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GENERALIZATIONS OF OSTROWSKI INEQUALITY VIA BIPARAMETRIC EULER HARMONIC IDENTITIES FOR MEASURES

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Dedicated to Professor Lars-Erik Persson on the occasion of his 65th birthday

Communicated by Z. Páles

ABSTRACT. Some generalizations of Ostrowski inequality are given by using biparametric Euler identities involving real Borel measures and harmonic sequences of functions.

1. INTRODUCTION

The following Ostrowski inequality, see [5], is well known:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\mathrm{d}t \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)M, \ a \leq x \leq b,$$

where $f : [a, b] \to \mathbb{R}$ is a differentiable function such that $|f'(x)| \leq M$, for every $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible. In other words, Ostrowski's inequality gives us an estimate for the deviation of the values of a smooth function from its mean value. It has been generalized in recent years in a number of ways. In this paper we shall present some new generalizations of Ostrowski-type inequalities by using biparametric Euler identities which involve real Borel measures and harmonic sequences of functions.

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For $a,b\in\mathbb{R}$ $a<b$, let $C[a,b]$ be the Banach space of all continuous functions $f:[a,b]\to\mathbb{R}$ with the max norm, and $M[a,b]$ the Banach space of all real Borel measures on $[a,b]$ with the total variation norm. In the rest of the paper we use the notation $\int_{[a,b]}F(s)\mathrm{d}\mu(s)$ to denote the Lebesgue integral of F over $[a,b]$ with respect to the measure μ, while for a given function $\varphi:[a,b]\to\mathbb{R}$ of bounded variation $\int_{[a,b]}F(s)\mathrm{d}\varphi(s)$ denotes Lebesgue–Stieltjes integral of F over $[a,b]$ with respect to φ. Also, by $\int_a^b F(s)\mathrm{d}s$ we denote the usual Lebesgue integral of F over $[a,b]$.

For $\mu\in M[a,b]$ define the function $\check{\mu}_n:[a,b]\to\mathbb{R}$, $n\geq 1$, by

$$\check{\mu}_n(t)=\frac{1}{(n-1)!}\int_{[a,t]}(t-s)^{n-1}d\mu(s).$$

For $n=1$,

$$\check{\mu}_1(t)=\int_{[a,t]}d\mu(s)=\mu([a,t]),\ a\leq t\leq b,$$

which means that $\check{\mu}_1$ is equal to the distribution function of μ.

Substituting $\check{\mu}_n(s)=\frac{1}{(n-1)!}\int_{[a,s]}(s-u)^{n-1}d\mu(u)$ in $\int_a^t\check{\mu}_n(s)ds$ and using the Fubini theorem we easily get the formula

$$\check{\mu}_{n+1}(t)=\int_a^t\check{\mu}_n(s)ds,\ a\leq t\leq b,\ n\geq 1.$$

It means that for $n\geq 1$, $\check{\mu}_{n+1}$ is differentiable at almost all points of $[a,b]$ and $\check{\mu}'_{n+1}=\check{\mu}_n$ almost everywhere on $[a,b]$ with respect to Lebesgue measure.

Substituting $\check{\mu}_1(s)=\int_{[a,s]}d\mu(u)$ in $\int_a^t(t-s)^{n-2}\check{\mu}_1(s)ds$ and using the Fubini theorem once again we easily get the following formula

$$\check{\mu}_n(t)=\frac{1}{(n-2)!}\int_a^t(t-s)^{n-2}\check{\mu}_1(s)ds,\ a\leq t\leq b,\ n\geq 2.$$

From this formula we get immediately that $\check{\mu}_n(a)=0,\ n\geq 2$.

Also, note that function $g(s)=(t-s)^{n-1}$ is nonincreasing on $[a,t]$ so that from the first expression for $\check{\mu}_n(t)$ we get the estimate

$$|\check{\mu}_n(t)|\leq\frac{(t-a)^{n-1}}{(n-1)!}\left\|\mu\right\|,\ a\leq t\leq b,\ n\geq 1,$$

where $\|\mu\|$ denotes the total variation of μ.

A sequence of functions $P_n:[a,b]\to\mathbb{R}$, $n\geq 1$, is called a μ-harmonic sequence of functions on $[a,b]$ if

$$P_1(t)=c+\check{\mu}_1(t),\ a\leq t\leq b,$$

for some $c\in\mathbb{R}$, and

$$P_{n+1}(t)=P_{n+1}(a)+\int_a^t P_n(s)ds,\ a\leq t\leq b,\ n\geq 1.$$

Since $P_{n+1},\ n\geq 1$ is defined as an indefinite Lebesgue integral of P_n, it is well known that $P_{n+1},\ n\geq 1$ is absolutely continuous function,

$$P'_{n+1}=P_n,\ \text{a.e. on } [a,b] \text{ with respect to Lebesgue measure,}$$

and for every $f \in C[a,b]$ we have

$$\int_{[a,b]} f(t) dP_{n+1}(t) = \int_a^b f(t) P_n(t) dt, \ n \geq 1.$$

The sequence $(\breve{\mu}_n, \ n \geq 1)$ is an example of a μ-harmonic sequence of functions on $[a,b]$.

