Banach J. Math. Anal. 4 (2010), no. 1, 28–52

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

# ON BOUNDEDNESS OF A CERTAIN CLASS OF HARDY–STEKLOV TYPE OPERATORS IN LEBESGUE SPACES

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*Dedicated to Professor Lars-Erik Persson on the occasion of his 65th birthday*

Communicated by L. Maligranda

ABSTRACT. $L_p$–$L_q$–boundedness of the map $f \to w(x) \int_{a(x)}^{b(x)} k(x,y)f(y)v(y)dy$ is described by different types of criteria expressed in terms of given parameters $0 < p, q < \infty$, strictly increasing boundaries $a(x)$ and $b(x)$, locally integrable weight functions $v, w$ and a positive continuous kernel $k(x,y)$ satisfying some growth conditions.

## 1. INTRODUCTION AND PRELIMINARIES

Let $0 < p < \infty$, $\|f\|_p := \left(\int_0^\infty |f(x)|^p dx\right)^{1/p}$ and $L_p$ denote the Lebesgue space of all measurable functions on $\mathbb{R}^+ := [0, \infty)$ such that $\|f\|_p < \infty$. Here and throughout the paper the mark $:=$ is applied for introducing new notations and quantities. For the same purposes we make use of the mark $=:$ .

*Date*: Received: 1 December 2009; Accepted: 21 January 2010.

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2000 *Mathematics Subject Classification.* Primary 26D10; Secondary 26D15, 26D07.

*Key words and phrases.* Integral operators, Lebesgue spaces, weights, boundedness.

The work was partially supported by the Russian Foundation for Basic Research (Projects 09-01-00093, 09-01-98516, 07-01-00054 and 09-01-98516-p_vostok_a) and by the Far-Eastern Branch of the Russian Academy of Sciences (Projects 09-I-OMH-02 and 09-II-CO-01-003).

Assume $w, v$ be locally integrable non-negative weight functions. We study $L_p - L_q$ boundedness of Hardy-Steklov type operator

$$\mathcal{K}f(x) := w(x) \int_{a(x)}^{b(x)} k(x,y) f(y) v(y) dy \tag{1.1}$$

with border functions $a(x)$ and $b(x)$ satisfying the conditions

$$\begin{array}{ll} \text{(i)} & a(x) \text{ and } b(x) \text{ are differentiable and strictly increasing on } (0,\infty); \\ \text{(ii)} & a(0) = b(0) = 0,\ a(x) < b(x) \text{ for } 0 < x < \infty,\ a(\infty) = b(\infty) = \infty, \end{array} \tag{1.2}$$

and a continuous kernel $k(x,y) > 0$ on $\mathcal{R} := \{(x,y)\colon x > 0, a(x) < y < b(x)\}$ satisfying at least one of two generalized *Oinarov's conditions* $\mathcal{O}_b$ and $\mathcal{O}_a$.

**Definition 1.1.** A kernel $k(x,y)$ is satisfying Oinarov's condition $\mathcal{O}_b$ if there exists a constant $D_1 \geq 1$ independent on $x$, $y$, $z$ such that for $z \leq x$ and $a(x) \leq y \leq b(z)$ we have

$$D_1^{-1} k(x,y) \leq k(x, b(z)) + k(z,y) \leq D_1 k(x,y). \tag{1.3}$$

**Definition 1.2.** We say $k(x,y) \in \mathcal{O}_a$ if there exists an independent on $x$, $y$, $z$ constant $D_2 \geq 1$ such that for $x \leq z$, $a(z) \leq y \leq b(x)$ it holds that

$$D_2^{-1} k(x,y) \leq k(x, a(z)) + k(z,y) \leq D_2 k(x,y). \tag{1.4}$$

Operators of the type (1.1) have been studied by many authors (see, for instance, [1] – [4], [5, 9, 11]). In the limiting cases $a(x) = 0$ or $b(x) = \infty$ the operator (1.1) is reduced to the Hardy type operators with only one variable boundary $a(x)$ or $b(x)$. This fact stands behind *a block-diagonal method*, which we use in this work for investigation of $\mathcal{K}$. The method consists of decomposition (1.1) into a sum of operators with non-overlapping domains and regulated by the following key lemma.

**Lemma 1.1.** [10, Lemma 1] *Let $U = \bigsqcup_k U_k$ and $V = \bigsqcup_k V_k$ be unions of non-overlapping measurable sets and $T = \sum_k T_k$, where $T_k\colon\ L_p(U_k) \to L_q(V_k)$. Then*

$$\|T\|_{L_p(U)\to L_q(V)} = \sup_k \|T_k\|_{L_p(U_k)\to L_q(V_k)},$$

*if $0 < p \leq q < \infty$. For $0 < q < p < \infty$ with $r := pq/(p-q)$ one has*

$$\|T\|_{L_p(U)\to L_q(V)} \approx \left( \sum_k \|T_k\|^r_{L_p(U_k)\to L_q(V_k)} \right)^{1/r}. \tag{1.5}$$

In (1.5) and throughout the paper we write $A \approx B$ instead of $A \ll B \ll A$ or $A = cB$, where relations like $A \ll B$ mean $A \leq cB$ with some constant $c$ depending only on parameters of summations and, possibly, on constants of equivalence in the inequalities of the type (1.3). We shall also make shortening $w_k(x,t) := k(x,t)w(x)$ and $v_k(t,y) := k(t,y)v(y)$.

Note that operators $T_k$ in Lemma 1.1 in our case have the forms

$$K_b f(x) = w(x) \int_{b(c)}^{b(x)} v_{k_b}(x,y) f(y) dy, \quad 0 \leq c \leq x \leq d \leq \infty, \tag{1.6}$$

or

$$K_a f(x) = w(x) \int_{a(x)}^{a(d)} v_{k_a}(x,y) f(y) dy, \quad 0 \le c \le x \le d \le \infty, \tag{1.7}$$

where $k_b(x,y)$ and $k_a(x,y)$ inherit properties of the original kernel function $k(x,y)$. In particular, if the kernel $k(x,y)$ in (1.1) is from Oinarov's class $\mathcal{O}_b$ then $k_b(x,y)$ is also equipped by (1.3) for $0 \le c \le z \le x \le d \le \infty$ and $0 \le b(c) \le y \le b(z)$. It is known (see [6, 7, 8]) that for $1 < q < p < \infty$ the $L_p - L_q$–boundedness of $K_b$ in this case is guaranteed by finiteness of the constants $B_{b,0}$, $B_{b,1}$, where

$$B_{b,0} := \left( \int_{b(c)}^{b(d)} \left[ \int_{b^{-1}(t)}^{d} w_k^q(x,t) dx \right]^{\frac{r}{q}} \left[ \int_{b(c)}^{t} v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}}, \tag{1.8}$$

$$B_{b,1} := \left( \int_{c}^{d} \left[ \int_{t}^{d} w^q(x) dx \right]^{\frac{r}{p}} \left[ \int_{b(c)}^{b(t)} v_k^{p'}(t,y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}}, \tag{1.9}$$

$p' := p/(p-1)$, $q' := q/(q-1)$ and $\|K_b\|_{L_p(b(c),b(d)) \to L_q(c,d)} \approx B_{b,0} + B_{b,1}$. Moreover, there are some known estimates for functionals (1.8) – (1.9) (see [8] or [12] for details):

$$B_{b,0} \approx \mathbb{B}_{b,0} := \left( \int_{b(c)}^{b(d)} \left[ \int_{t}^{b(d)} \left\{ \int_{b^{-1}(y)}^{d} w_k^q(x,y) dx \right\}^{p'} v^{p'}(y) dy \right]^{\frac{r}{q'}} \right.$$

$$\left. \times \left[ \int_{b^{-1}(t)}^{d} w_k^q(x,t) dx \right]^{p' - \frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}}, \tag{1.10}$$

$$B_{b,1} \approx \mathbb{B}_{b,1} := \left( \int_{c}^{d} \left[ \int_{c}^{t} \left\{ \int_{b(c)}^{b(x)} v_k^{p'}(x,y) dy \right\}^{q} w^q(x) dx \right]^{\frac{r}{p}} \right.$$

$$\left. \times \left[ \int_{b(c)}^{b(t)} v_k^{p'}(t,y) dy \right]^{q - \frac{r}{p}} w^q(t) dt \right)^{\frac{1}{r}}. \tag{1.11}$$

Similar characterizations and estimates are true for $K_a$ if $k_a(x,y)$ is satisfying (1.4) as ever $0 \le c \le x \le z \le d \le \infty$ and $0 \le a(z) \le y \le a(d)$. Namely, for $1 < q < p < \infty$ the operator $K_a$ is bounded from $L_p$ to $L_q$ if and only if $B_{a,0}, B_{a,1} < \infty$, where

$$B_{a,0} := \left( \int_{a(c)}^{a(d)} \left[ \int_{c}^{a^{-1}(t)} w_k^q(x,t) dx \right]^{\frac{r}{q}} \left[ \int_{t}^{a(d)} v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}}, \tag{1.12}$$

$$B_{a,1} := \left( \int_{c}^{d} \left[ \int_{c}^{t} w^q(x) dx \right]^{\frac{r}{p}} \left[ \int_{a(t)}^{a(d)} v_k^{p'}(t,y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}} \tag{1.13}$$

and $\|K_a\|_{L_p(a(c),a(d))\to L_q(c,d)} \approx B_{a,0}+B_{a,1}$. We have also

$$B_{a,0} \approx \mathbb{B}_{a,0} := \left( \int_{a(c)}^{a(d)} \left[ \int_{a(c)}^{t} \left\{ \int_{c}^{a^{-1}(y)} w_k^q(x,y)dx \right\}^{p'} v^{p'}(y)dy \right]^{\frac{r}{q'}} \times \left[ \int_{c}^{a^{-1}(t)} w_k^q(x,t)dx \right]^{p'-\frac{r}{q'}} v^{p'}(t)dt \right)^{\frac{1}{r}}, \tag{1.14}$$

$$B_{a,1} \approx \mathbb{B}_{a,1} := \left( \int_{c}^{d} \left[ \int_{t}^{d} \left\{ \int_{a(x)}^{a(d)} v_k^{p'}(x,y)dy \right\}^{q} w^q(x)dx \right]^{\frac{r}{p}} \times \left[ \int_{a(t)}^{a(d)} v_k^{p'}(t,y)dy \right]^{q-\frac{r}{p}} w^(t)dt \right)^{\frac{1}{r}}. \tag{1.15}$$

If $k(x,y)=1$ in (1.1) then, obviously, $A_{b,0}=A_{b,1}$ and $B_{b,0}=B_{b,1}$. Besides, properties of (1.6) and (1.7) are described by functionals similar to $B_{b,0}=B_{b,1}$ and by alternative boundedness constants even in more general case when $0<q<p<\infty$ and

$$\bar{K}_b = w(x)\int_a^{b(x)} f(y)v(y)dy, \qquad 0\le a\le b(x)<\infty, \quad x\in[c,d) \tag{1.16}$$

or

$$\bar{K}_a = w(x)\int_{a(x)}^{b} f(y)v(y)dy, \qquad 0<a(x)\le b\le\infty, \quad x\in(c,d). \tag{1.17}$$

**Lemma 1.2.** [11, Lemma 1] *Let $0<q<p<\infty$, $p>1$ and the operator $\bar{K}_b$ be defined by (1.16). Then $\bar{K}_b$ is bounded from $L_p(a,b(d))$ to $L_q(c,d)$ if and only if $\bar{B}_b<\infty$ or $\bar{\mathbb{B}}_b<\infty$, where $\|\bar{K}_b\|_{L_p(a,b(d))\to L_q(c,d)} \approx \bar{B}_b \approx \bar{\mathbb{B}}_b$ and*

$$\bar{B}_b := \left( \int_c^d \left[ \int_t^d w^q(x)dx \right]^{\frac{r}{p}} \left[ \int_a^{b(t)} v^{p'}(y)dy \right]^{\frac{r}{p'}} w^q(t)dt \right)^{\frac{1}{r}}, \tag{1.18}$$

$$\bar{\mathbb{B}}_b := \left( \int_c^d \left[ \int_c^t \left\{ \int_a^{b(x)} v^{p'}(y)dy \right\}^{q} w^q(x)dx \right]^{\frac{r}{p}} \times \left[ \int_a^{b(t)} v^{p'}(y)dy \right]^{q-\frac{r}{p}} w^q(t)dt \right)^{\frac{1}{r}}. \tag{1.19}$$

**Lemma 1.3.** [11, Lemma 2] *Let $0<q<p<\infty$, $p>1$ and the operator $\bar{B}_a$ be defined by (1.17). Then $\bar{K}_a : L_p(a(c),b)) \to L_q(c,d)$ iff $\bar{B}_a<\infty$ or $\bar{\mathbb{B}}_a<\infty$ with*

$\|\bar{K}_a\|_{L_p(a(c),b)\to L_q(c,d)} \approx \bar{B}_a \approx \bar{\mathbb{B}}_a,$ *where*

$$\bar{B}_a := \left(\int_c^d \left[\int_c^t w^q(x)dx\right]^{\frac{r}{p}} \left[\int_{a(t)}^b v^{p'}(y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}} \tag{1.20}$$

$$\bar{\mathbb{B}}_a := \left(\int_c^d \left[\int_t^d \left\{\int_{a(x)}^b v^{p'}(y)dy\right\}^q w^q(x)dx\right]^{\frac{r}{p}} \times \left[\int_{a(t)}^b v^{p'}(y)dy\right]^{q-\frac{r}{p}} w^q(t)dt\right)^{\frac{1}{r}}. \tag{1.21}$$

In this work we adduce several results describing the boundedness of (1.1) with kernel $k(x,y)$ satisfying Oinarov's type conditions $\mathcal{O}_b$ and/or $\mathcal{O}_a$. The results are obtained with using enumerated characteristics (1.8) – (1.15), (1.18) – (1.21) and under conception of *the fairway function* introduced in [9]. We use the notion of the fairway in its original form (Definition 2.1) for the boundedness criteria in case $k(x,y) = 1$ (Theorem 2.1) and modify it in a proper way for deriving new necessary and sufficient conditions for the boundedness of (1.1) when $k(x,y)$ is from Oinarov's type classes (Theorem 2.5 or Theorem 2.6).

