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GOOD ℓ_2-SUBSPACES OF L_p, $p > 2$

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Abstract. We give an alternate proof of the result due to Haydon, Odell and Schlumprecht that subspaces of L_p, $p > 2$, which are isomorphic to ℓ_2 contain subspaces which are well isomorphic to ℓ_2 and well complemented.

1. Introduction and preliminaries

In a recent paper Haydon, Odell and Schlumprecht show that any subspace of L_p, $p > 2$, which is isomorphic to ℓ_2 contains a subspace on which the L_p norm behaves similarly to its behavior on the span of independent, mean 0, Gaussian variables. (See [2, Section 6].) Using this subspace they obtain a well complemented subspace $(1 + \epsilon)$-isomorphic to ℓ_2. In order to find this subspace the authors use types [3] and random measures [1].

In this note we show that the same result can be produced without as much machinery by using a version of the Central Limit Theorem for martingales [4]. In the proof of Lemma 6.6 of [2] the Central Limit Theorem also plays an essential role. Below we assume that (Ω, P) is an atomless probability space and denote by $\mathbf{E}(\cdot|\mathcal{F})$ the conditional expectation with respect to the σ-algebra $\mathcal{F}$. $N(\mu, \sigma^2)$ denotes, as usual, the normal distribution with center at μ and variance σ^2. Below $\mathcal{F}_{n,0} = \{\Omega, \emptyset\}$, the trivial σ-algebra. We follow the convention of suppressing the measure space variable, i.e., $\{f > r\} = \{\omega : f(\omega) > r\}$.

Theorem 1.1. [4, VII.8 Theorem 4] *Suppose that for each n, $(f_{n,k})_{k=1}^n$ is a square integrable martingale difference sequence adapted to $(\mathcal{F}_{n,k})_{k=1}^n$ satisfying*

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the Lindeberg condition: for every $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{E}(f_{n,k}^2 1_{\{|f_{n,k}|>\epsilon\}}|\mathcal{F}_{n,k-1}) = 0,$$

in probability. If

$$\lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{E}(f_{n,k}^2|\mathcal{F}_{n,k-1}) = \sigma^2 \textit{ in probability}$$

then

$$\lim_{n\to\infty} \sum_{k=1}^{n} f_{n,k} = N(0,\sigma^2) \textit{ in distribution.}$$

Our argument uses the limiting conditional variance as in [2] and we borrow a few facts from that paper. If (x_n) is a weakly null sequence in L_p such that (x_n^2) converges weakly in $L_{p/2}$ to v, then v is the *limiting conditional variance* of (x_n). Recall that a subset U of $L_p(\Omega, P)$ is said to be p-uniformly integrable if for every $\epsilon > 0$ there is a δ such that if A is measurable and $P(A) < \delta$ then for all $f \in U$, $\mathbf{E}(|f|^p 1_A) < \epsilon$. An equivalent definition is that there exists K such that for all $f \in U$, $\mathbf{E}(|f|^p 1_{\{|f|>K|\}}) < \epsilon$. Also it is easy to see that the negation is equivalent to: there is an $\epsilon > 0$ and a sequence (f_j) in U and a sequence of disjoint measurable sets (A_j) such that $\mathbf{E}(|f_j|^p 1_{A_j}) \geq \epsilon$ for all j.

Lemma 1.2. [2, Lemma 5.3] *Let (x_n) be a martingale difference sequence which is p-uniformly integrable. Then set of linear combinations of (x_n) with coefficients in the unit sphere of ℓ_2 is p-uniformly integrable.*

Our goal is to prove the following.

Theorem 1.3. *Suppose that X is a subspace of L_p for some p, $2 < p < \infty$, which is isomorphic to ℓ_2. Then for every $\epsilon > 0$ there is a sequence (z_n) in X such that for all (c_n) in ℓ_2,*

$$(1-\epsilon)\left(\sum_{n=1}^{\infty} c_n^2\right) \leq \left\|\sum_{n=1}^{\infty} c_n z_n\right\| \leq (1+\epsilon)\left(\sum_{n=1}^{\infty} c_n^2\right)$$

and $[z_n]$ is complemented in L_p with a projection of norm less than $(1+\epsilon)\gamma_p$, where γ_p is the norm of a symmetric Gaussian random variable. Moreover, if (x_n) is a normalized sequence in X equivalent to the unit vector basis of ℓ_2 with limiting conditional variance v, then (z_n) can be chosen with limiting conditional variance v.

This result is implicit in the proof of Theorem 6.8 of [2]. The proof given here can also be used to prove that result since it produces a stabilized ℓ_2-sequence with the required limiting conditional variance.