Assume that $(P_n, \ n \geq 1)$ is a μ-harmonic sequence of functions on $[a,b]$. Define P_n^*, for $n \geq 1$, to be a periodic function of period 1, related to P_n as

$$P_n^*(t) = \frac{P_n(a + (b-a)t)}{(b-a)^n}, \ 0 \leq t < 1,$$

and

$$P_n^*(t+1) = P_n^*(t), \ t \in \mathbb{R}.$$

Thus, for $n \geq 2$, P_n^* is continuous on $\mathbb{R} \backslash \mathbb{Z}$ and has a jump of

$$\alpha_n = \frac{P_n(a) - P_n(b)}{(b-a)^n}$$

at every $k \in \mathbb{Z}$, whenever $\alpha_n \neq 0$. Note that for $n \geq 1$, $\left(P_{n+1}^*\right)' = P_n^*$ a.e. on $\mathbb{R}$.

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. In a recent paper [1] the following identity has been proved:

$$\mu([a,b]) f(x) = \int_{[a,b]} f_x(t) d\mu(t) + S_n(x) + R_n(x), \tag{1.1}$$

where

$$S_m(x) = \sum_{k=1}^{m} P_k(x) \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] + \sum_{k=2}^{m} \left[P_k(a) - P_k(b) \right] f^{(k-1)}(x),$$

for $1 \leq m \leq n$, with convention $S_1(x) = P_1(x) \left[f(b) - f(a) \right]$, and

$$f_x(t) = \begin{cases} f(a+x-t), & a \leq t \leq x \\ f(b+x-t), & x < t \leq b \end{cases},$$

while

$$R_n(x) = -(b-a)^n \int_{[a,b]} P_n^* \left(\frac{x-t}{b-a} \right) df^{(n-1)}(t)$$

for every $x \in [a,b]$.

Identity (1.1) is called the generalized Euler harmonic identity. It has been used in [1] to prove some generalizations of Ostrowski's inequality. The reader can find further references to some recent results on generalizations and applications of Euler identities in [2], [4] and [3].

The aim of this paper is to generalize formula (1.1) by replacing the sequence $\left(P_n^* \left(\frac{x-t}{b-a}\right), \ n \geq 1\right)$ with a more general sequence of functions, and using them to prove some further generalizations of Ostrowski's inequality.

2. Biparametric Euler harmonic identities

For $\mu \in M[a,b]$ let $(P_n,\ n \geq 1)$ be a μ-harmonic sequence of functions on $[a,b]$. For $x, y \in [a,b]$, $x \leq y$, define function $K_n : [a,b]^3 \to \mathbb{R}$, for $n \geq 1$, by

$$K_n(x,y,t) = \begin{cases} P_n(b-a+x-y+t), & a \leq t \leq a+y-x \\ P_n(x-y+t), & a+y-x < t \leq b \end{cases}, \tag{2.1}$$

for $y - x < b - a$, and

$$K_n(a,b,t) = \begin{cases} P_n(t), & a \leq t < b \\ P_n(a), & t = b \end{cases}. \tag{2.2}$$

Thus, for $n \geq 2$, $K_n(x,y,\cdot)$ is continuous on $[a,b] \setminus \{a+y-x\}$ and has a jump of $P_n(a) - P_n(b)$ at $a+y-x$. Note that $K_n(x,y,\cdot),\ n \geq 1$ is a function of bounded variation and for $n \geq 1$

$$K_{n+1}'(x,y,\cdot) = K_n(x,y,\cdot) \text{ a.e. on } [a,b] \text{ with respect to Lebesgue measure.}$$

Also note that $K_n(x,y,a) = K_n(x,y,b) = P_n(b+x-y),\ n \geq 1$.

Lemma 2.1. *For every $f \in C[a,b]$ and $n \geq 2$ we have*

$$\int_{[a,b]} f(t) dK_n(x,y,t) = \int_a^b f(t) K_{n-1}(x,y,t) dt + f(a+y-x) \left[P_n(a) - P_n(b)\right].$$

Proof. Follows directly from properties of Lebesgue-Stieltjes integral of continuous function f over $[a,\ b]$ with respect to K_n, and given properties of the function K_n. Namely, the function $K_n(x,y,\cdot),\ n \geq 2$ is almost everywhere differentiable on $[a,b]$ and its derivative is equal to $K_{n-1}(x,y,\cdot)$ a.e. on $[a,b]$ with respect to Lebesgue measure. Further, it has a jump at $a+y-x$ of magnitude $P_n(a) - P_n(b)$, which proves our assertion. □

Lemma 2.2. *For every $\mu \in M[a,b]$ and $f \in C[a,b]$ we have*

$$\int_{[a,b]} f(t) dK_1(x,y,t) = \int_{[a,b]} f_{x,y}(t) d\mu(t) - f(a+y-x)\mu([a,b]), \tag{2.3}$$

where

$$f_{x,y}(t) = \begin{cases} f(y-x+t), & a \leq t \leq b+x-y \\ f(a-b+y-x+t), & b+x-y < t \leq b \end{cases}. \tag{2.4}$$