Throughout the paper we assume products of the form $0\cdot\infty$ be equal to 0. The notation $\|T\|_{L_p(a,b)\to L_q(c,d)}$ means the norm of an operator $T$ from $L_p(a,b)$ to $L_q(c,d)$. $\mathbb{Z}$ denotes the set of all integers and $\chi_E$ stands for a characteristic function (indicator) of a subset $E\subset\mathbb{R}^+$.

## 2. Main result

We start from the case $k(x,y) = 1$.

**Definition 2.1.** [9, Deinition 1] Given boundary functions $a(x)$ and $b(x)$, satisfying the conditions (1.2), a number $p\in(1,\infty)$ and a weight function $v(x)$ such that $0 < v(x) < \infty$ a.e. $x\in\mathbb{R}^+$ and $v^{p'}(x)$ is locally integrable on $\mathbb{R}^+$, we define the *fairway-function* $\sigma(x)$ such that $a(x) < \sigma(x) < b(x)$ and

$$\int_{a(x)}^{\sigma(x)} v^{p'}(y)dy = \int_{\sigma(x)}^{b(x)} v^{p'}(y)dy \quad \text{for all } x\in\mathbb{R}^+.$$

Under given conditions (1.2) on boundary functions $a(x)$ and $b(x)$ it is possible to prove that the fairway $\sigma$ is differentiable and strictly increasing function.

Put

$$\Delta(t) := [a(t), b(t)],\ \ \delta(t) := [b^{-1}(\sigma(t)), a^{-1}(\sigma(t))],\ \ \theta(t) := (\sigma^{-1}(a(t)), \sigma^{-1}(b(t))).$$

The following statement contains two forms of criteria for the boundedness of the operator (1.1) with $k(x,y) = 1$.

**Theorem 2.1.** *Let the operator $\mathcal{K}$ of the form (1.1) be given with the boundary functions $a(x)$ and $b(x)$ satisfying the conditions (1.2). $\mathcal{K}$ is bounded from $L_p$ to*

$L_q$ *for* $1 < p \le q < \infty$ *if and only if*

$$\mathcal{A}_M := \sup_{t>0} \mathcal{A}_M(t) = \sup_{t>0} \left( \int_{\delta(t)} w^q(x)dx \right)^{\frac{1}{q}} \left( \int_{\Delta(t)} v^{p'}(y)dy \right)^{\frac{1}{p'}} < \infty$$

*or if and only if*

$$\mathcal{A}_T := \sup_{t>0} \left( \int_{\theta(t)} \left[ \int_{\Delta(x)} v^{p'}(y)dy \right]^q w^q(x)dx \right)^{\frac{1}{q}} \left( \int_{\Delta(t)} v^{p'}(y)dy \right)^{-\frac{1}{p}} < \infty.$$

*Moreover,* $\mathcal{A}_M \approx \mathcal{A}_T \approx \|\mathcal{K}\|_{L_p \to L_q}$.

*If* $0 < q < p < \infty$, $p > 1$ *then* $\mathcal{K}$ *is bounded if and only if* $\mathcal{B}_{MR} < \infty$ *or if and only if* $\mathcal{B}_{PS} < \infty$, *where*

$$\mathcal{B}_{MR} := \left( \int_0^\infty \left[ \int_{\delta(t)} w^q(x)dx \right]^{\frac{r}{p}} \left[ \int_{\Delta(t)} v^{p'}(y)dy \right]^{\frac{r}{p'}} w^q(t)dt \right)^{\frac{1}{r}},$$

$$\mathcal{B}_{PS} := \left( \int_0^\infty \left[ \int_{\theta(t)} \left\{ \int_{\Delta(x)} v^{p'}(y)dy \right\}^q w^q(x)dx \right]^{\frac{r}{p}} \left[ \int_{\Delta(t)} v^{p'}(y)dy \right]^{q-\frac{r}{p}} w^q(t)dt \right)^{\frac{1}{r}}$$

*and* $\mathcal{B}_{MR} \approx \mathcal{B}_{PS} \approx \|\mathcal{K}\|_{L_p \to L_q}$.

Less general form of this statement can be found in [9] and [11]. The dual form of Theorem 2.1 reads

**Theorem 2.2.** *Let the operator* $\mathcal{K}$ *of the form* (1.1) *be given with* $k(x,y) = 1$ *and* $a(x), b(x)$ *satisfying the conditions* (1.2). *Then* $\mathcal{K} : L_p \to L_q$ *for* $1 < p \le q < \infty$ *iff* $\mathcal{A}_M^* < \infty$ *or iff* $\mathcal{A}_T^* < \infty$, *where*

$$\mathcal{A}_M^* := \sup_{t>0} \left( \int_{a(\psi(t))}^{b(\psi(t))} v^{p'}(y)dy \right)^{\frac{1}{p'}} \left( \int_{b^{-1}(t)}^{a^{-1}(t)} w^q(x)dx \right)^{\frac{1}{q}},$$

$$\mathcal{A}_T^* := \sup_{t>0} \left( \int_{\psi^{-1}(b^{-1}(t))}^{\psi^{-1}(a^{-1}(t))} \left[ \int_{b^{-1}(y)}^{a^{-1}(y)} w^q(x)dx \right]^{p'} v^{p'}(y)dy \right)^{\frac{1}{p'}} \left( \int_{b^{-1}(t)}^{a^{-1}(t)} w^q(x)dx \right)^{-\frac{1}{q'}},$$

$$\int_{b^{-1}(y)}^{\psi(y)} w^q(x)dx = \int_{\psi(y)}^{a^{-1}(y)} w^q(x)dx \quad \text{for all } y \in \mathbb{R}^+ \tag{2.1}$$

*and* $\mathcal{A}_M^* \approx \mathcal{A}_T^* \approx \|\mathcal{K}\|_{L_p \to L_q}$.

*If* $0 < q < p < \infty$, $p > 1$ *one has* $\mathcal{K} : L_p \to L_q$ *iff* $\mathcal{B}_{MR}^* < \infty$ *or* $\mathcal{B}_{PS}^* < \infty$, *where*

$$\mathcal{B}_{MR}^* := \left( \int_0^\infty \left[ \int_{a(\psi(t))}^{b(\psi(t))} v^{p'}(y)dy \right]^{\frac{r}{q'}} \left[ \int_{b^{-1}(t)}^{a^{-1}(t)} w^q(x)dx \right]^{\frac{r}{q}} v^{p'}(t)dt \right)^{\frac{1}{r}},$$

$$\mathcal{B}^*_{PS} := \left(\int_0^\infty \left[\int_{\psi^{-1}(b^{-1}(t))}^{\psi^{-1}(a^{-1}(t))} \left\{\int_{b^{-1}(y)}^{a^{-1}(y)} w^q(x)dx\right\}^{p'} v^{p'}(y)dy\right]^{\frac{r}{q'}} \right.$$
$$\left. \times \left[\int_{b^{-1}(t)}^{a^{-1}(t)} w^q(x)dx\right]^{p'-\frac{r}{q'}} v^{p'}(t)dt\right)^{\frac{1}{r}}$$

*with fairway-function $\psi(t)$ defined by (2.1) and $\mathcal{B}^*_{MR} \approx \mathcal{B}^*_{PS} \approx \|\mathcal{K}\|_{L_p\to L_q}$.*

The next statement was obtained for $\mathcal{K}$ with $k(x,y)$ from Oinarov's class $\mathcal{O}_a$.

**Theorem 2.3.** [9, Theorem 2] *Let the boundaries $a(x)$, $b(x)$ of the operator (1.1) be satisfying (1.2) and $k(x,y)\in\mathcal{O}_a$. If $1<p\le q<\infty$, then*

$$\|\mathcal{K}\|_{L_p\to L_q} \approx \mathcal{A}_{a,0} + \mathcal{A}_{a,1}, \tag{2.2}$$

*where*

$$\mathcal{A}_{a,0} := \sup_{s>0}\sup_{s\le t\le a^{-1}(b(s))} \left(\int_s^t w_k^q(x,a(t))dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v^{p'}(y)dy\right)^{\frac{1}{p'}}, \tag{2.3}$$

$$\mathcal{A}_{a,1} := \sup_{s>0}\sup_{s\le t\le a^{-1}(b(s))} \left(\int_s^t w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v_k^{p'}(t,y)dy\right)^{\frac{1}{p'}}. \tag{2.4}$$

*If $1<q<p<\infty$, then*

$$\|\mathcal{K}\|_{L_p\to L_q} \approx \left(\sum_k \left[\left(\mathcal{B}^*_{k,1}\right)^r + \left(\mathcal{B}^*_{k,2}\right)^r + \left(\mathcal{B}^*_{k,3}\right)^r + \left(\mathcal{B}^*_{k,4}\right)^r\right]\right)^{\frac{1}{r}}, \tag{2.5}$$

*where*

$$\mathcal{B}^*_{k,1} := \left(\int_{a(\xi_k)}^{a(\xi_{k+1})} \left[\int_{\xi_k}^{a^{-1}(t)} w_k^q(x,t)dx\right]^{\frac{r}{q}} \left[\int_t^{a(\xi_{k+1})} v^{p'}(y)dy\right]^{\frac{r}{q'}} v^{p'}(t)dt\right)^{\frac{1}{r}},$$

$$\mathcal{B}^*_{k,2} := \left(\int_{\xi_k}^{\xi_{k+1}} \left[\int_{\xi_k}^{t} w^q(x)dx\right]^{\frac{r}{p}} \left[\int_{a(t)}^{a(\xi_{k+1})} v_k^{p'}(t,y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}},$$

$$\mathcal{B}^*_{k,3} := \left(\int_{b(\xi_k)}^{b(\xi_{k+1})} \left[\int_{b^{-1}(t)}^{\xi_{k+1}} w_k^q(x,a(\xi_{k+1}))dx\right]^{\frac{r}{q}} \left[\int_{b(\xi_k)}^{t} v^{p'}(y)dy\right]^{\frac{r}{q'}} v^{p'}(t)dt\right)^{\frac{1}{r}},$$

$$\mathcal{B}^*_{k,4} := \left(\int_{\xi_k}^{\xi_{k+1}} \left[\int_t^{\xi_{k+1}} w^q(x)dx\right]^{\frac{r}{p}} \left[\int_{b(\xi_k)}^{b(t)} v_k^{p'}(\xi_{k+1},y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}}$$

*and $\xi_0=1,\ \xi_k=(a^{-1}\circ b)^k(\xi_0),\ k\in\mathbb{Z}$.*

Similar result is true for $\mathcal{K}$ with $k(x,y)\in\mathcal{O}_b$.

