{x, y\})^\lambda}]^p dy \\ & < (\frac{r^2 + s^2}{\lambda^2})^p \int_a^\infty [1 - \frac{s^2}{r^2 + s^2} (\frac{a}{x})^\frac{\lambda}{s}] \phi_r(x) f^p(x) dx. \end{aligned}$$

*Remark 3.5.* (i) For  $\alpha = 1, p = q = 2$ , inequality (3.1) deduces to (1.2). (ii) Inequality (2.8) is a refinement of (2.10), because of

$$I_\lambda(a) < k_\lambda(r) \|f\|_{p, \tilde{\mu} \cdot \phi_r} \|g\|_{q, \kappa \cdot \psi_s} \leq k_\lambda(r) \|f\|_{p, \phi_r}, \|g\|_{q, \psi_s}.$$

(iii) When  $a \rightarrow 0^+$ , (2.10) deduces to a Hilbert-type integral inequality in  $(0, \infty)$  with a best constant factor  $k_\lambda(r)$  as:

$$I_\lambda(0) \leq k_\lambda(r) \{ \int_0^\infty \phi_r(x) f^p(x) dx \}^\frac{1}{p} \{ \int_0^\infty \psi_s(y) g^q(y) dy \}^\frac{1}{q}.$$

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