Proof. Define $I, J : C[a,b] \times M[a,b] \to \mathbb{R}$ by

$$I(f,\mu) = \int_{[a,b]} f(t) dK_1(x,y,t)$$

and

$$J(f,\mu) = \int_{[a,b]} f_{x,y}(t) d\mu(t) - f(a+y-x)\mu([a,b]).$$

Then I and J are continuous bilinear functionals with

$$|I(f,\mu)| \leq \|f\| \, \|\mu\|, \qquad |J(f,\mu)| \leq 2 \|f\| \, \|\mu\|.$$

Let us prove that $I(f,\mu) = J(f,\mu)$ for every $f \in C[a,b]$ and every $\mu \in M[a,b]$.

Since $P_1(t) = c + \mu([a,t]),\ a \leq t \leq b$, for some constant c, and obviously the integral on the left hand side of (2.3) is independent of the choice of the constant

c, we may assume that $c = 0$. Therefore, from (2.1) and (2.2) we easily see that for $n = 1$

$$K_1(x,y,t) = \begin{cases} \mu([a, b-a+x-y+t]), & a \le t \le a+y-x \\ \mu([a, x-y+t]), & a+y-x < t \le b \end{cases}, \tag{2.5}$$

for $y - x < b - a$, and

$$K_1(a,b,t) = \begin{cases} \mu([a,t]), & a \le t < b \\ \mu(\{a\}), & t = b \end{cases}. \tag{2.6}$$

(1) For $z \in [a,b]$ let $\mu = \delta_z$ be the Dirac measure at z, i.e., the measure defined by

$$\int_{[a,b]} f(t) d\delta_z(t) = f(z).$$

If $z \in [a,b]$ and $a \le z \le b + x - y$, from (2.5) and (2.6) we get

$$K_1(x,y,t) = \begin{cases} 0, & a+y-x < t < z+y-x \\ 1, & (a \le t \le a+y-x) \text{ or } (z+y-x \le t \le b) \end{cases},$$

for $y - x < b - a$, and

$$K_1(a,b,t) = 1, \ a \le t \le b.$$

Now, by a simple calculation we have

$$\begin{aligned} I(f,\delta_z) &= f(y-x+z) - f(a+y-x) \\ &= \int_{[a,b]} f(y-x+t) d\delta_z(t) - f(a+y-x)\delta_z([a,b]) \\ &= \int_{[a,b]} f_{x,y}(t) d\delta_z(t) - f(a+y-x)\delta_z([a,b]) = J(f,\delta_z), \end{aligned}$$

for $y - x < b - a$, and

$$\begin{aligned} I(f,\delta_z) &= I(f,\delta_a) = 0 = f(b) - f(a+b-a) \\ &= \int_{[a,b]} f(b) d\delta_a(t) - f(a+b-a)\delta_z([a,b]) \\ &= \int_{[a,b]} f_{a,b}(t) d\delta_a(t) - f(a+y-x)\delta_a([a,b]) = J(f,\delta_a) = J(f,\delta_z), \end{aligned}$$

for $y - x = b - a$. Similarly, if $z \in [a,b]$ and $b + x - y < z \le b$, from (2.5) and (2.6) we find

$$K_1(x,y,t) = \begin{cases} 0, & (a \le t < a+y-x-b+z) \text{ or } (a+y-x < t \le b) \\ 1, & a+y-x-b+z \le t \le a+y-x \end{cases},$$

for $y - x < b - a$, and

$$K_1(a,b,t) = \begin{cases} 0, & (a \le t < z) \text{ or } (t = b) \\ 1, & z \le t < b \end{cases}.$$

Now, by analogous calculation we have

$$\begin{aligned} I(f,\delta_z) &= f(a-b+y-x+z)-f(a+y-x) \\ &= \int_{[a,b]} f(a-b+y-x+t)d\delta_z(t)-f(a+y-x)\delta_z([a,b]) \\ &= \int_{[a,b]} f_{x,y}(t)d\delta_z(t)-f(a+y-x)\delta_z([a,b])=J(f,\delta_z), \end{aligned}$$

for $y-x<b-a$, and

$$\begin{aligned} I(f,\delta_z) &= f(z)-f(b) \\ &= \int_{[a,b]} f(t)d\delta_z(t)-f(a+b-a)\delta_z([a,b]) \\ &= \int_{[a,b]} f_{a,b}(t)d\delta_z(t)-f(a+y-x)\delta_z([a,b])=J(f,\delta_z), \end{aligned}$$

for $y-x=b-a$. Therefore, for every $f\in C\,[a,b]$ and every $z\in[a,b]$ we have $I(f,\delta_z)=J(f,\delta_z)$.