**Theorem 2.4.** [9, Theorem 3] *Let the operator* $\mathcal{K}$ *be defined by* (1.1) *with* $a(x)$, $b(x)$ *satisfying* (1.2) *and let* $k(x,y) \in \mathcal{O}_b$. *If* $1 < p \le q < \infty$, *then*

$$\|\mathcal{K}\|_{L_p \to L_q} \approx \mathcal{A}_{b,0} + \mathcal{A}_{b,1}, \tag{2.6}$$

*where*

$$\mathcal{A}_{b,0} := \sup_{t>0} \sup_{b^{-1}(a(t)) \le s \le t} \left( \int_s^t w_k^q(x, b(s)) dx \right)^{\frac{1}{q}} \left( \int_{a(t)}^{b(s)} v^{p'}(y) dy \right)^{\frac{1}{p'}}, \tag{2.7}$$

$$\mathcal{A}_{b,1} := \sup_{t>0} \sup_{b^{-1}(a(t)) \le s \le t} \left( \int_s^t w^q(x) dx \right)^{\frac{1}{q}} \left( \int_{a(t)}^{b(s)} v_k^{p'}(s, y) dy \right)^{\frac{1}{p'}}. \tag{2.8}$$

*If* $1 < q < p < \infty$, *then*

$$\|\mathcal{K}\|_{L_p \to L_q} \approx \left( \sum_k \left[ \mathcal{B}_{k,1}^r + \mathcal{B}_{k,2}^r + \mathcal{B}_{k,3}^r + \mathcal{B}_{k,4}^r \right] \right)^{\frac{1}{r}}, \tag{2.9}$$

*where*

$$\mathcal{B}_{k,1} := \left( \int_{a(\xi_k)}^{a(\xi_{k+1})} \left[ \int_{\xi_k}^{a^{-1}(t)} w_k^q(x, b(\xi_k)) dx \right]^{\frac{r}{q}} \left[ \int_t^{a(\xi_{k+1})} v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}},$$

$$\mathcal{B}_{k,2} := \left( \int_{\xi_k}^{\xi_{k+1}} \left[ \int_{\xi_k}^t w^q(x) dx \right]^{\frac{r}{p}} \left[ \int_{a(t)}^{a(\xi_{k+1})} v_k^{p'}(\xi_k, y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}},$$

$$\mathcal{B}_{k,3} := \left( \int_{b(\xi_k)}^{b(\xi_{k+1})} \left[ \int_{b^{-1}(t)}^{\xi_{k+1}} w_k^q(x, t) dx \right]^{\frac{r}{q}} \left[ \int_{b(\xi_k)}^t v^{p'}(y) dy \right]^{\frac{r}{q'}} v^{p'}(t) dt \right)^{\frac{1}{r}},$$

$$\mathcal{B}_{k,4} := \left( \int_{\xi_k}^{\xi_{k+1}} \left[ \int_t^{\xi_{k+1}} w^q(x) dx \right]^{\frac{r}{p}} \left[ \int_{b(\xi_k)}^{b(t)} v_k^{p'}(t, y) dy \right]^{\frac{r}{p'}} w^q(t) dt \right)^{\frac{1}{r}}$$

*and* $\xi_0 = 1$, $\xi_k = (a^{-1} \circ b)^k (\xi_0)$, $k \in \mathbb{Z}$.

Note that (2.6) was derived earlier in [3] and [4].

Double supremums in (2.3), (2.4), (2.7), (2.8) and a discrete form of (2.5), (2.9) gave a motivation for searching new necessary and sufficient boundedness conditions of $\mathcal{K}$ with more convenient forms.

Let functions $\phi(x)$ and $\rho(y)$ on $\mathbb{R}^+ \cup \{+\infty\}$, where $a(x) \le \phi(x) \le b(x)$ and $b^{-1}(y) \le \rho(y) \le a^{-1}(y)$, be *fairway*-functions satisfying the following

**Definition 2.2.** Given boundary functions $a(x)$ and $b(x)$ satisfying the conditions (1.2), numbers $p, q \in (1, \infty)$, a continuous kernel $0 < k(x,y) < \infty$ a.e on $\mathcal{R}$ and weight functions $0 < v, w < \infty$ a.e. on $\mathbb{R}^+$ such that for any fixed $x > 0$ the function $v_k^{p'}(x,y)$ is locally integrable on $\mathbb{R}^+$ with respect to the variable $y$ as well as for any $y > 0$ the function $w_k^q(x,y)$ is locally integrable on $\mathbb{R}^+$ with respect to $x$, we define two *fairways* – the functions $\phi(x)$ and $\rho(y)$ such that $a(x) < \phi(x) < b(x)$, $b^{-1}(y) < \rho(y) < a^{-1}(y)$ and

$$\int_{a(x)}^{\phi(x)} v_k^{p'}(x,y)dy = \int_{\phi(x)}^{b(x)} v_k^{p'}(x,y)dy \quad \text{for all } x > 0, \tag{2.10}$$

$$\int_{b^{-1}(y)}^{\rho(y)} w_k^q(x,y)dx = \int_{\rho(y)}^{a^{-1}(y)} w_k^q(x,y)dx \quad \text{for all } y > 0. \tag{2.11}$$

By assumptions of the definition $\phi(x)$ and $\rho(y)$ are continuous functions. Put

$$\Theta(t) := \Theta^-(t) \cup \Theta^+(t), \quad \Theta^-(t) := [b^{-1}(t), \rho(t)), \quad \Theta^+(t) := [\rho(t), a^{-1}(t)),$$
$$\vartheta(t) := \vartheta^-(t) \cup \vartheta^+(t), \quad \vartheta^-(t) := [a(\rho(t)), t), \quad \vartheta^+(t) := [t, b(\rho(t))),$$
$$\delta(t) := \delta^-(t) \cup \delta^+(t), \quad \delta^-(t) := [b^{-1}(\phi(t)), t), \quad \delta^+(t) := [t, a^{-1}(\phi(t))),$$
$$\Delta(t) := \Delta^-(t) \cup \Delta^+(t), \quad \Delta^-(t) := [a(t), \phi(t)), \quad \Delta^+(t) := [\phi(t), b(t))$$

and denote

$$\mathcal{A}_\rho^\pm := \sup_{t>0} \mathcal{A}_\rho^\pm(t) = \sup_{t>0} \left(\int_{\Theta(t)} w_k^q(x,t)dx\right)^{\frac{1}{q}} \left(\int_{\vartheta^\pm(t)} v^{p'}(y)dy\right)^{\frac{1}{p'}},$$

$$\mathcal{A}_\phi^\pm := \sup_{t>0} \mathcal{A}_\phi^\pm(t) = \sup_{t>0} \left(\int_{\delta^\pm(t)} w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{\Delta(t)} v_k^{p'}(t,y)dy\right)^{\frac{1}{p'}},$$

$$\mathcal{B}_\rho^\pm := \left(\int_0^\infty \mathcal{B}_\rho^\pm(t)dt\right)^{\frac{1}{r}} = \left(\int_0^\infty \left[\int_{\Theta(t)} w_k^q(x,t)dx\right]^{\frac{r}{q}} \left[\int_{\vartheta^\pm(t)} v^{p'}(y)dy\right]^{\frac{r}{q'}} v^{p'}(t)dt\right)^{\frac{1}{r}},$$

$$\mathcal{B}_\phi^\pm := \left(\int_0^\infty \mathcal{B}_\phi^\pm(t)dt\right)^{\frac{1}{r}} = \left(\int_0^\infty \left[\int_{\delta^\pm(t)} w^q(x)dx\right]^{\frac{r}{p}} \left[\int_{\Delta(t)} v_k^{p'}(t,y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}},$$

$$\mathcal{A}_\rho := \sup_{t>0} \mathcal{A}_\rho(t) = \sup_{t>0} \left(\int_{\Theta(t)} w_k^q(x,t)dx\right)^{\frac{1}{q}} \left(\int_{\vartheta(t)} v^{p'}(y)dy\right)^{\frac{1}{p'}},$$

$$\mathcal{A}_\phi := \sup_{t>0} \mathcal{A}_\phi(t) = \sup_{t>0} \left(\int_{\delta(t)} w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{\Delta(t)} v_k^{p'}(t,y)dy\right)^{\frac{1}{p'}},$$

$$\mathcal{B}_\rho := \left(\int_0^\infty \mathcal{B}_\rho(t)dt\right)^{\frac{1}{r}} = \left(\int_0^\infty \left[\int_{\Theta(t)} w_k^q(x,t)dx\right]^{\frac{r}{q}} \left[\int_{\vartheta(t)} v^{p'}(y)dy\right]^{\frac{r}{q'}} v^{p'}(t)dt\right)^{\frac{1}{r}},$$

$$\mathcal{B}_\phi := \left(\int_0^\infty \mathcal{B}_\phi(t)dt\right)^{\frac{1}{r}} = \left(\int_0^\infty \left[\int_{\delta(t)} w^q(x)dx\right]^{\frac{r}{p}} \left[\int_{\Delta(t)} v_k^{p'}(t,y)dy\right]^{\frac{r}{p'}} w^q(t)dt\right)^{\frac{1}{r}}.$$

New boundedness conditions for $\mathcal{K}$ with $k(x,y) \in \mathcal{O}_b$ are proved in Section 3 and read

**Theorem 2.5.** *Let the operator $\mathcal{K}$ be defined by (1.1) with the border functions $a(x)$, $b(x)$ satisfying (1.2) and a continuous positive kernel $k(x,y)$ on $\mathcal{R}$ from the Oinarov's type class $\mathcal{O}_b$. Suppose that the functions $\rho(y)$, $\phi(x)$ on $\mathbb{R}^+$ are strictly increasing fairways from Definition 2.2. If $1 < p \le q < \infty$, then*

$$\mathcal{A}_\rho^- + \mathcal{A}_\phi^+ \ll \|\mathcal{K}\|_{L_p \to L_q} \ll \mathcal{A}_\rho + \mathcal{A}_\phi. \tag{2.12}$$

*If $1 < q < p < \infty$, then*

$$\mathcal{B}_\rho^- + \mathcal{B}_\phi^+ \ll \|\mathcal{K}\|_{L_p \to L_q} \ll \mathcal{B}_\rho + \mathcal{B}_\phi. \tag{2.13}$$

Analogously we obtain a similar result for $\mathcal{K}$ with $k(x,y)$ satisfying the condition (1.4).

**Theorem 2.6.** *Let the operator $\mathcal{K}$ be defined by (1.1) with $a(x)$, $b(x)$ satisfying (1.2) and a continuous kernel $k(x,y) > 0$ on $\mathcal{R}$ from the class $\mathcal{O}_a$. Suppose that $\rho(y)$, $\phi(x)$ on $\mathbb{R}^+$ are strictly increasing fairways satisfying Definition 2.2. If $1 < p \le q < \infty$, then*

$$\mathcal{A}_\rho^+ + \mathcal{A}_\phi^- \ll \|\mathcal{K}\|_{L_p \to L_q} \ll \mathcal{A}_\rho + \mathcal{A}_\phi.$$

*If $1 < q < p < \infty$, then*

$$\mathcal{B}_\rho^+ + \mathcal{B}_\phi^- \ll \|\mathcal{K}\|_{L_p \to L_q} \ll \mathcal{B}_\rho + \mathcal{B}_\phi.$$

In conclusion of the section we provide several cases when the results of Theorems 2.5 and 2.6 became of a criterion form.

**Theorem 2.7.** *Let the operator $\mathcal{K}$ be defined by (1.1) with $a(x)$, $b(x)$ satisfying (1.2) and a continuous kernel $k(x,y) > 0$ on $\mathcal{R}$. Suppose that the functions $\rho(x)$, $\phi(x)$ on $\mathbb{R}^+$ are strictly increasing fairways from Definition 2.2.*

**(a)** *If $k(x,y) \in \mathcal{O}_b$ and*

$$\int_{\vartheta^-(t)} v^{p'}(y)dy \approx \int_{\vartheta(t)} v^{p'}(y)dy, \qquad t > 0,$$

$$\int_{\delta^+(t)} w^q(x)dx \approx \int_{\delta(t)} w^q(x)dx, \qquad t > 0,$$

*then*

$$\|\mathcal{K}\|_{L_p \to L_q} \approx \begin{cases} \mathcal{A}_\rho + \mathcal{A}_\phi, & 1 < p \le q < \infty, \\ \mathcal{B}_\rho + \mathcal{B}_\phi, & 1 < q < p < \infty, \end{cases} \tag{2.14}$$

**(b)** *If $k(x,y) \in \mathcal{O}_a$ and*

$$\int_{\vartheta^+(t)} v^{p'}(y)dy \approx \int_{\vartheta(t)} v^{p'}(y)dy, \qquad t > 0,$$

$$\int_{\delta^-(t)} w^q(x)dx \approx \int_{\delta(t)} w^q(x)dx, \qquad t > 0,$$

*then the estimate (2.14) holds.*

**(c)** *If $k(x,y) \in \mathcal{O}_a \cap \mathcal{O}_b$, then the equivalence (2.14) is true.*

Proof of Theorem 2.7 easy follows from Theorems 2.5 and 2.6.