2. Proof of the Theorem

The proof is mostly reductions to a simple situation where the Central Limit Theorem can be easily applied. Except for the results cited above the proof uses only basic measure theory and functional analysis.

Proof. Let (x_n) be a normalized sequence in X equivalent to the unit vector basis of ℓ_2 with limiting conditional variance v. Because (x_n) is weakly null, we may assume by passing to a subsequence and perturbing that (x_n) is a martingale difference with respect to some increasing family of σ-algebras, $(\mathcal{G}_n)$. We may assume that the simple functions with respect to $\cup\mathcal{G}_n$ are dense in $L_p(\Omega, P)$ for all p, $1 \leq p < \infty$.

Our first step is to replace (x_n) by a sequence of blocks (y_k) of (x_n) so that (y_k) is uniformly p-integrable. (See Lemma 5.4 of [HOS] for a similar argument.)

Let

$$a = \sup\{\epsilon : \text{ there exists } (n_j) \text{ and disjoint measurable sets } (A_j) \text{ such that } \|x_{n_j|A_j}\| \geq \epsilon\}.$$

Then choose a subsequence (n_j) and a sequence of disjoint sets (A_j) so that the supremum is achieved, i.e., $\lim \|x_{n_j|A_j}\| = a$. Moreover by taking an appropriate subsequence of the σ-algebras and relabeling we may assume that A_j is in G_{n_j} for each j. It follows that $(x_{n_j}1_{\Omega\setminus A_j})$ is a martingale difference. Moreover by the choice of (A_{n_j}), $(x_{n_j}1_{\Omega\setminus A_j})$ must be p-uniformly integrable.

Let (K_m) be finite subsets of $\mathbb{N}$ such that $\max K_m < \min K_{m+1}$ for all m and strictly increasing cardinality. Let $y_m = |K_m|^{-1/2}\sum_{j\in K_m} x_{n_j}$ for all m, where $|F|$ denotes the cardinality of F.

Observe that

$$\||K_m|^{-1/2}\sum_{j\in K_m} x_{n_j}1_{A_j}\| = |K_m|^{-1/2}\left(\sum_{j\in K_m}\|x_{n_j|A_j}\|^p\right)^{1/p} \leq a|K_m|^{(2-p)/(2p)}.$$

Because $p > 2$, this goes to zero as m increases. A norm perturbation of a p-uniformly integrable sequence is also p-uniformly integrable and thus it follows from Lemma 1.2 that (y_m) is p-uniformly integrable. Further notice that (y_m^2) converges weakly to v. Indeed

$$y_m^2 = |K_m|^{-1}\sum_{j\in K_m} x_{n_j}^2 + |K_m|^{-1}\sum_{r,s\in K_m, r\neq s} 2x_{n_s}x_{n_r}.$$

By our assumption on the family $(\mathcal{G}_n)$, every element in $L_{p/2}^*$ can be approximated in norm as close as required by $\mathcal{G}_n$-measurable simple functions for n sufficiently large. Thus for m sufficiently large for all $r, s \in K_m, r \neq s$, $x_{n_r}x_{n_s}$ is orthogonal to $L_{p/2}(\Omega, \mathcal{G}_n, P)^*$ and hence $|K_m|^{-1}\sum_{r,s\in K_m,r\neq s} 2x_{n_s}x_{n_r}$ tends to 0 weakly.

Our next step is to show that we may assume

> (*) v is measurable with respect to some finite (cardinality) σ-algebra $\mathcal{G}_0$ and that (y_n) is a p-uniformly integrable martingale difference sequence such that $\mathbf{E}(y_n|\mathcal{G}_0) = 0$ for all n.

By another application of Lemma 1.2 the set S of all linear combinations of (y_n) with coefficients in the sphere of ℓ_2 is p-uniformly integrable. Let $\epsilon > 0$ and choose $\delta > 0$ such that for all $z \in S$, if B is measurable with $P(B) < \delta$ then $E(|z|^p 1_B) < \epsilon$. Choose $M_0 > 0$ such that $P(\{v \le M_0\}) < \delta/2$ and $M_1 > 0$ such that $P(\{v > M_1\}) < \delta/2$. Let $A = \{v \le M_0\} \cup \{v > M_1\}$. Then if $z \in S$, $\mathbf{E}(|z|^p 1_A) < \epsilon$. Consequently we may (and do) assume by a norm perturbation that $y_n 1_A = 0$ and that v and $1/v$ are bounded on the support of v.