(2) Every discrete measure $\mu\in M[a,b]$, with finite support, is a linear combination of Dirac measures, i.e., it has the form $\mu=\sum_{k=1}^{n}c_k\delta_{x_k}$, for some real numbers c_k, and $x_k\in[a,b]$. By linearity of I and J, we get

$$\begin{aligned} I(f,\mu) &= I(f,\sum_{k=1}^{n}c_k\delta_{x_k})=\sum_{k=1}^{n}c_kI(f,\delta_{x_k}) \\ &= \sum_{k=1}^{n}c_kJ(f,\delta_{x_k})=J(f,\sum_{k=1}^{n}c_k\delta_{x_k})=J(f,\mu). \end{aligned}$$

for every $f\in C[a,b]$ and every discrete measure $\mu\in M[a,b]$ with finite support.

(3) Let $\mathcal{T}$ be the minimal topology on $M[a,b]$ such that linear functionals $\mu\mapsto\int Fd\mu$ are continuous, for every bounded Borel function $F:[a,b]\to\mathbb{R}$. By the definition we see that $\mathcal{T}$ contains the weak* topology on $M[a,b]$ and is contained in the weak topology on $M[a,b]$. Further, the curve $x\mapsto\delta_x$ is bounded and $\mathcal{T}$-measurable since $x\mapsto\int Fd\delta_x=F(x)$ is measurable by assumption. Therefore, the integral $\int\delta_x d\mu(x)$ exists in the $\mathcal{T}$ topology, for every $\mu\in M[a,b]$. It is easy to see that this integral is equal to μ, i.e. $\int\delta_x d\mu(x)=\mu$, for every measure $\mu\in M[a,b]$, which means that μ is a $\mathcal{T}$-limit of a sequence of discrete measures with finite support. Thus, we conclude that the subspace of all discrete measures with finite support is $\mathcal{T}$-dense in $M[a,b]$, and therefore the functionals $I(f,\cdot)$ and $J(f,\cdot)$ are equal, for every $f\in C[a,b]$, since they are equal on a dense subspace and they are $\mathcal{T}$-continuous. This completes the proof. □

Theorem 2.3. *For $\mu\in M[a,b]$ let $(P_n,\ n\geq1)$ be a μ-harmonic sequence of functions on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n\geq1$. Then we have*

$$\int_{[a,b]} f_{x,y}(t)d\mu(t)-\mu(\{a\})f(a+y-x)+S_n(x,y)=R_n(x,y),$$

for every $x, y \in [a, b]$, $x \le y$, *where* $f_{x,y}(t)$ *is defined by (2.4),*

$$\begin{aligned} S_n(x,y) &= \sum_{k=1}^{n}(-1)^k P_k(b+x-y)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] \\ &\quad +\sum_{k=1}^{n}(-1)^k f^{(k-1)}(a+y-x)\left[P_k(b)-P_k(a)\right] \end{aligned}$$

and

$$R_n(x,y) = (-1)^n \int_{[a,b]} K_n(x,y,t)\, df^{(n-1)}(t).$$

Proof. For $1 \le k \le n$ consider the integral

$$R_k(x,y) = (-1)^k \int_{[a,b]} K_k(x,y,t)\, df^{(k-1)}(t).$$

Integrating by parts we get

$$\begin{aligned} R_k(x,y) &= (-1)^k K_k(x,y,t)\, f^{(k-1)}(t)\Big|_a^b \\ &\quad -(-1)^k \int_{[a,b]} f^{(k-1)}(t) dK_k(x,y,t)\,. \end{aligned} \tag{2.7}$$

For every $k \ge 2$, by Lemma 2.1, we get

$$\begin{aligned} R_k(x,y) &= (-1)^k P_k(b+x-y)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] \\ &\quad -(-1)^k f^{(k-1)}(a+y-x)\left[P_k(a)-P_k(b)\right] \\ &\quad -(-1)^k \int_a^b f^{(k-1)}(t) K_{k-1}(x,y,t)\, dt \\ &= (-1)^k P_k(b+x-y)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] \\ &\quad +(-1)^k f^{(k-1)}(a+y-x)\left[P_k(b)-P_k(a)\right] \\ &\quad +R_{k-1}(x,y), \end{aligned} \tag{2.8}$$

since

$$K_k(x,y,a) = K_k(x,y,b) = P_k(b+x-y).$$

By Lemma 2.2, for $k = 1$, (2.7) becomes

$$\begin{aligned} R_1(x,y) &= -P_1(b+x-y)\left[f(b)-f(a)\right] + \int_{[a,b]} f(t) dK_1(x,y,t) \\ &= -P_1(b+x-y)\left[f(b)-f(a)\right] - f(a+y-x)\mu([a,b]) \\ &\quad + \int_{[a,b]} f_{x,y}(t) d\mu(t) \end{aligned} \tag{2.9}$$

where $f_{x,y}(t)$ is defined by (2.4). From (2.8) and (2.9) it follows, by iteration