## 3. Proof of Theorem 2.5

We start from *the lower estimate* in (2.12). Let $1 < p \le q < \infty$. We have from (2.6)

$$\|\mathcal{K}\|_{L_p\to L_q} \approx \sup_{t>0} \sup_{b^{-1}(a(t))\le s\le t} \left[\mathcal{A}_0(s,t) + \mathcal{A}_1(s,t)\right], \tag{3.1}$$

where

$$\mathcal{A}_0(s,t) := \left(\int_s^t w_k^q(x,b(s))dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v^{p'}(y)dy\right)^{\frac{1}{p'}},$$

$$\mathcal{A}_1(s,t) := \left(\int_s^t w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(t)}^{b(s)} v_k^{p'}(s,y)dy\right)^{\frac{1}{p'}}.$$

Using (2.11) we find that

$$\mathcal{A}_\rho^-(t) = 2^{1/q}\left(\int_{\Theta^-(t)} w_k^q(x,t)dx\right)^{\frac{1}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y)dy\right)^{\frac{1}{p'}}$$

$$= 2^{1/q}\left(\int_{b^{-1}(\rho^{-1}(s))}^{s} w_k^q(x,\rho^{-1}(s))dx\right)^{\frac{1}{q}} \left(\int_{a(s)}^{\rho^{-1}(s)} v^{p'}(y)dy\right)^{\frac{1}{p'}}$$

$$= 2^{1/q}A_0(b^{-1}(\rho^{-1}(s)),s) \le \sup_{t>0}\sup_{b^{-1}(a(t))\le s\le t}\mathcal{A}_0(s,t).$$

On the strength of (3.1) it implies $\mathcal{A}_\rho^- \ll \|\mathcal{K}\|_{L_p\to L_q}$. Analogously,

$$\mathcal{A}_\phi^+(t) \overset{(2.10)}{=} 2^{1/p'}\left(\int_{\delta^+(t)} w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{\Delta^+(t)} v_k^{p'}(t,y)dy\right)^{\frac{1}{p'}}$$

$$= 2^{1/p'}A_1(t,a^{-1}(\phi(t))) \le \sup_{t>0}\sup_{b^{-1}(a(t))\le s\le t}\mathcal{A}_1(s,t)$$

implies $\mathcal{A}_\phi^+ \ll \|\mathcal{K}\|_{L_p\to L_q}$, and the lower estimate in (2.12) is proved.

For *the upper estimate* in (2.12) we put $\tau_0 := \rho(a(t))$ and write

$$\sup_{b^{-1}(a(t))\le s\le t}\mathcal{A}_0(s,t) \le \sup_{b^{-1}(a(t))\le s\le \rho(a(t))<t}\mathcal{A}_0(s,t) + \sup_{\rho(a(t))\le s\le t}\mathcal{A}_0(s,t)$$

$$\le \sup_{b^{-1}(\rho^{-1}(\tau_0))\le s\le\tau_0}\left(\int_s^{a^{-1}(\rho^{-1}(\tau_0))} w_k^q(x,b(s))dx\right)^{\frac{1}{q}} \left(\int_{\rho^{-1}(\tau_0)}^{b(s)} v^{p'}(y)dy\right)^{\frac{1}{p'}}$$

$$+\sup_{\rho(a(t))\le s\le t}\left(\int_s^{a^{-1}(\rho^{-1}(s))} w_k^q(x,b(s))dx\right)^{\frac{1}{q}} \left(\int_{a(s)}^{b(s)} v^{p'}(y)dy\right)^{\frac{1}{p'}} =: H_1(\tau_0)+H_2(t).$$

Indeed, if $b^{-1}(a(t)) \le s \le \rho(a(t)) < t$, that is $b^{-1}(\rho^{-1}(\tau_0)) \le s \le \tau_0$, then $(s,t) = (s, a^{-1}(\rho^{-1}(\tau_0)))$ and $(a(t),b(s)) = (\rho^{-1}(\tau_0), b(s))$. If $\rho(a(t)) \le s \le t$, then $(s,t) \subset (s, a^{-1}(\rho^{-1}(s)))$ and $(a(t),b(s)) \subset (a(s),b(s))$.

To estimate $H_1(\tau_0)$ we use (1.3) with $y=\rho^{-1}(\tau_0)$, $z=s$ and obtain

$$H_1(\tau_0) \ll \left(\int_{b^{-1}(\rho^{-1}(\tau_0))}^{a^{-1}(\rho^{-1}(\tau_0))} w_k^q(x,\rho^{-1}(\tau_0))dx\right)^{\frac{1}{q}} \left(\int_{\rho^{-1}(\tau_0)}^{b(\tau_0)} v^{p'}(y)dy\right)^{\frac{1}{p'}}$$
$$= \left(\int_{b^{-1}(z)}^{a^{-1}(z))} w_k^q(x,z)dx\right)^{\frac{1}{q}} \left(\int_{z}^{b(\rho(z))} v^{p'}(y)dy\right)^{\frac{1}{p'}} = \mathcal{A}_\rho^+(z) \le \mathcal{A}_\rho^+.$$

Since $s\le x$ and $a(x)\le \rho^{-1}(s)\le b(s)$ in $H_2(t)$ we obtain by using (1.3) with $z=s$ and $y=\rho^{-1}(s)$ :

$$H_2(t) \ll \sup_{\rho(a(t))\le s\le t} \left(\int_{s}^{a^{-1}(\rho^{-1}(s))} w_k^q(x,\rho^{-1}(s))dx\right)^{\frac{1}{q}} \left(\int_{a(s)}^{b(s)} v^{p'}(y)dy\right)^{\frac{1}{p'}}$$
$$= \sup_{a(t)\le z\le \rho^{-1}(t)} \left(\int_{\rho(z)}^{a^{-1}(z)} w_k^q(x,z)dx\right)^{\frac{1}{q}} \left(\int_{a(\rho(z))}^{b(\rho(z))} v^{p'}(y)dy\right)^{\frac{1}{p'}} \le \mathcal{A}_\rho.$$

Thus,

$$\sup_{t>0}\ \sup_{b^{-1}(a(t))\le s\le t} \mathcal{A}_0(s,t) \ll \mathcal{A}_\rho. \tag{3.2}$$

Analogously, we put $\tau_1 := \phi^{-1}(a(t))$ and write

$$\sup_{b^{-1}(a(t))\le s\le t} \mathcal{A}_1(s,t) \le \sup_{b^{-1}(a(t))\le s\le \phi^{-1}(a(t))<t} \mathcal{A}_1(s,t) + \sup_{\phi^{-1}(a(t))\le s\le t} \mathcal{A}_1(s,t)$$
$$\le \sup_{b^{-1}(\phi(\tau_1))\le s\le \tau_1} \left(\int_{s}^{a^{-1}(\phi(\tau_1))} w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{\phi(\tau_1)}^{b(s)} v_k^{p'}(s,y)dy\right)^{\frac{1}{p'}}$$
$$+ \sup_{\phi^{-1}(a(t))\le s\le t} \left(\int_{s}^{a^{-1}(\phi(s))} w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{a(s)}^{b(s)} v_k^{p'}(s,y)dy\right)^{\frac{1}{p'}} =: H_3(\tau_1)+H_4(t).$$

Obviously, $H_4(t) \le \sup_{\phi^{-1}(a(t))\le s\le t} \mathcal{A}_\phi^+(s) \le \mathcal{A}_\phi^+$. For $H_3(\tau_1)$ we apply (1.3) with $z=s\le \tau_1 = x$ and $a(\tau_1)<\phi(\tau_1)\le y\le b(s)$ :

$$\begin{aligned} H_3(\tau_1) &\ll \sup_{b^{-1}(\phi(\tau_1))\le s\le \tau_1} \left(\int_{s}^{a^{-1}(\phi(\tau_1))} w^q(x)dx\right)^{\frac{1}{q}} \left(\int_{\phi(\tau_1)}^{b(s)} v_k^{p'}(\tau_1,y)dy\right)^{\frac{1}{p'}} \\ &\le \mathcal{A}_\phi(\tau_1) \le \mathcal{A}_\phi. \end{aligned}$$

Thus,

$$\sup_{t>0}\ \sup_{b^{-1}(a(t))\le s\le t} \mathcal{A}_1(s,t) \ll \mathcal{A}_\phi. \tag{3.3}$$

By combining (3.1), (3.2) and (3.3) we obtain the upper estimate in (2.12).

Now we consider the case $1<q<p<\infty$. Let us prove first *the upper estimate* in (2.13). To this end we take a point sequence $\{\xi_k\}_{k\in\mathbb{Z}} \subset (0,\infty)$ such that

$$\xi_0 = 1, \qquad \xi_k = (a^{-1}\circ b)^k(1), \qquad k\in\mathbb{Z},$$

and put

$$\eta_k = a(\xi_k) = b(\xi_{k-1}), \qquad \Delta_k = [\xi_k, \xi_{k+1}), \qquad \delta_k = [\eta_k, \eta_{k+1}), \qquad k \in \mathbb{Z}.$$

Breaking the semiaxis $(0, \infty)$ by points of the sequence $\{\xi_k\}_{k\in\mathbb{Z}}$ we decompose the operator $\mathcal{K}$ into the sum

$$\mathcal{K} = \mathcal{T} + \mathcal{S} \tag{3.4}$$

of block-diagonal operators $\mathcal{T}$ and $\mathcal{S}$ such that

$$\mathcal{T} = \sum_{k\in\mathbb{Z}} T_k, \qquad \mathcal{S} = \sum_{k\in\mathbb{Z}} S_k, \tag{3.5}$$

where

$$T_k f(x) = w(x) \int_{a(x)}^{a(\xi_{k+1})} v_k(x, y) f(y) dy, \qquad T_k : L_p(\delta_k) \to L_q(\Delta_k),$$

$$S_k f(x) = w(x) \int_{b(\xi_k)}^{b(x)} v_k(x, y) f(y) dy, \qquad S_k : L_p(\delta_{k+1}) \to L_q(\Delta_k).$$

Kernels $k(x, y)$ of the operators $T_k$ and $S_k$ satisfy the condition (1.3) for $z \le x$, $x \in [\xi_k, \xi_{k+1}]$ and

$$a(x) \le y \le b(\xi_k), \qquad b(\xi_k) \le y \le b(z), \tag{3.6}$$

respectively.

To estimate a norm of the operator $S_k$ we take into account two key points $s_\rho := b^{-1}(\rho^{-1}(\xi_{k+1}))$, $s_\phi := \phi^{-1}(b(\xi_k)) = \phi^{-1}(a(\xi_{k+1}))$ and consider three only possible variants:

$$\text{(i)} \quad s_\rho < s_\phi, \qquad \text{(ii)} \quad s_\rho = s_\phi, \qquad \text{(iii)} \quad s_\rho > s_\phi.$$

In the case **(i)** we have

$$S_k f = \sum_{i=1}^{3} S_{k,i} f + \sum_{i=1}^{3} H_{k,i} f, \tag{3.7}$$

where

$$S_{k,1} f = \chi_{[\xi_k, s_\rho]} S_k f, \quad L_p(b(\xi_k), b(s_\rho)) \to L_q(\xi_k, s_\rho),$$

$$H_{k,1} f = \chi_{[s_\rho, s_\phi]} S_k \left( f \chi_{[b(\xi_k), b(s_\rho)]} \right), \quad L_p(b(\xi_k), b(s_\rho)) \to L_q(s_\rho, s_\phi),$$

$$S_{k,2} f = \chi_{[s_\rho, s_\phi]} S_k \left( f \chi_{[b(s_\rho), b(s_\phi)]} \right), \quad L_p(b(s_\rho), b(s_\phi)) \to L_q(s_\rho, s_\phi),$$

$$H_{k,2} f = \chi_{[s_\phi, \xi_{k+1}]} S_k \left( f \chi_{[b(\xi_k), b(s_\rho)]} \right), \quad L_p(b(\xi_k), b(s_\rho)) \to L_q(s_\phi, \xi_{k+1}),$$

$$H_{k,3} f = \chi_{[s_\phi, \xi_{k+1}]} S_k \left( f \chi_{[b(s_\rho), b(s_\phi)]} \right), \quad L_p(b(s_\rho), b(s_\phi)) \to L_q(s_\phi, \xi_{k+1}),$$

$$S_{k,3} f = \chi_{[s_\phi, \xi_{k+1}]} S_k \left( f \chi_{[b(s_\phi), b(\xi_{k+1})]} \right), \quad L_p(b(s_\phi), b(\xi_{k+1})) \to L_q(s_\phi, \xi_{k+1}).$$

