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HYERS–ULAM STABILITY OF A POLYNOMIAL EQUATION

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Abstract. The aim of this paper is to prove the stability in the sense of Hyers–Ulam stability of a polynomial equation. More precisely, if x is an approximate solution of the equation $x^n + \alpha x + \beta = 0$, then there exists an exact solution of the equation near to x.

1. Introduction and preliminaries

The basic problem of the stability of functional equations asks whether an approximate solution of the Cauchy functional equation $f(x+y) = f(x) + f(y)$ can be approximated by a solution of this equation [12]. In 1940, S. M. Ulam [16] posed the following problem concerning the stability of functional equations: Give conditions in order for a linear mapping near an approximately linear mapping to exist. The problem for the case of approximately additive mappings was solved by D. H. Hyers [3] when G_1 and G_2 are Banach spaces. Since then, the stability problems of functional equations have been extensively investigated by several mathematicians (cf. [2, 4, 11, 13]).

C. Alsina and R. Ger [1] remarked that the differential equation $y' = y$ has the Hyers–Ulam stability. The result of C. Alsina and R. Ger has been generalized by T. Miura, S.-E. Takahasi and H. Choda [10], by T. Miura [8], and also by S.-E. Takahasi, T. Miura and S. Miyajima [14]. Furthermore, the result of Hyers–Ulam stability for first-order linear differential equations has been generalized by T. Miura, S. Miyajima and S. -E. Takahasi [9], by S.-E. Takahasi, H. Takagi, T. Miura and S. Miyajima [15], and also by S.-M. Jung [5, 6]. Recently, G. Wang,

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M. Zhou and L. Sun [17] discussed the Hyers–Ulam stability of the first-order nonhomogeneous linear differential equation.

Motivated by and connected to the results mentioned above and [7], we consider stability problems for a polynomial equation. In this paper, we will investigate the Hyers–Ulam stability of the following polynomial equation:

$$x^n + \alpha x + \beta = 0 \tag{1.1}$$

where $x \in [-1, 1]$.

We say that equation (1.1) has the Hyers–Ulam stability if there exists a constant $K > 0$ with the following property: for every $\varepsilon > 0, y \in [-1, 1]$, if

$$|y^n + \alpha y + \beta| \leq \varepsilon,$$

then there exists some $z \in [-1, 1]$ satisfying

$$z^n + \alpha z + \beta = 0$$

such that $|y - z| < K\varepsilon$. We call such K a Hyers–Ulam stability constant for equation (1.1).

2. Main Results

Now, the main result of this work is given in the following theorem.

Theorem 2.1. *If $|\alpha| > n, |\beta| < |\alpha| - 1$ and $y \in [-1, 1]$ satisfies the inequality*

$$|y^n + \alpha y + \beta| \leq \varepsilon$$

then there exists a solution $v \in [-1, 1]$ of equation (1.1) such that

$$|y - v| \leq K\varepsilon$$

Where $K > 0$ is a constant.

Proof. Let $\varepsilon > 0$ and $y \in [-1, 1]$ such that

$$|y^n + \alpha y + \beta| \leq \varepsilon$$

We will show that there exists a constant K independent of ε and v such that $|y - v| < K\varepsilon$ for some $v \in [-1, 1]$ satisfying $x^n + \alpha x + \beta = 0$.

If we set

$$g(x) = \frac{1}{\alpha}(-\beta - x^n), x \in [-1, 1]$$

then

$$|g(x)| = |\frac{1}{\alpha}(-\beta - x^n)| \leq 1$$

Let $X = [-1, 1], d(x, y) = |x - y|$, then (X, d) is a complete metric space, and g map X into X.

Next, we will show that g is a contraction mapping from X to X. For any $x, y \in X$, one have

$$\begin{aligned} d(g(x), g(y)) &= |\frac{1}{\alpha}(-\beta - x^n) - \frac{1}{\alpha}(-\beta - y^n)| \\ &\leq \frac{1}{|\alpha|}|x^n - y^n| \end{aligned}$$

$$= \frac{1}{|\alpha|}|x-y||x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1}|$$

Since $|\alpha| > n, x, y \in [-1,1], x \neq y$, we obtain

$$d(g(x),g(y)) \leq \gamma\, d(x,y)$$

Where $\gamma = \frac{n}{|\alpha|} \in (0,1)$.

Thus g is a contraction mapping from X to X, by S. Banach's contraction mapping theorem, there exists unique $v \in X$, such that

$$g(v) = v$$

Hence equation (1.1) has a solution on [-1, 1].

Finally, we show that equation (1.1) has the Hyers–Ulam stability. Let us introduce the abbreviations $K = \frac{1}{|\alpha|(1-\gamma)}$, then

$$\begin{aligned}|y-v| &= |y-g(y)+g(y)-g(v)|\\ &\leq |y-g(y)|+|g(y)-g(v)|\\ &\leq |y-\frac{1}{\alpha}(-\beta-y^n)|+\gamma|y-v|\\ &= \frac{1}{|\alpha|}|y^n+\alpha y+\beta|+\gamma|y-v|\end{aligned}$$

thus, we come to the inequalities

$$\begin{aligned}|y-v| &\leq \frac{1}{|\alpha|(1-\gamma)}|y^n+\alpha y+\beta|\\ &\leq K\varepsilon.\end{aligned}$$

which completes the proof. □

By applying a similar argument of the proof of Theorem 2.1, it is easy to see the following theorem holds.

Theorem 2.2. *Let (X,d) be a complete metric linear space, T be a contraction mapping from X to X, then $(T-I)x = 0$ has the Hyers–Ulam stability. That is, for every $\epsilon > 0$, if*

$$d(Tx-x,0) \leq \epsilon,$$

then there exists an unique $z \in X$ satisfying

$$Tz - z = 0$$

with

$$d(x,z) \leq K\varepsilon$$

for some $K > 0$.

Proof. Since (X, d) is a complete metric linear space and there exist $\gamma \in (0,1)$ such that $d(Tx, Ty) \le \gamma d(x, y)$ for all $x \ne y, x, y \in X$, by S. Banach's contraction mapping theorem, there exists unique $z \in X$, such that $Tz = z$, hence equation $Tv - v = 0$ has a solution on X.

For every $\epsilon > 0$, if $d(Tx - x, 0) \le \epsilon$, then

$$\begin{aligned} d(x, z) &= d(x - Tx + Tx - Tz, 0) \\ &= d(Tx - x, Tx - Tz) \\ &\le d(Tx - x, 0) + d(Tx - Tz, 0) \\ &\le \varepsilon + \gamma d(x, z) \end{aligned}$$

thus, we obtain the inequalities

$$d(x, z) \le \frac{\varepsilon}{(1 - \gamma)}$$

which completes the proof. $\square$

It is easy to see that by the similar technique, we can discuss the Hyers–Ulam stability of the polynomial equation defined on any finite interval $[a, b]$. Unfortunately, we could not prove the Hyers–Ulam stability of the polynomial equation defined on an infinite interval. It is an interesting open problem whether the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ has the Hyers–Ulam stability for the case it has some solutions in $[a, b]$.

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