$$\begin{aligned} R_n(x,y) &= \sum_{k=1}^{n}(-1)^k P_k(b+x-y)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] \\ &+\sum_{k=1}^{n}(-1)^k f^{(k-1)}(a+y-x)\left[P_k(b)-P_k(a)\right] \\ &-f(a+y-x)\mu(\{a\})+\int_{[a,b]} f_{x,y}(t)d\mu(t) \end{aligned}$$

since

$$f(a+y-x)\mu([a,b])=f(a+y-x)\left[P_1(b)-P_1(a)+\mu(\{a\})\right],$$

which proves our assertion. □

Corollary 2.4. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n\geq 1$. Then we have*

$$\int_{[a,b]} f_{x,y}(t)d\mu(t)+\check{S}_n(x,y)=\check{R}_n(x,y).$$

for every $x,y\in[a,b]$, $x\leq y$, where

$$\begin{aligned} \check{S}_n(x,y) &= \sum_{k=1}^{n}(-1)^k \check{\mu}_k(b+x-y)\left[f^{(k-1)}(b)-f^{(k-1)}(a)\right] \\ &+\sum_{k=1}^{n}(-1)^k f^{(k-1)}(a+y-x)\check{\mu}_k(b), \end{aligned}$$

$$\check{R}_n(x,y)=(-1)^n\int_{[a,b]}\check{K}_n(x,y,t)\,df^{(n-1)}(t)$$

and

$$\check{K}_n(x,y,t)=\begin{cases} \check{\mu}_n(b-a+x-y+t), & a\leq t\leq a+y-x \\ \check{\mu}_n(x-y+t), & a+y-x<t\leq b \end{cases}$$

for $y-x<b-a$, while

$$\check{K}_n(a,b,t)=\begin{cases} \check{\mu}_n(t), & a\leq t<b \\ \check{\mu}_n(a), & t=b \end{cases}.$$

Proof. Apply the theorem above to the special case $P_n=\check{\mu}_n$, $n\geq 1$, and note that $\check{\mu}_k(a)=0$ for $k\geq 2$. □

Corollary 2.5. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation for some $n\geq 1$. Then we have*

$$\int_a^b f(t)dt+\bar{S}_n(x,y)=\bar{R}_n(x,y).$$

for every $x, y \in [a, b]$, $x \leq y$, *where*

$$\bar{S}_n(x,y) = \sum_{k=1}^{n} \frac{(-1)^k}{k!}(b-a+x-y)^k \left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + \sum_{k=1}^{n} \frac{(-1)^k}{k!}(b-a)^k f^{(k-1)}(a+y-x),$$

$$\bar{R}_n(x,y) = (-1)^n \int_{[a,b]} \bar{K}_n(x,y,t)\, df^{(n-1)}(t)$$

and

$$\bar{K}_n(x,y,t) = \begin{cases} \frac{1}{n!}(b-2a+x-y+t)^n, & a \leq t \leq a+y-x \\ \frac{1}{n!}(x-y+t-a)^n, & a+y-x < t \leq b \end{cases}$$

for $y - x < b - a$, *while*

$$\bar{K}_n(a,b,t) = \begin{cases} \frac{1}{n!}(t-a)^n, & a < t < b \\ 0, & (t=a) \ \textit{or} \ (t=b) \end{cases}.$$

Proof. Apply Corollary 2.4 in the special case when μ is the Lebesgue measure on $[a, b]$. In this case

$$\check{\mu}_k(t) = \frac{(t-a)^k}{k!}, \ k \geq 1$$

and

$$\int_{[a,b]} f_{x,y}(t)d\mu(t) = \int_a^b f_{x,y}(t)dt = \int_a^b f(t)dt.$$

□

3. Generalizations of Ostrowski's inequality

In this section we use the identity obtained in Theorem 2.3 to prove a number of Ostrowski-type inequalities which hold for a class of functions f whose derivatives $f^{(n-1)}$ are either L-Lipschitzian on $[a, b]$ or continuous and of bounded variation on $[a, b]$. Analogous results are obtained for a class of functions f possessing derivatives $f^{(n)}$ in $L_p[a, b]$, $1 \leq p \leq \infty$. Throughout this section we use the same notations as above.

Lemma 3.1. *For every* μ*-harmonic sequence* $(P_n,\ n \geq 1)$ *and* $f \in C[a, b]$ *we have*

$$\int_a^b f(K_n(x,y,t))dt = \int_a^b f(P_n(t))dt.$$

Proof. Follows from (2.1) and (2.2) using simple calculations,

$$\begin{aligned} &\int_a^b f(K_n(x,y,t))dt \\ &= \int_a^{a+y-x} f(P_n(b-a+x-y+t))dt + \int_{a+y-x}^b f(P_n(x-y+t))dt \\ &= \int_{b+x-y}^b f(P_n(t))dt + \int_a^{b+x-y} f(P_n(t))dt = \int_a^b f(P_n(t))dt. \end{aligned}$$

□

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a, b]$ for some $n \geq 1$. Then*

$$\left| \int_{[a,b]} f_{x,y}(t) d\mu(t) - \mu(\{a\}) f(a+y-x) + S_n(x,y) \right| \leq L \int_a^b |P_n(t)| \, dt,$$

for every $x, y \in [a, b]$, $x \leq y$.