By (1.8), (1.9) and (1.11) we obtain

$$\|S_{k,1}\|^r_{L_p(b(\xi_k),b(s_\rho))\to L_q(\xi_k,s_\rho)} \approx B^r_{b,0} + B^r_{b,1} \approx B^r_{b,0} + \mathbb{B}^r_{b,1} \tag{3.8}$$
$$= \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{b^{-1}(t)}^{s_\rho} w_k^q(x,t)dx\right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{t} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(t)dt$$
$$+\int_{\xi_k}^{s_\rho}\left(\int_{\xi_k}^{t}\left[\int_{b(\xi_k)}^{b(x)} v_k^{p'}(x,y)dy\right]^q w^q(x)dx\right)^{\frac{r}{p}} \left[\int_{b(\xi_k)}^{b(t)} v_k^{p'}(t,y)dy\right]^{q-\frac{r}{p}} w^q(t)dt.$$

Since $t \geq b(\xi_k) > a(s_\rho)$ in $B_{b,0}$ then $[b^{-1}(t), s_\rho] \subset [b^{-1}(t), a^{-1}(t)]$, and $[b(\xi_k), t] \subseteq [a(\rho(t)), t]$ because of $t \leq b(s_\rho) = \rho^{-1}(\xi_{k+1}) \implies \rho(t) \leq \xi_{k+1} \implies a(\rho(t)) \leq a(\xi_{k+1}) = b(\xi_k)$. Therefore,

$$B^r_{b,0} \leq \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{\Theta(t)} w_k^q(x,t)dx\right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(t)dt \leq \int_{\delta_{k+1}} \mathcal{B}^-_\rho(t)dt. \tag{3.9}$$

To estimate $\mathbb{B}_{b,1}$ note that in view of (1.3) we have $k(x,y) \ll k(t,y)$, where $x \leq t$, $a(t) < b(\xi_k) \leq y \leq b(x)$ and, hence,

$$\mathbb{B}^r_{b,1} \ll \int_{\xi_k}^{s_\rho} \left(\int_{\xi_k}^{t} w^q(x)dx\right)^{\frac{r}{p}} \left(\int_{b(\xi_k)}^{b(t)} k^{p'}(t,y)v^{p'}(y)dy\right)^{\frac{r}{p'}} w^q(t)dt.$$

Here $[\xi_k, t] \subset [b^{-1}(\phi(t)), t]$ since $t \leq s_\rho < s_\phi = \phi^{-1}(b(\xi_k)) \implies \phi(t) < b(\xi_k) \implies b^{-1}(\phi(t)) < \xi_k$. Obviously, $[b(\xi_k), b(t)] \subset [a(t), b(t)]$. Therefore,

$$\mathbb{B}^r_{b,1} \ll \int_{\xi_k}^{s_\rho} \left(\int_{\delta^-(t)} w^q(x)dx\right)^{\frac{r}{p}} \left(\int_{\Delta(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt \leq \int_{\Delta_k} \mathcal{B}^-_\phi(t)dt. \tag{3.10}$$

To estimate $\|H_{k,1}\|_{L_p(b(\xi_k),b(s_\rho))\to L_q(s_\rho,s_\phi)}$ we decompose the operator $H_{k,1}$ by using (1.3) into the following sum:

$$H_{k,1}f(x) = w_k(x, b(s_\rho)) \int_{b(\xi_k)}^{b(s_\rho)} f(y)v(y)dy + w(x) \int_{b(\xi_k)}^{b(s_\rho)} v_k(s_\rho, y)f(y)dy$$
$$=: H^w_{k,1}f(x) + H^v_{k,1}f(x).$$

By Hölder's inequality and (1.3)

$$\|H^w_{k,1}\|^r_{L_p(b(\xi_k),b(s_\rho))\to L_q(s_\rho,s_\phi)} = \left(\int_{s_\rho}^{s_\phi} w_k^q(x, b(s_\rho))dx\right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{b(s_\rho)} v^{p'}(y)dy\right)^{\frac{r}{p'}}$$
$$\overset{(1.3)}{\ll} \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{b(\xi_k)}^{t} v^{p'}(y)dy\right)^{\frac{r}{q'}} \left(\int_{b^{-1}(t)}^{s_\phi} w_k^q(x,t)dx\right)^{\frac{r}{q}} v^{p'}(t)dt$$
$$< \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{\Theta(t)} w_k^q(x,t)dx\right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(t)dt \leq \int_{\delta_{k+1}} \mathcal{B}^-_\rho(t)dt, \tag{3.11}$$

since $a(s_\phi) < t$ and $t \le b(s_\rho) = \rho^{-1}(\xi_{k+1}) \Longrightarrow \rho(t) \le \xi_{k+1} \Longrightarrow a(\rho(t)) \le a(\xi_{k+1}) = b(\xi_k)$. Analogously,

$$\|H^v_{k,1}\|^r_{L_p(b(\xi_k),b(s_\rho))\to L_q(s_\rho,s_\phi)} = \left(\int_{s_\rho}^{s_\phi} w^q(x)dx\right)^{\frac{r}{q}}\left(\int_{b(\xi_k)}^{b(s_\rho)} v_k^{p'}(s_\rho,y)dy\right)^{\frac{r}{p'}}$$
$$\ll \int_{s_\rho}^{s_\phi}\left(\int_{s_\rho}^{t} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{b(\xi_k)}^{b(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt$$
$$< \int_{s_\rho}^{s_\phi}\left(\int_{\delta^-(t)} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{\Delta(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt \le \int_{\Delta_k} \mathcal{B}^-_\phi(t)dt \tag{3.12}$$

on the strength of $a(t) < a(\xi_{k+1}) = b(\xi_k)$ and $t \le s_\phi \Longrightarrow \phi(t) \le b(\xi_k) \Longrightarrow b^{-1}(\phi(t)) \le \xi_k < s_\rho$. For $S_{k,2}$ we use (1.8), (1.9) and (1.10), (1.11):

$$\|S_{k,2}\|^r_{L_p(b(s_\rho),b(s_\phi))\to L_q(s_\rho,s_\phi)} \approx B^r_{b,0} + B^r_{b,1} \approx \mathbb{B}^r_{b,0} + \mathbb{B}^r_{b,1} \tag{3.13}$$
$$= \int_{b(s_\rho)}^{b(s_\phi)}\left(\int_t^{b(s_\phi)}\left[\int_{b^{-1}(y)}^{s_\phi} w_k^q(x,y)dx\right]^{p'} v^{p'}(y)dy\right)^{\frac{r}{q'}}\left[\int_{b^{-1}(t)}^{s_\phi} w_k^q(x,t)dx\right]^{p'-\frac{r}{q'}} v^{p'}(t)dt$$
$$+ \int_{s_\rho}^{s_\phi}\left(\int_{s_\rho}^{t}\left[\int_{b(s_\rho)}^{b(x)} v_k^{p'}(x,y)dy\right]^{q} w^q(x)dx\right)^{\frac{r}{p}}\left[\int_{b(s_\rho)}^{b(t)} v_k^{p'}(t,y)dy\right]^{q-\frac{r}{p}} w^q(t)dt.$$

Since $a(x) < b(\xi_k) < b(s_\rho) \le t \le y = b(z)$ with $z = b^{-1}(y) \le x$ in $\mathbb{B}_{b,0}$ then $k(x,y) = k(x,b(z)) \ll k(x,t)$. Therefore,

$$\mathbb{B}^r_{b,0} \ll \int_{b(s_\rho)}^{b(s_\phi)}\left(\int_t^{b(s_\phi)} v^{p'}(y)dy\right)^{\frac{r}{q'}}\left(\int_{b^{-1}(t)}^{s_\phi} w_k^q(x,t)dx\right)^{\frac{r}{q}} v^{p'}(t)dt.$$

Since $s_\phi < \xi_{k+1}$ then $a(s_\phi) < \rho^{-1}(s_\phi) < \rho^{-1}(\xi_{k+1}) = b(s_\rho) \le t \Longrightarrow s_\phi < a^{-1}(t)$ and $s_\phi < \rho(t) \Longrightarrow b(s_\phi) < b(\rho(t))$. Thus, $[t, b(s_\phi)] \subset [t, b(\rho(t))]$, $[b^{-1}(t), s_\phi] \subset [b^{-1}(t), a^{-1}(t)]$. Hence,

$$\mathbb{B}^r_{b,0} \ll \int_{b(s_\rho)}^{b(s_\phi)}\left(\int_{\Theta(t)} w_k^q(x,t)dx\right)^{\frac{r}{q}}\left(\int_{\vartheta^+(t)} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(t)dt \le \int_{\delta_{k+1}} \mathcal{B}^+_\rho(t)dt. \tag{3.14}$$

Since $x \le t$, $a(t) < b(\xi_k) < b(s_\rho) \le y \le b(x)$ we have $k(x,y) \ll k(t,y)$ in $\mathbb{B}_{b,1}$. Therefore, in view of $\xi_k < s_\rho \le t \le s_\phi < \xi_{k+1} \Longrightarrow \phi(t) \le b(\xi_k) \Longrightarrow b^{-1}(\phi(t)) \le \xi_k < s_\rho$ and $a(t) < b(\xi_k)$ it holds that

$$\mathbb{B}^r_{b,1} \ll \int_{s_\rho}^{s_\phi}\left(\int_{s_\rho}^{t} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{b(s_\rho)}^{b(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt \tag{3.15}$$
$$< \int_{s_\rho}^{s_\phi}\left(\int_{\delta^-(t)} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{\Delta(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt \le \int_{\Delta_k} \mathcal{B}^-_\phi(t)dt.$$

By using (1.3) we decompose $H_{k,2}$ into the sum

$$H_{k,2}f(x) = w_k(x, b(s_\rho)) \int_{b(\xi_k)}^{b(s_\rho)} f(y)v(y)dy + w(x) \int_{b(\xi_k)}^{b(s_\rho)} v_k(s_\rho, y)f(y)dy =: H^w_{k,2}f(x) + H^v_{k,2}f(x).$$

By Hölder's inequality and (1.3)

$$\begin{aligned}\|H^w_{k,2}\|^r_{L_p(b(\xi_k),b(s_\rho))\to L_q(s_\phi,\xi_{k+1})} &< \left(\int_{s_\rho}^{\xi_{k+1}} w_k^q(x, b(s_\rho))dx\right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{b(s_\rho)} v^{p'}(y)dy\right)^{\frac{r}{p'}} \\ &\ll \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{b(\xi_k)}^{t} v^{p'}(y)dy\right)^{\frac{r}{q'}} \left(\int_{b^{-1}(t)}^{\xi_{k+1}} w_k^q(x,t)dx\right)^{\frac{r}{q}} v^{p'}(t)dt \\ &< \int_{b(\xi_k)}^{b(s_\rho)} \left(\int_{\Theta(t)} w_k^q(x,t)dx\right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(t)dt \le \int_{\delta_{k+1}} \mathcal{B}^-_\rho(t)dt, \qquad (3.16)\end{aligned}$$

since $t$ is still not greater then $\rho^{-1}(\xi_{k+1}) = b(s_\rho)$. By Hölder's inequality and (1.3)

$$\begin{aligned}\|H^v_{k,2}\|^r_{L_p(b(\xi_k),b(s_\rho))\to L_q(s_\phi,\xi_{k+1})} &= \left(\int_{s_\phi}^{\xi_{k+1}} w^q(x)dx\right)^{\frac{r}{q}} \left(\int_{b(\xi_k)}^{b(s_\rho)} v_k^{p'}(s_\rho, y)dy\right)^{\frac{r}{p'}} \\ &\ll \int_{s_\phi}^{\xi_{k+1}} \left(\int_t^{\xi_{k+1}} w^q(x)dx\right)^{\frac{r}{p}} \left(\int_{b(\xi_k)}^{b(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt \\ &< \int_{s_\phi}^{\xi_{k+1}} \left(\int_{\delta^+(t)} w^q(x)dx\right)^{\frac{r}{p}} \left(\int_{\Delta(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt \le \int_{\Delta_k} \mathcal{B}^+_\phi(t)dt \qquad (3.17)\end{aligned}$$

on the strength of $a(t) \le b(\xi_k)$ and $s_\phi \le t \implies a(\xi_{k+1}) \le \phi(t) \implies \xi_{k+1} \le a^{-1}(\phi(t))$. To estimate $S_{k,3}$ we use again (1.8), (1.9) and (1.10):

$$\|S_{k,3}\|^r_{L_p(b(s_\phi),b(\xi_{k+1}))\to L_q(s_\phi,\xi_{k+1})} \approx B^r_{b,0} + B^r_{b,1} \approx \mathbb{B}^r_{b,0} + B^r_{b,1} \qquad (3.18)$$

$$\begin{aligned}&= \int_{b(s_\phi)}^{b(\xi_{k+1})} \left(\int_t^{b(\xi_{k+1})} \left[\int_{b^{-1}(y)}^{\xi_{k+1}} w_k^q(x,y)dx\right]^{p'} v^{p'}(y)dy\right)^{\frac{r}{q'}} \left[\int_{b^{-1}(t)}^{\xi_{k+1}} w_k^q(x,t)dx\right]^{p'-\frac{r}{q'}} v^{p'}(t)dt \\ &\quad + \int_{s_\phi}^{\xi_{k+1}} \left(\int_t^{\xi_{k+1}} w^q(x)dx\right)^{\frac{r}{p'}} \left(\int_{b(s_\phi)}^{b(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p}} w^q(t)dt.\end{aligned}$$