Let $\rho > 0$. Let R_0 and R_1 be integers such that $(1+\rho)^{R_0} \le M_0$ and $(1+\rho)^{R_1} > M_1$. For each r, $R_0 \le r < R_1$, let $A_r = \{(1+\rho)^r \le v < (1+\rho)^{r+1}\}$. Let

$$w = 1_{\Omega \setminus \operatorname{supp} v} + \sum_{r=R_0}^{R_1 - 1} \frac{(1+\rho)^r}{v} 1_{A_r}.$$

Notice that for all r, $1 \le r < \infty$,

$$\mathbf{E}(|f|^r w^{r/2}) \le \mathbf{E}(|f|^r) \le (1+\rho)^{r/2} \mathbf{E}(|f|^r w^{r/2})$$

for all $f \in L_r(\Omega, P)$. Thus multiplication by $w^{1/2}$ is a $(1+\rho)^{1/2}$-isomorphism from $L_r(\Omega, P)$ onto $L_r(\Omega, P)$. Moreover $(y_m w^{1/2})$ converges weakly to 0 and $(y_m^2 w)$ converges weakly to $v_0 = \sum_{r=R_0}^{R_1-1} (1+\rho)^r 1_{A_r}$. The first assertion is immediate and the second follows by approximating w by simple functions. Another perturbation gives our required reduction (*). Indeed let $\mathcal{F}_0$ be the finite σ-algebra generated by v_0 and for each $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the σ-algebra generated by $\mathcal{F}_0$ and $\mathcal{G}_n$. Because $(y_n w^{1/2})$ converges weakly to 0, there is a subsequence $(y_{n_j} w^{1/2})$ and a L_p-norm perturbation (z_j), i.e., $\|y_{n_j} w^{1/2} - z_j\|_p \to 0$, such that (z_j) is a martingale difference relative to (F_{m_j}). Moreover

$$\|y_{n_j}^2 w - z_j^2\|_1 \le \|y_{n_j} w^{1/2} - z_j\|_2 \|y_{n_j} w^{1/2} + z_j\|_2 \to 0.$$

Hence the weak limit of (z_j^2) is v_0.

To summarize, we can now assume that (y_n) is a martingale difference sequence where y_n is $\mathcal{G}_n$ measurable for all n, $\mathcal{G}_0 \subset \ldots \mathcal{G}_n \subset \mathcal{G}_{n+1} \subset \cdots$, $|\mathcal{G}_n| < \infty$ for all n, (y_n^2) converges weakly to v, v is measurable with respect to the σ-algebra $\mathcal{G}_0$ and the set of all linear combinations of (y_n) with coefficients in the unit sphere of ℓ_2 is p-uniformly integrable. Because (y_n^2) converges weakly to v and the σ-algebras $(\mathcal{G}_n)$ are finite, by passing to a subsequence we may assume that $\|\mathbf{E}(y_n^2|\mathcal{G}_{n-1}) - v\|_p < 2^{-n}$ for all n.

Next we will apply the Central Limit Theorem to the restriction of (y_m) to each of the level sets of v. Let $v = \sum_{r=1}^R a_r^2 1_{A_r}$, with $a_r > 0$ for all r, $a_r \ne a_s$ and $A_r \cap A_s = \emptyset$ if $s \ne r$. Fix r and consider the probability space (A_r, P_r) where $P_r(S) = P(S)/P(A_r)$ and the corresponding expectation is $\mathbf{E}_r$ and let (z_n) be the restriction of (y_n/a_r) to A_r. Clearly (z_n^2) converges weakly to 1_{A_r} and (z_n) is a p-uniformly integrable martingale difference.

Let $\epsilon > 0, \delta > 0$, and $M \subset \mathbb{N}$, $M \ne \emptyset$, and choose K such that $\mathbf{E}_r(|z_n|^p 1_{\{|z_n| \ge K\}}) < \delta^{p/2}$. Then if $c = |M|^{-1/2}$ and $|M| > K^2 \epsilon^{-2}$, by Hölder's and

Chebychev's inequalities

$$\mathbf{E}_r(|cz_n|^2 1_{\{|cz_n|\geq\epsilon\}}) \leq c^2(\mathbf{E}_r(|z_n|^p 1_{\{|cz_n|\geq\epsilon\}}))^{2/p} P_r(\{|cz_n| \geq \epsilon\})^{(p-2)/p} \leq (\mathbf{E}_r(|z_n|^p 1_{\{|cz_n|\geq\epsilon\}}))^{2/p} c^p \epsilon^{-(p-2)} \mathbf{E}_r(|z_n|^p)^{(p-2)/p}$$

and thus

$$\sum_{n\in M} \mathbf{E}_r(|cz_n|^2 1_{\{|cz_n|\geq\epsilon\}}) < \delta|M||M|^{-p/2}\epsilon^{-(p-2)}\mathrm{max}_{n\in M}\mathbf{E}_r(|z_n|^p)^{(p-2)/p}.$$

It follows that if (M_j) is a sequence of non empty subsets of $\mathbb{N}$ with $\max M_j < \min M_{j+1}$ and $\lim_{j\to\infty}|M_j| = \infty$ then for every $\epsilon > 0$,

$$\sum_{n\in M_j} \mathbf{E}_r(|M_j|^{-1}|z_n|^2 1_{\{|M_j|^{-1/2}|z_n|\geq\epsilon\}}|\mathcal{G}_{n-1})$$

converges in probability to 0. Thus the Lindeberg condition is satisfied.