Proof. By Lemma 3.1 we have

$$\begin{aligned} |R_n(x,y)| &= \left| \int_{[a,b]} K_n(x,y,t) \, df^{(n-1)}(t) \right| \\ &\leq L \int_a^b |K_n(x,y,t)| \, dt \\ &= L \int_a^b |P_n(t)| \, dt. \end{aligned}$$

Therefore, our assertion follows from Theorem 2.3. □

Corollary 3.3. *If f is L-Lipschitzian on $[a, b]$, then for every $x, y \in [a, b]$, $x \leq y$, and $c \in \mathbb{R}$ we have*

$$\begin{aligned} & \left| \int_{[a,b]} f_{x,y}(t) d\mu(t) - \mu([a,b]) f(a+y-x) \right. \\ & \left. -[c + \check{\mu}_1(b+x-y)] \, [f(b) - f(a)] \right| \\ \leq \; & L \int_a^b |c + \check{\mu}_1(t)| dt. \end{aligned}$$

Proof. Put $n = 1$ in the theorem above. □

Corollary 3.4. *If f is L-Lipschitzian on $[a, b]$ and $\mu \geq 0$, then for every $x, y, z \in [a, b]$, $x \leq y$, we have*

$$\begin{aligned} & \left| \int_{[a,b]} f_{x,y}(t) d\mu(t) - \mu([a,b]) f(a+y-x) \right. \\ & \left. -[\check{\mu}_1(b+x-y) - \check{\mu}_1(z)] \, [f(b) - f(a)] \right| \\ \leq \; & L \left[(2z - a - b)\check{\mu}_1(z) - 2\check{\mu}_2(z) + \check{\mu}_2(b) \right]. \end{aligned}$$

Proof. Put $c = -\check{\mu}_1(z)$ in Corollary 3.3 and note that in this case

$$\int_a^b |\check{\mu}_1(t) - \check{\mu}_1(z)| \, dt = (2z - a - b)\check{\mu}_1(z) - 2\check{\mu}_2(z) + \check{\mu}_2(b).$$

□

Corollary 3.5. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a, b]$ for some $n \geq 1$. Then for $\mu \geq 0$ we have*

$$\left| \int_{[a,b]} f_{x,y}(t) d\mu(t) + \check{S}_n(x,y) \right| \leq L \check{\mu}_{n+1}(b) \leq \frac{(b-a)^n}{n!} L \, \|\mu\|,$$

for every $x, y \in [a, b]$, $x \leq y$.

Proof. Apply the theorem above to the μ-harmonic sequence $(\check{\mu}_n,\ n \geq 1)$. Then

$$\int_a^b |\check{\mu}_n(t)|\, dt = \int_a^b \check{\mu}_n(t)dt = \check{\mu}_{n+1}(b) \leq \frac{(b-a)^n}{n!} \|\mu\| .$$

□

Corollary 3.6. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a,b]$ for some $n \geq 1$. Then we have*

$$\left| \int_a^b f(t)dt + \bar{S}_n(x,y) \right| \leq L\frac{(b-a)^{n+1}}{(n+1)!},$$

for every $x, y \in [a,b]$, $x \leq y$.

Proof. Apply the corollary above to the Lebesgue measure on $[a,b]$. □

Corollary 3.7. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a,b]$ for some $n \geq 1$. Then*

$$|f(y-x+z) + T_n(x,y,z)| \leq L\frac{(b-z)^n}{n!},$$

for every $x, y, z \in [a,b]$, $x \leq y$ and $z \leq b+x-y$, where

$$\begin{aligned} T_n(x,y,z) &= \sum_{k=1}^{n} (-1)^k \frac{(b+x-y-z)^{k-1}}{(k-1)!} \left[f^{(k-1)}(b) - f^{(k-1)}(a) \right] \\ &\quad + \sum_{k=1}^{n} (-1)^k \frac{(b-z)^{k-1}}{(k-1)!} f^{(k-1)}(a+y-x). \end{aligned}$$

Proof. Apply Corollary 3.5 to $\mu = \delta_z$, $z \leq b+x-y$. □

Corollary 3.8. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is L-Lipschitzian on $[a,b]$ for some $n \geq 1$. Then*

$$\left| f(a-b+y-x+z) + T_n^2(x,y,z) \right| \leq L\frac{(b-z)^n}{n!},$$

for every $x, y, z \in [a,b]$, $x \leq y$ and $b+x-y < z \leq b$, where

$$T_n^2(x,y,z) = \sum_{k=1}^{n} (-1)^k \frac{(b-z)^{k-1}}{(k-1)!} f^{(k-1)}(a+y-x).$$

Proof. Apply Corollary 3.5 to $\mu = \delta_z$, $b+x-y < z \leq b$. □

Theorem 3.9. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then*

$$\left| \int_{[a,b]} f_{x,y}(t)d\mu(t) - \mu(\{a\})f(a+y-x) + S_n(x,y) \right| \leq \sup_{t\in[a,b]} |P_n(t)|\, V_a^b(f^{(n-1)}),$$

for every $x, y \in [a,b]$, $x \leq y$.