As before $k(x,y) \ll k(x,t)$ for $a(x) \le b(\xi_k) < b(s_\phi) \le t \le y = b(z)$ with $z = b^{-1}(y) \le x$ in $\mathbb{B}_{b,0}$, and also $b(\xi_{k+1}) < b(\rho(t))$ since $\rho^{-1}(\xi_{k+1}) < t$. Therefore,

$$\begin{aligned}\mathbb{B}^r_{b,0} &\ll \int_{b(s_\phi)}^{b(\xi_{k+1})} \left(\int_t^{b(\xi_{k+1})} v^{p'}(y)dy\right)^{\frac{r}{q'}} \left(\int_{b^{-1}(t)}^{\xi_{k+1}} w_k^q(x,t)dx\right)^{\frac{r}{q}} v^{p'}(t)dt \\ &< \int_{b(s_\phi)}^{b(\xi_{k+1})} \left(\int_{\Theta(t)} w_k^q(x,t)dx\right)^{\frac{r}{q}} \left(\int_{\vartheta^+(t)} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(t)dt \le \int_{\delta_{k+1}} \mathcal{B}^+_\rho(t)dt. \qquad (3.19)\end{aligned}$$

The estimate

$$B^r_{b,1}\ll\int_{s_\phi}^{\xi_{k+1}}\bigg(\int_{\delta^+(t)}w^q(x)dx\bigg)^{\frac{r}{p}}\bigg(\int_{\Delta(t)}v_k^{p'}(t,y)dy\bigg)^{\frac{r}{p'}}w^q(t)dt\le\int_{\Delta_k}\mathcal{B}^+_\phi(t)dt \quad (3.20)$$

follows by $a(t)\le a(\xi_{k+1})=b(\xi_k)<b(s_\phi)$ and $s_\phi\le t\Longrightarrow a(\xi_{k+1})\le\phi(t)\Longrightarrow \xi_{k+1}\le a^{-1}(\phi(t))$. For the last norm $\|H_{k,3}\|_{L_p(b(s_\rho),b(s_\phi))\to L_q(s_\phi,\xi_{k+1})}$ we make a decomposition

$$H_{k,3}f(x)=w_k(x,b(s_\phi))\int_{b(s_\rho)}^{b(s_\phi)}f(y)v(y)dy+w(x)\int_{b(s_\rho)}^{b(s_\phi)}v_k(s_\phi,y)f(y)dy$$
$$=:H^w_{k,3}f(x)+H^v_{k,3}f(x),$$

since $k(x,y)\approx k(x,b(s_\phi))+k(s_\phi,y)$ for $s_\phi\le x$ and $a(x)\le y\le b(s_\phi)$. By Hölder's inequality

$$\|H^w_{k,3}\|^r_{L_p(b(s_\rho),b(s_\phi))\to L_q(s_\phi,\xi_{k+1})}=\bigg(\int_{s_\phi}^{\xi_{k+1}}w_k^q(x,b(s_\phi))dx\bigg)^{\frac{r}{q}}\bigg(\int_{b(s_\rho)}^{b(s_\phi)}v^{p'}(y)dy\bigg)^{\frac{r}{p'}}$$
$$\ll\int_{b(s_\rho)}^{b(s_\phi)}\bigg(\int_t^{b(s_\phi)}v^{p'}(y)dy\bigg)^{\frac{r}{q'}}\bigg(\int_{b^{-1}(t)}^{\xi_{k+1}}w_k^q(x,t)dx\bigg)^{\frac{r}{q}}v^{p'}(t)dt$$
$$<\int_{b(s_\rho)}^{b(s_\phi)}\bigg(\int_{\Theta(t)}w_k^q(x,t)dx\bigg)^{\frac{r}{q}}\bigg(\int_{\vartheta^+(t)}v^{p'}(y)dy\bigg)^{\frac{r}{q'}}v^{p'}(t)dt\le\int_{\delta_{k+1}}\mathcal{B}^+_\rho(t)dt, \quad (3.21)$$

since $\rho^{-1}(s_\phi)<\rho^{-1}(\xi_{k+1})=b(s_\rho)<b(s_\phi)\le t\Longrightarrow s_\phi<\rho(t)\Longrightarrow b(s_\phi)<b(\rho(t))$. We have also

$$\|H^v_{k,3}\|^r_{L_p(b(s_\rho),b(s_\phi))\to L_q(s_\phi,\xi_{k+1})}=\bigg(\int_{s_\phi}^{\xi_{k+1}}w^q(x)dx\bigg)^{\frac{r}{q}}\bigg(\int_{b(s_\rho)}^{b(s_\phi)}v_k^{p'}(s_\phi,y)dy\bigg)^{\frac{r}{p'}}$$
$$\ll\int_{s_\phi}^{\xi_{k+1}}\bigg(\int_t^{\xi_{k+1}}w^q(x)dx\bigg)^{\frac{r}{p}}\bigg(\int_{b(s_\rho)}^{b(t)}v_k^{p'}(t,y)dy\bigg)^{\frac{r}{p'}}w^q(t)dt$$
$$<\int_{s_\phi}^{\xi_{k+1}}\bigg(\int_{\delta^+(t)}w^q(x)dx\bigg)^{\frac{r}{p}}\bigg(\int_{\Delta(t)}v_k^{p'}(t,y)dy\bigg)^{\frac{r}{p'}}w^q(t)dt\le\int_{\Delta_k}\mathcal{B}^+_\phi(t)dt, \quad (3.22)$$

since $s_\phi\le t\Longrightarrow a(\xi_{k+1})\le\phi(t)\Longrightarrow\xi_{k+1}\le a^{-1}(\phi(t))$ and $a(t)\le b(\xi_k)<b(s_\phi)$. Thus, by (3.7) – (3.22) it holds for the case (i) that

$$\|S_k\|^r_{L_p(\delta_{k+1})\to L_q(\Delta_k)}\ll\int_{\delta_{k+1}}\mathcal{B}_\rho(t)dt+\int_{\Delta_k}\mathcal{B}_\phi(t)dt. \quad (3.23)$$

In the case **(iii)** we have

$$S_kf=\sum_{i=1}^3 S_{k,i}f+\sum_{i=1}^2 H_{k,i}f, \quad (3.24)$$

where

$$S_{k,1}f=\chi_{[\xi_k,s_\phi]}S_kf,\qquad L_p(b(\xi_k),b(s_\phi))\to L_q(\xi_k,s_\phi),$$
$$H_{k,1}f=\chi_{[s_\phi,s_\rho]}S_k\left(f\chi_{[b(\xi_k),b(s_\phi)]}\right),\qquad L_p(b(\xi_k),b(s_\phi))\to L_q(s_\phi,s_\rho),$$

$$S_{k,2}f = \chi_{[s_\phi,s_\rho]}S_k\left(f\chi_{[b(s_\phi),b(s_\rho)]}\right), \quad L_p(b(s_\phi),b(s_\rho)) \to L_q(s_\phi,s_\rho),$$
$$H_{k,2}f = \chi_{[s_\rho,\xi_{k+1}]}S_k\left(f\chi_{[b(\xi_k),b(s_\rho)]}\right), \quad L_p(b(\xi_k),b(s_\rho)) \to L_q(s_\rho,\xi_{k+1}),$$
$$S_{k,3}f = \chi_{[s_\rho,\xi_{k+1}]}S_k\left(f\chi_{[b(s_\rho),b(\xi_{k+1})]}\right), \quad L_p(b(s_\rho),b(\xi_{k+1})) \to L_q(s_\rho,\xi_{k+1}).$$

The estimate

$$\|S_{k,1}\|^r_{L_p(b(\xi_k),b(s_\phi))\to L_q(\xi_k,s_\phi)} \ll \int_{\delta_{k+1}}\mathcal{B}^-_\rho(t)dt + \int_{\Delta_k}\mathcal{B}^-_\phi(t)dt \tag{3.25}$$

can be obtained analogously to $\|S_{k,1}\|_{L_p(b(\xi_k),b(s_\rho))\to L_q(\xi_k,s_\rho)}$ in the case (i). The next operator $H_{k,1}$ should be decomposed by (1.3) into the sum

$$H_{k,1}f(x) = w_k(x,b(s_\phi))\int_{b(\xi_k)}^{b(s_\phi)} f(y)v(y)dy + w(x)\int_{b(\xi_k)}^{b(s_\phi)} v_k(s_\phi,y)f(y)dy$$
$$=: H^w_{k,1}f(x) + H^v_{k,1}f(x).$$

Applying Hölder's inequality and (1.3) we obtain

$$\|H^w_{k,1}\|^r_{L_p(b(\xi_k),b(s_\phi))\to L_q(s_\phi,s_\rho)} = \left(\int_{s_\phi}^{s_\rho} w_k^q(x,b(s_\phi))dx\right)^{\frac{r}{q}}\left(\int_{b(\xi_k)}^{b(s_\phi)} v^{p'}(y)dy\right)^{\frac{r}{p'}}$$
$$\ll \int_{b(\xi_k)}^{b(s_\phi)}\left(\int_{b(\xi_k)}^{t} v^{p'}(y)dy\right)^{\frac{r}{q'}}\left(\int_{b^{-1}(t)}^{s_\rho} w_k^q(x,t)dx\right)^{\frac{r}{q}} v^{p'}(t)dt$$
$$< \int_{b(\xi_k)}^{b(s_\phi)}\left(\int_{\Theta(t)} w_k^q(x,t)dx\right)^{\frac{r}{q}}\left(\int_{\vartheta^-(t)} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(t)dt \le \int_{\delta_{k+1}}\mathcal{B}^-_\rho(t)dt, \tag{3.26}$$

since $a(s_\rho) < t$ and $t < b(s_\rho) = \rho^{-1}(\xi_{k+1}) \Longrightarrow \rho(t) < \xi_{k+1} \Longrightarrow a(\rho(t)) < a(\xi_{k+1}) = b(\xi_k)$. Further,

$$\|H^v_{k,1}\|^r_{L_p(b(\xi_k),b(s_\phi))\to L_q(s_\phi,s_\rho)} = \left(\int_{s_\phi}^{s_\rho} w^q(x)dx\right)^{\frac{r}{q}}\left(\int_{b(\xi_k)}^{b(s_\phi)} v_k^{p'}(s_\phi,y)dy\right)^{\frac{r}{p'}}$$
$$\ll \int_{s_\phi}^{s_\rho}\left(\int_t^{s_\rho} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{b(\xi_k)}^{b(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt$$
$$< \int_{s_\phi}^{s_\rho}\left(\int_{\delta^+(t)} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{\Delta(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt \le \int_{\Delta_k}\mathcal{B}^+_\phi(t)dt \tag{3.27}$$

on the strength of $a(t) < a(\xi_{k+1}) = b(\xi_k)$ and $a(s_\rho) < b(\xi_k) = \phi(s_\phi) \le \phi(t) \Longrightarrow s_\rho < a^{-1}(\phi(t))$. For $S_{k,2}$ we use (1.8), (1.9) with $c = s_\phi$ and $d = s_\rho$ :

$$\|S_{k,2}\|^r_{L_p(b(s_\phi),b(s_\rho))\to L_q(s_\phi,s_\rho)} \approx B^r_{b,0} + B^r_{b,1} \tag{3.28}$$
$$= \int_{b(s_\phi)}^{b(s_\rho)}\left(\int_{b^{-1}(t)}^{s_\rho} w_k^q(x,t)dx\right)^{\frac{r}{q}}\left(\int_{b(s_\phi)}^{t} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(t)dt$$
$$+ \int_{s_\phi}^{s_\rho}\left(\int_t^{s_\rho} w^q(x)dx\right)^{\frac{r}{p}}\left(\int_{b(s_\phi)}^{b(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt.$$