Our assumption that $\|\mathbf{E}(y_n^2|\mathcal{G}_{n-1}) - v\|_p < 2^{-n}$ for all n implies that $\sum_{n\in M_j}\mathbf{E}_r(y_n^2|\mathcal{G}_{n-1})$ converges in probability, P_r, to 1. By the Central Limit Theorem $|M_j|^{-1/2}\sum_{n\in M_j} z_n$ converges in distribution to $\mathcal{N}(0,1)$.

Let $w_j = |M_j|^{-1/2}\sum_{n\in M_j} z_n$ for all j. We claim that for any $\epsilon_1 > 0$ there is a J such that if w is a linear combination of $(w_j)_{j\geq J}$ with coefficients in the sphere of ℓ_2, then $(1+\epsilon_1)^{-1}\mathbf{E}_r(|g_r|^p) \leq \mathbf{E}_r(|w|^p) \leq (1+\epsilon_1)\mathbf{E}_r(|g_r|^p)$. Here g_r is normally distributed with mean 0 and variance 1 on (Ω, P_r).

Suppose that for some $\epsilon_0 > 0$ there is no such J. Then we can find a sequence (x_k) of linear combinations of $(w_j)_{j\geq J_k}$ with coefficients in the sphere of ℓ_2, $\max J_k < \min J_{k+1}$ and ϵ_0 such that

$$(1+\epsilon_0)^{-1}\mathbf{E}_r(|g_r|^p) > \mathbf{E}_r(|x_k|^p)$$

or

$$\mathbf{E}_r(|x_k|^p) > (1+\epsilon_0)\mathbf{E}_r(|g_r|^p),$$

for all k. Because (z_n) is p-uniformly integrable, so is (x_k). The argument above shows that (x_k) converges in distribution to $\mathcal{N}(0,1)$. Thus $\lim_{k\to\infty}\mathbf{E}_r(|x_k|^p) = \mathbf{E}_r(|g_r|^p)$, a contradiction.

There are only finitely many sets A_r in the representation of v as a simple function, so we can choose sets M_j and J as above so that $(1+\epsilon_1)^{-1}\mathbf{E}_r(|g_r|^p) \leq \mathbf{E}_r(|w|^p) \leq (1+\epsilon_1)\mathbf{E}_r(|g_r|^p)$ for all r and all linear combinations of $(w_j)_{j\geq J}$ with coefficients in the sphere of ℓ_2. Thus if $u_j = |M_j|^{-1/2}\sum_{n\in M_j} y_n$ for $j \geq J$ and u is a linear combination of $(u_j)_{j\geq J}$ with coefficients in the sphere of ℓ_2, then

$$\sum_{r=1}^R (1+\epsilon_1)^{-1} a_r^p P(A_r)\mathbf{E}_r(|g_r|^p) \leq \sum_{r=1}^R \mathbf{E}(|u|^p 1_{A_r}) \leq \sum_{r=1}^R (1+\epsilon_1) a_r^p P(A_r)\mathbf{E}_r(|g_r|^p).$$

The orthogonal projection from L_p onto the closed span of $(u_j)_{j\geq J}$ is bounded by $\sup\{\|u\|_p/\|u\|_2 : u \in [u_j : j \geq J]\} \leq (1+\epsilon_1)^{1/p}\|v\|_p\gamma_p$. With a suitable choice of ϵ_1 the sequence $(u_j/\|u_j\|_p)_{j\geq J}$ satisfies the conclusion of Theorem 1.3. $\square$

Remark 2.1. In the last estimate in the proof the comparison is to $\|v\|_p^p \gamma_p^p$ but in fact is really an approximation in distribution. Thus the argument above gives a fairly explicit limiting distribution for the elements of the subspace and the approximating basic sequence is obtained by at most two ℓ_2-averages of the original basic sequence. If $1 \leq p < 2$ and $(x_n) \subset L_p$ is equivalent to unit vector basis of ℓ_r, $p < r < 2$, is there a sequence of linear combinations of (x_n) which are close in distribution to a sum of multiples of disjointly supported r-stable random variables?.

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