Proof. We have

$$\begin{aligned} |R_n(x,y)| &= \left| \int_{[a,b]} K_n(x,y,t)\, df^{(n-1)}(t) \right| \\ &\le \sup_{t\in[a,b]} |K_n(x,y,t)|\, V_a^b(f^{(n-1)}) \\ &= \sup_{t\in[a,b]} |P_n(t)|\, V_a^b(f^{(n-1)}). \end{aligned}$$

Therefore, our assertion follows from Theorem 2.3. □

Corollary 3.10. *If f is a continuous function of bounded variation on $[a,b]$, then for every $x,y\in[a,b]$, $x\le y$, and $c\in\mathbb{R}$ we have*

$$\begin{aligned} & \left| \int_{[a,b]} f_{x,y}(t)d\mu(t) - \mu([a,b])f(a+y-x) \right. \\ & \left. -[c+\check{\mu}_1(b+x-y)]\,[f(b)-f(a)] \right| \\ \le\; & \sup_{t\in[a,b]} |c+\check{\mu}_1(t)|\, V_a^b(f), \end{aligned}$$

for every $x,y\in[a,b]$, $x\le y$.

Proof. Put $n=1$ in the theorem above. □

Corollary 3.11. *If f is a continuous function of bounded variation on $[a,b]$ and $\mu\ge 0$, then for every $x,y,z\in[a,b]$, $x\le y$, we have*

$$\begin{aligned} & \left| \int_{[a,b]} f_{x,y}(t)d\mu(t) - \mu([a,b])f(a+y-x) \right. \\ & \left. -[\check{\mu}_1(b+x-y)-\check{\mu}_1(z)]\,[f(b)-f(a)] \right| \\ \le\; & \frac{1}{2}\left[\check{\mu}_1(b)-\check{\mu}_1(a)+|\check{\mu}_1(a)+\check{\mu}_1(b)-2\check{\mu}_1(z)|\right] V_a^b(f). \end{aligned}$$

Proof. Put $c=-\check{\mu}_1(z)$ in Corollary 3.10. Then

$$\begin{aligned} \sup_{t\in[a,b]} |c+\check{\mu}_1(t)| &= \sup_{t\in[a,b]} |\check{\mu}_1(t)-\check{\mu}_1(z)| \\ &= \frac{1}{2}\left[\check{\mu}_1(b)-\check{\mu}_1(a)+|\check{\mu}_1(a)+\check{\mu}_1(b)-2\check{\mu}_1(z)|\right]. \end{aligned}$$

□

Corollary 3.12. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n\ge 1$. Then for $\mu\ge 0$ we have*

$$\begin{aligned} \left| \int_{[a,b]} f_{x,y}(t)d\mu(t) + \check{S}_n(x,y) \right| &\le \check{\mu}_n(b)V_a^b(f^{(n-1)}) \\ &\le \frac{(b-a)^{n-1}}{(n-1)!} V_a^b(f^{(n-1)})\,\|\mu\|, \end{aligned}$$

for every $x,y\in[a,b]$, $x\le y$.

Proof. Apply the theorem above to the μ-harmonic sequence $(\check{\mu}_n,\ n \geq 1)$. Then

$$\sup_{t\in[a,b]} \check{\mu}_n(t) = \check{\mu}_n(b) \leq \frac{(b-a)^{n-1}}{(n-1)!} \|\mu\| .$$

□

Corollary 3.13. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then*

$$\left| \int_a^b f(t)dt + \bar{S}_n(x,y) \right| \leq \frac{(b-a)^n}{n!} V_a^b(f^{(n-1)}),$$

for every $x, y \in [a,b]$, $x \leq y$.

Proof. Apply the corollary above to the Lebesgue measure on $[a,b]$. □

Corollary 3.14. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then*

$$|f(y-x+z) + T_n(x,y,z)| \leq \frac{(b-z)^{n-1}}{(n-1)!} V_a^b(f^{(n-1)}),$$

for every $x, y, z \in [a,b]$, $x \leq y$ and $a \leq z \leq b + x - y$, where $T_n(x,y,z)$ is from Corollary 3.7.

Proof. Apply Corollary 3.12 to $\mu = \delta_z$, $z \leq b + x - y$. □

Corollary 3.15. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[a,b]$ for some $n \geq 1$. Then*

$$\left| f(a-b+y-x+z) + T_n^2(x,y,z) \right| \leq \frac{(b-z)^{n-1}}{(n-1)!} V_a^b(f^{(n-1)}),$$

for every $x, y, z \in [a,b]$, $x \leq y$ and $b + x - y < z \leq b$, where $T_n^2(x,y,z)$ is from Corollary 3.8.