Since $a(s_\rho) < b(s_\phi) \le t \le b(s_\rho) = \rho^{-1}(\xi_{k+1})$ we have in $B_{b,0}$ that $s_\rho < a^{-1}(t)$ and $\rho(t) \le \xi_{k+1} \Longrightarrow a(\rho(t)) \le a(\xi_{k+1}) < b(s_\phi)$. Therefore,

$$B_{b,0}^r \ll \int_{b(s_\phi)}^{b(s_\rho)} \left(\int_{\Theta(t)} w_k^q(x,t)dx\right)^{\frac{r}{q}} \left(\int_{\vartheta^-(t)} v^{p'}(y)dy\right)^{\frac{r}{q'}} v^{p'}(t)d \le \int_{\delta_{k+1}} \mathcal{B}_\rho^-(t)dt. \tag{3.29}$$

In $B_{b,1}$ in view of $a(s_\rho) < b(\xi_k) = \phi(s_\phi) < \phi(t) \Longrightarrow s_\rho < a^{-1}(\phi(t))$ and $a(t) < b(s_\phi)$ we have

$$B_{b,1}^r \ll \int_{s_\phi}^{s_\rho} \left(\int_{\delta^+(t)} w^q(x)dx\right)^{\frac{r}{p}} \left(\int_{\Delta(t)} v_k^{p'}(t,y)dy\right)^{\frac{r}{p'}} w^q(t)dt \le \int_{\Delta_k} \mathcal{B}_\phi^+(t)dt. \tag{3.30}$$

Analogously to the case (i) it holds that

$$\|H_{k,2}\|_{L_p(b(\xi_k),b(s_\rho))\to L_q(s_\rho,\xi_{k+1})} \ll \int_{\delta_{k+1}} \mathcal{B}_\rho^-(t)dt + \int_{\Delta_k} \mathcal{B}_\phi^+(t)dt \tag{3.31}$$

and

$$\|S_{k,3}\|^r_{L_p(b(s_\rho),b(\xi_{k+1}))\to L_q(s_\rho,\xi_{k+1})} \ll \int_{\delta_{k+1}} \mathcal{B}_\rho^+(t)dt + \int_{\Delta_k} \mathcal{B}_\phi^+(t)dt. \tag{3.32}$$

Now from (3.24) – (3.32) we have the estimate (3.23) for the case (iii) too. The case **(ii)** is clear from (i) either (iii). Now we obtain from (3.5) by Lemma 1.1 that

$$\|\mathcal{S}\| \approx \left(\sum_k \|S_k\|^r_{L_p(\delta_{k+1})\to L_q(\Delta_k)}\right)^{\frac{1}{r}} \ll \mathcal{B}_\rho + \mathcal{B}_\phi. \tag{3.33}$$

To estimate the norm of the operator $T_k$ we decompose it by (1.3), (3.6) into the sum

$$\begin{aligned} T_k f(x) \ &\approx\ w_k(x,b(\xi_k)) \int_{a(x)}^{b(\xi_k)} f(y)v(y)dy \\ &\quad + w(x) \int_{a(x)}^{b(\xi_k)} v_k(\xi_k,y)f(y)dy =: T_k^w f(x) + T_k^v f(x). \end{aligned} \tag{3.34}$$

Then, we find points $t_\rho := a^{-1}(\rho^{-1}(\xi_k))$, $t_\phi := \phi^{-1}(b(\xi_k))$ and make two more decompositions:

$$T_k^w f = T_k^w \left(f\chi_{[a(\xi_k),\rho^{-1}(\xi_k)]}\right) + T_k^w \left(f\chi_{[\rho^{-1}(\xi_k),b(\xi_k)]}\right) =: T_{k,1}^w f + T_{k,2}^w f, \tag{3.35}$$

$$T_k^v f = T_k^v f\chi_{[\xi_k,t_\phi]} + T_k^v f\chi_{[t_\phi,\xi_{k+1}]} =: T_{k,1}^v f + T_{k,2}^v f. \tag{3.36}$$

Denote

$$\|T_{k,1}^w\| := \|T_{k,1}^w\|_{L_p(a(\xi_k),\rho^{-1}(\xi_k))\to L_q(\xi_k,t_\rho)},\ \|T_{k,2}^w\| := \|T_{k,2}^w\|_{L_p(\rho^{-1}(\xi_k),b(\xi_k))\to L_q(\xi_k,\xi_{k+1})},$$

$$\|T_{k,1}^v\| := \|T_{k,1}^v\|_{L_p(a(\xi_k),b(\xi_k))\to L_q(\xi_k,t_\phi)},\ \|T_{k,2}^v\| := \|T_{k,2}^v\|_{L_p(a(t_\phi),b(\xi_k))\to L_q(t_\phi,\xi_{k+1})}.$$

By duality and (1.19) it follows from Lemma 1.2 with $c = a(\xi_k)$, $d = \rho^{-1}(\xi_k)$, $a = \xi_k$, $b(x) \to a^{-1}(y)$, $v(y) \to w(x)k(x, b(\xi_k))$, $w(x) \to v(y)$ and $q = p'$, $p = q'$ that

$$\|T^w_{k,1}\|^r \approx \int_{a(\xi_k)}^{\rho^{-1}(\xi_k)} \left( \int_{a(\xi_k)}^{t} \left[ \int_{\xi_k}^{a^{-1}(y)} w_k^q(x, b(\xi_k))dx \right]^{p'} v^{p'}(y)dy \right)^{\frac{r}{q'}} \times \left( \int_{\xi_k}^{a^{-1}(t)} w_k^q(x, b(\xi_k))dx \right)^{p' - \frac{r}{q'}} v^{p'}(t)dt.$$

Since $\xi_k \le x$ and $a(x) \le y \le t \le \rho^{-1}(\xi_k) < b(\xi_k)$ then $k(x, b(\xi_k)) \ll k(x, t)$. Therefore,

$$\|T^w_{k,1}\|^r \ll \int_{a(\xi_k)}^{\rho^{-1}(\xi_k)} \left( \int_{a(\xi_k)}^{t} v^{p'}(y)dy \right)^{\frac{r}{q'}} \left( \int_{\xi_k}^{a^{-1}(t)} w_k^q(x, t)dx \right)^{\frac{r}{q}} v^{p'}(t)dt$$
$$< \int_{a(\xi_k)}^{\rho^{-1}(\xi_k)} \left( \int_{\Theta(t)} w_k^q(x, t)dx \right)^{\frac{r}{q}} \left( \int_{\vartheta^-(t)} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(t)dt \le \int_{\delta_k} \mathcal{B}^-_\rho(t)dt \qquad (3.37)$$

on the strength of $t < b(\xi_k) \Longrightarrow b^{-1}(t) < \xi_k$ and $t \le \rho^{-1}(\xi_k) \Longrightarrow a(\rho(t)) \le a(\xi_k)$. Further, again by duality and (1.18) we obtain from Lemma 1.2 with $c = \rho^{-1}(\xi_k)$, $d = b(\xi_k)$, $a = \xi_k$, $b(x) \to a^{-1}(y)$, $v(y) \to w(x)k(x, b(\xi_k))$, $w(x) \to v(y)$ and $q = p'$, $p = q'$ in view of $a^{-1}(\rho^{-1}(\xi_k)) = t_\rho$ that

$$\|T^w_{k,2}\|^r \approx \int_{\rho^{-1}(\xi_k)}^{b(\xi_k)} \left( \int_{\xi_k}^{a^{-1}(t)} w_k^q(x, b(\xi_k))dx \right)^{\frac{r}{q}} \left( \int_{t}^{b(\xi_k)} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(t)dt.$$

Since $\xi_k \le x$ and $a(x) \le t \le b(\xi_k)$ then $k(x, b(\xi_k)) \ll k(x, t)$ and, therefore,

$$\|T^w_{k,2}\|^r \ll \int_{\rho^{-1}(\xi_k)}^{b(\xi_k)} \left( \int_{\xi_k}^{a^{-1}(t)} w_k^q(x, t)dx \right)^{\frac{r}{q}} \left( \int_{t}^{b(\xi_k)} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(t)dt$$
$$< \int_{\rho^{-1}(\xi_k)}^{b(\xi_k)} \left( \int_{\Theta(t)} w_k^q(x, t)dx \right)^{\frac{r}{q}} \left( \int_{\vartheta^+(t)} v^{p'}(y)dy \right)^{\frac{r}{q'}} v^{p'}(t)dt \le \int_{\delta_k} \mathcal{B}^+_\rho(t)dt \qquad (3.38)$$

on the strength of $t \le b(\xi_k) \Longrightarrow b^{-1}(t) \le \xi_k$ and $t \ge \rho^{-1}(\xi_k) \Longrightarrow b(\xi_k) \le b(\rho(t))$. Now we have by (1.20) from Lemma 1.3 and (1.3)

$$\|T^v_{k,1}\|^r \approx \int_{\xi_k}^{t_\phi} \left( \int_{\xi_k}^{t} w^q(x)dx \right)^{\frac{r}{p}} \left( \int_{a(t)}^{b(\xi_k)} v_k^{p'}(\xi_k, y)dy \right)^{\frac{r}{p'}} w^q(t)dt$$
$$\overset{(1.3)}{\ll} \int_{\xi_k}^{t_\phi} \left( \int_{\xi_k}^{t} w^q(x)dx \right)^{\frac{r}{p}} \left( \int_{a(t)}^{b(\xi_k)} v_k^{p'}(t, y)dy \right)^{\frac{r}{p'}} w^q(t)dt$$
$$< \int_{\xi_k}^{t_\phi} \left( \int_{\delta^-(t)} w^q(x)dx \right)^{\frac{r}{p}} \left( \int_{\Delta(t)} v_k^{p'}(t, y)dy \right)^{\frac{r}{p'}} w^q(t)dt \le \int_{\Delta_k} \mathcal{B}^-_\phi(t)dt, \qquad (3.39)$$

since $\phi(t) \le b(\xi_k) \Longrightarrow b^{-1}(\phi(t)) \le \xi_k$ and $b(\xi_k) \le b(t)$. Finally, by (1.21) from Lemma 1.3 and (1.3) we obtain

$$\|T^v_{k,2}\|^r \approx \int_{t_\phi}^{\xi_{k+1}} \left( \int_t^{\xi_{k+1}} \left[ \int_{a(x)}^{b(\xi_k)} v_k^{p'}(\xi_k, y)dy \right]^q w^q(x)dx \right)^{\frac{r}{p}} \times \left( \int_{a(t)}^{b(\xi_k)} v_k^{p'}(\xi_k, y)dy \right)^{q-\frac{r}{p}} w^q(t)dt$$
$$\ll \int_{t_\phi}^{\xi_{k+1}} \left( \int_t^{\xi_{k+1}} w^q(x)dx \right)^{\frac{r}{p}} \left( \int_{a(t)}^{b(\xi_k)} v_k^{p'}(t, y)dy \right)^{\frac{r}{p'}} w^q(t)dt$$
$$< \int_{t_\phi}^{\xi_{k+1}} \left( \int_{\delta^+(t)} w^q(x)dx \right)^{\frac{r}{p}} \left( \int_{\Delta(t)} v_k^{p'}(t, y)dy \right)^{\frac{r}{p'}} w^q(t)dt \le \int_{\Delta_k} \mathcal{B}^+_\phi(t)dt, \tag{3.40}$$

since $a(\xi_{k+1}) = b(\xi_k) \le \phi(t) \Longrightarrow \xi_{k+1} \le a^{-1}(\phi(t))$ and $b(\xi_k) < b(t)$. Now from (3.34) – (3.40) we obtain

$$\|T_k\|^r_{L_p(\delta_k)\to L_q(\Delta_k)} \ll \int_{\delta_k} \mathcal{B}_\rho(t)dt + \int_{\Delta_k} \mathcal{B}_\phi(t)dt.$$

Therefore, from (3.5) by Lemma 1.1

$$\|\mathcal{T}\| \approx \left( \sum_k \|T_k\|^r_{L_p(\delta_k)\to L_q(\Delta_k)} \right)^{\frac{1}{r}} \ll \mathcal{B}_\rho + \mathcal{B}_\phi.$$

Thus and from (3.33) the upper estimate in (2.13) follows in view of (3.4).

*The lower estimate.* Suppose that the inequality

$$\|\mathcal{K}f\|_q \le \|\mathcal{K}\|\|f\|_p \tag{3.41}$$

holds. To prove

$$\|\mathcal{K}\|_{L_p\to L_q} \gg \mathcal{B}^-_\rho \tag{3.42}$$

we take a point sequence $\{\xi_k\}_{k\in\mathbb{Z}} \subset (0,\infty)$ such that

$$\xi_0 = 1, \qquad \xi_k = (a^{-1}\circ b)^k(1), \qquad k \in \mathbb{Z}, \tag{3.43}$$

and denote

$$W_\rho(t) := \int_{b^{-1}(t)}^{a^{-1}(t)} w_k^q(x, t)dx, \quad V^-_\rho(t) := \int_{a(\rho(t))}^t v^{p'}(y)dy.$$

Note that $[W_\rho(t)]^{r/pq}[V^-_\rho(t)]^{r/pq'}[v(t)]^{p'-1} = \mathcal{B}^-_\rho(t)^{1/p}$. If we put

$$f_\rho(t) := [W_\rho(t)]^{r/pq}[V^-_\rho(t)]^{r/pq'}[v(t)]^{p'-1}$$

then $\|f_\rho\|_p = \left(\mathcal{B}_\rho^-\right)^{r/p}$. Thus, on the strength of $\sqcup_k[\xi_k, \xi_{k+1}) = (0, \infty)$ and (3.41) we have

$$\|\mathcal{K}\| \left(\mathcal{B}_\rho^-\right)^{r/p} \overset{(3.41)}{\geq} \|\mathcal{K}f_\rho\|_q = \left(\sum_k \int_{\xi_k}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x)dx\right)^{\frac{1}{q}}$$
$$\geq 2^{-1/q} \left(\sum_k \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x)dx\right)^{\frac{1}{q}}. \qquad (3.44)$$

Using the explicit form of the operator $\mathcal{K}$ we find that

$$\int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x)dx = \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} w^q(x) \left(\int_{a(x)}^{b(x)} v_k(x,y)f_\rho(y)dy\right)^q dx$$
$$= \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} w^q(x) \left(\int_{a(x)}^{b(x)} v_k(x,t)f_\rho(t)dt\right) \left(\int_{a(x)}^{b(x)} v_k(x,y)f_\rho(y)dy\right)^{q-1} dx$$
$$\geq \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} f_\rho(t)v(t) \left(\int_{b^{-1}(t)}^{\min\{a^{-1}(t),\xi_{k+1}\}} k(x,t)w^q(x) \left[\int_{a(x)}^{b(x)} v_k(x,y)f_\rho(y)dy\right]^{q-1} dx\right) dt$$
$$\geq \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} f_\rho(t)v(t) \left(\int_{b^{-1}(t)}^{\rho(t)} k(x,t)w^q(x) \left[\int_{a(x)}^{t} v_k(x,y)f_\rho(y)dy\right]^{q-1} dx\right) dt$$

[applying (1.3) with $z = b^{-1}(t)$]

$$\gg \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} f_\rho(t)v(t) \left(\int_{b^{-1}(t)}^{\rho(t)} w_k^q(x,t) \left[\int_{a(x)}^{t} f_\rho(y)v(y)dy\right]^{q-1} dx\right) dt$$
$$\geq \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} f_\rho(t)v(t) \left(\int_{b^{-1}(t)}^{\rho(t)} w_k^q(x,t)dx\right) \left(\int_{a(\rho(t))}^{t} f_\rho(y)v(y)dy\right)^{q-1} dt$$
$$= \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} [W_\rho(t)]^{r/pq}[V_\rho^-(t)]^{r/pq'} \left(\int_{b^{-1}(t)}^{\rho(t)} w_k^q(x,t)dx\right)$$
$$\times \left(\int_{a(\rho(t))}^{t} [W_\rho(y)]^{r/pq}[V_\rho^-(y)]^{r/pq'} v^{p'}(y)dy\right)^{q-1} v^{p'}(t)dt$$
$$\overset{(2.10)}{\gg} \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} [V_\rho^-(t)]^{r/pq'}[W_\rho(t)]^{r/pq+1}$$
$$\times \left(\int_{a(\rho(t))}^{t} \left[\int_{b^{-1}(t)}^{\rho(t)} w_k^q(z,y)dz\right]^{\frac{r}{pq}} [V_\rho^-(y)]^{r/pq'} v^{p'}(y)dy\right)^{q-1} v^{p'}(t)dt.$$

It follows from (1.3) that $k(z,y) \gg k(z,t)$, since $a(z) \le a(\rho(t)) \le y \le t = b(\tau)$, $\tau = b^{-1}(t) \le z$. Therefore, in view of (2.10)

$$\int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x)dx \gg \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} [V_\rho^-(t)]^{r/pq'}[W_\rho(t)]^{r/q} \times \left(\int_{a(\rho(t))}^t \left[\int_{a(\rho(t))}^y v^{p'}(z)dz\right]^{r/pq'} v^{p'}(y)dy\right)^{q-1} v^{p'}(t)dt \approx \int_{\rho^{-1}(\xi_k)}^{\rho^{-1}(\xi_{k+1})} \mathcal{B}_\rho^-(t)dt.$$

Since $\sqcup_k[\rho^{-1}(\xi_k), \rho^{-1}(\xi_{k+1})) = (0,\infty)$, it yields

$$\left(\sum_k \int_{b^{-1}(\rho^{-1}(\xi_k))}^{\xi_{k+1}} (\mathcal{K}f_\rho)^q(x)dx\right)^{\frac{1}{q}} \gg \left(\mathcal{B}_\rho^-\right)^{r/q}.$$

Thus, by (3.44) we obtain (3.42).

To prove

$$\|\mathcal{K}\|_{L_p\to L_q} \gg \mathcal{B}_\phi^+$$

we use the dual to (3.41) inequality

$$\left(\int_0^\infty \left[\int_{b^{-1}(y)}^{a^{-1}(y)} w_k(x,y)g(x)dx\right]^{p'} v^{p'}(y)dy\right)^{\frac{1}{p'}} \le \|\bar{\mathcal{K}}\| \left(\int_0^\infty g^{q'}(x)dx\right)^{\frac{1}{q'}}. \tag{3.45}$$

Note that here the kernel $k(x,y)$ is satisfying the following from (1.3) condition of the form: if $y \le z$ and $b^{-1}(z) \le x \le a^{-1}(y)$ then

$$D^{-1}k(x,y) \le k(x,z) + k(b^{-1}(z),y) \le Dk(x,y). \tag{3.46}$$

Break the semiaxis $(0,\infty)$ by the point sequence (3.43) and put

$$W_\phi^+(t) := \int_t^{a^{-1}(\phi(t))} w^q(x)dx, \qquad V_\phi(t) := \int_{a(t)}^{b(t)} v_k^{p'}(t,y)dy.$$

Note that $[W_\phi^+(t)]^{r/pq'}[V_\phi(t)]^{r/p'q'}[w(t)]^{q-1} = \mathcal{B}_\phi^+(t)^{1/q'}$. If we take

$$g_\phi(t) = [W_\phi^+(t)]^{r/pq'}[V_\phi(t)]^{r/p'q'}[w(t)]^{q-1}$$

then

$$\left(\int_0^\infty g_\phi^{q'}(x)dx\right)^{\frac{1}{q'}} = \left(\mathcal{B}_\phi^+\right)^{r/q'}.$$

Thus, by $\sqcup_k[\phi(\xi_k), \phi(\xi_{k+1})) = (0,\infty)$ and (3.45) we have

$$\begin{aligned} \|\bar{\mathcal{K}}\| \left(\mathcal{B}_\phi^+\right)^{r/q'} &\ge \|\bar{\mathcal{K}}g_\phi\|_{p'} = \left(\sum_k \int_{\phi(\xi_k)}^{\phi(\xi_{k+1})} (\bar{\mathcal{K}}g_\phi)^{p'}(y)dy\right)^{\frac{1}{p'}} \\ &\ge 2^{-1/p'} \left(\sum_k \int_{\phi(\xi_k)}^{b(\xi_{k+1})} (\bar{\mathcal{K}}g_\phi)^{p'}(y)dy\right)^{\frac{1}{p'}}, \end{aligned} \tag{3.47}$$

where $\bar{\mathcal{K}}g(y) := v(y)\int_{b^{-1}(y)}^{a^{-1}(y)} k(x,y)g(x)w(x)dx$. We find that

$$\int_{\phi(\xi_k)}^{b(\xi_{k+1})} (\bar{\mathcal{K}}g_\phi)^{p'}(y)dy = \int_{\phi(\xi_k)}^{b(\xi_{k+1})} v^{p'}(y)\left(\int_{b^{-1}(y)}^{a^{-1}(y)} w_k(x,y)g_\phi(x)dx\right)^{p'} dy$$

$$= \int_{\phi(\xi_k)}^{b(\xi_{k+1})} v^{p'}(y)\left(\int_{b^{-1}(y)}^{a^{-1}(y)} w_k(t,y)g_\phi(t)dt\right)\left(\int_{b^{-1}(y)}^{a^{-1}(y)} w_k(x,y)g_\phi(x)dx\right)^{p'-1} dx$$

$$\geq \int_{\xi_k}^{\xi_{k+1}} g_\phi(t)w(t)\left(\int_{\max\{a(t),\phi(\xi_k)\}}^{b(t)} k(t,y)v^{p'}(y)\left[\int_{b^{-1}(y)}^{a^{-1}(y)} w_k(x,y)g_\phi(x)dx\right]^{p'-1} dy\right) dt$$

$$\geq \int_{\xi_k}^{\xi_{k+1}} g_\phi(t)w(t)\left(\int_{\phi(t)}^{b(t)} k(t,y)v^{p'}(y)\left[\int_{t}^{a^{-1}(y)} w_k(x,y)g_\phi(x)dx\right]^{p'-1} dy\right) dt$$

[applying (3.46) with $z = b(t)$]

$$\begin{aligned}
&\gg \int_{\xi_k}^{\xi_{k+1}} g_\phi(t)w(t)\left(\int_{\phi(t)}^{b(t)} v_k^{p'}(t,y)\left[\int_t^{a^{-1}(y)} g_\phi(x)w(x)dx\right]^{p'-1} dy\right) dt\\
&\geq \int_{\xi_k}^{\xi_{k+1}} g_\phi(t)w(t)\left(\int_{\phi(t)}^{b(t)} v_k^{p'}(t,y)dy\right)\left(\int_t^{a^{-1}(\phi(t))} g_\phi(x)w(x)dx\right)^{p'-1} dt\\
&= \int_{\xi_k}^{\xi_{k+1}} [W_\phi^+(t)]^{r/pq'}[V_\phi(t)]^{r/p'q'}\left(\int_{\phi(t)}^{b(t)} v_k^{p'}(t,y)dy\right)\\
&\quad\times\left(\int_t^{a^{-1}(\phi(t))} [W_\phi^+(x)]^{r/pq'}[V_\phi(x)]^{r/p'q'}w^q(x)dx\right)^{p'-1} w^q(t)dt\\
&\overset{(2.11)}{\gg} \int_{\xi_k}^{\xi_{k+1}} [W_\phi^+(t)]^{r/pq'}[V_\phi(t)]^{r/p'q'+1}\\
&\quad\times\left(\int_t^{a^{-1}(\phi(t))}\left[\int_{\phi(t)}^{b(t)} v_k^{p'}(x,z)dz\right]^{\frac{r}{p'q'}}[W_\phi^+(x)]^{r/pq'}w^q(x)dx\right)^{p'-1} w^q(t)dt.
\end{aligned}$$

It follows from (3.46) that $k(x,z) \gg k(t,z)$ because $b^{-1}(\tau) = t \leq x \leq a^{-1}(\phi(t)) \leq a^{-1}(z)$ and $z \leq \tau = b(t)$. Therefore, in view of (2.11)

$$\int_{\phi(\xi_k)}^{b(\xi_{k+1})} (\bar{\mathcal{K}}g_\phi)^{p'}(y)dy \gg \int_{\xi_k}^{\xi_{k+1}} [W_\phi^+(t)]^{r/pq'}[V_\phi(t)]^{r/p'}$$

$$\times\left(\int_t^{a^{-1}(\phi(t))}\left[\int_x^{a^{-1}(\phi(t))} w^q(z)dz\right]^{r/pq'} w^q(x)dx\right)^{p'-1} w^q(t)dt \approx \int_{\xi_k}^{\xi_{k+1}} \mathcal{B}_\phi^+(t)dt.$$

Thus and from (3.47)

$$\|\mathcal{K}\|_{L_p\to L_q} = \|\bar{\mathcal{K}}\|_{L_{q'}\to L_{p'}} \gg \mathcal{B}_\phi^+. \tag{3.48}$$

By combining (3.42) and (3.48) we obtain the lower estimate in (2.13).

The assertions of Theorem 2.6 can be proved analogously by using (2.2), (1.12), (1.13), (1.14), (1.15) and lemmas 1.2, 1.3.

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