Proof. Apply Corollary 3.12 to $\mu = \delta_z$, $b + x - y < z \leq b$. □

Theorem 3.16. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is integrable for some $n \geq 1$. Then*

$$\left| \int_{[a,b]} f_{x,y}(t)d\mu(t) - \mu(\{a\})f(a+y-x) + S_n(x,y) \right| \leq \sup_{t\in[a,b]} |P_n(t)| \cdot \|f^{(n)}\|_1,$$

for every $x, y \in [a,b]$, $x \leq y$.

Proof. Note that in this case

$$V_a^b(f^{(n-1)}) = \int_a^b \left| f^{(n)}(t) \right| dt = \|f^{(n)}\|_1,$$

and apply Theorem 3.9. □

Theorem 3.17. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)} \in L_\infty[a, b]$ for some $n \geq 1$. Then*

$$\left| \int_{[a,b]} f_{x,y}(t) d\mu(t) - \mu(\{a\}) f(a + y - x) + S_n(x, y) \right| \leq \int_a^b |P_n(t)|\, dt \cdot \|f^{(n)}\|_\infty,$$

for every $x, y \in [a, b]$, $x \leq y$.

Proof. In this case $f^{(n-1)}$ is L-Lipschitzian with $L = \|f^{(n)}\|_\infty$. □

Theorem 3.18. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a, b]$ for some $n \geq 1$ and $1 < p < \infty$. Then*

$$\left| \int_{[a,b]} f_{x,y}(t) d\mu(t) - \mu(\{a\}) f(a + y - x) + S_n(x, y) \right| \leq \|P_n\|_q \|f^{(n)}\|_p,$$

for every $x, y \in [a, b]$, $x \leq y$, where $1/p + 1/q = 1$.

Proof. By applying the Hölder inequality we have

$$\begin{aligned} |R_n(x, y)| &\leq \int_a^b |K_n(x, y, t)| \left| f^{(n)}(t) \right| dt \\ &\leq \left(\int_a^b |K_n(x, y, t)|^q \, dt \right)^{1/q} \|f^{(n)}\|_p \\ &= \left(\int_a^b |P_n(t)|^q \, dt \right)^{1/q} \|f^{(n)}\|_p, \end{aligned}$$

which proves our assertion. □

Corollary 3.19. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a, b]$ for some $n \geq 1$ and $1 < p < \infty$. Then*

$$\left| \int_{[a,b]} f_{x,y}(t) d\mu(t) + \check{S}_n(x, y) \right| \leq \frac{\|\mu\| \, \|f^{(n)}\|_p}{(n-1)!} \frac{(b-a)^{n-1+1/q}}{[(n-1)q + 1]^{1/q}},$$

for every $x, y \in [a, b]$, $x \leq y$, where $1/p + 1/q = 1$.

Proof. Apply the theorem above to the μ-harmonic sequence $(\check{\mu}_n, \ n \geq 1)$ and note that

$$\begin{aligned} \int_a^b |\check{\mu}_n(t)|^q \, dt &\leq \left[\frac{\|\mu\|}{(n-1)!} \right]^q \int_a^b (t - a)^{(n-1)q} \, dt \\ &= \left[\frac{\|\mu\|}{(n-1)!} \right]^q \frac{(b-a)^{(n-1)q+1}}{(n-1)q + 1}. \end{aligned}$$

□

Corollary 3.20. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a, b]$ for some $n \geq 1$ and $1 < p < \infty$. Then*

$$\left| \int_a^b f(t) dt + \bar{S}_n(x, y) \right| \leq \frac{\|f^{(n)}\|_p}{n!} \frac{(b-a)^{n+1/q}}{[nq + 1]^{1/q}},$$

for every $x, y \in [a, b]$, $x \leq y$, where $1/p + 1/q = 1$.

Proof. Apply the theorem above to the Lebesgue measure on $[a, b]$. □

Corollary 3.21. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a, b]$ for some $n \geq 1$ and $1 < p < \infty$. Then*

$$|f(y - x + z) + T_n(x, y, z)| \leq \frac{\|f^{(n)}\|_p}{(n-1)!} \frac{(b-z)^{n-1+1/q}}{[(n-1)q+1]^{1/q}},$$

for every $x, y, z \in [a, b]$, $x \leq y$ and $a \leq z \leq b + x - y$, where $T_n(x, y, z)$ is from Corollary 3.7.

Proof. Apply Corollary 3.19 to $\mu = \delta_z$, $a \leq z \leq b + x - y$. □

Corollary 3.22. *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)} \in L_p[a, b]$ for some $n \geq 1$ and $1 < p < \infty$. Then*

$$\left|f(a - b + y - x + z) + T_n^2(x, y, z)\right| \leq \frac{\|f^{(n)}\|_p}{(n-1)!} \frac{(b-z)^{n-1+1/q}}{[(n-1)q+1]^{1/q}},$$

for every $x, y, z \in [a, b]$, $x \leq y$ and $b + x - y < z \leq b$, where $T_n^2(x, y, z)$ is from Corollary 3.8.

Proof. Apply Corollary 3.19 to $\mu = \delta_z$, $b + x - y < z \leq b$. □

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