Banach J. Math. Anal. 3 (2009), no. 1, 52–60

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

A FUNCTIONAL METHOD APPLIED TO OPERATOR EQUATIONS

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Communicated by T. Riedel

ABSTRACT. We consider second order hyperbolic equations with unbounded operator's coefficients possessing time dependent domain of definition in a Hilbert space. Existence and uniqueness of the strong generalized solution are studied. The proofs rely on a generalization of the well known energy integral method. First, we derive a priori estimates for the strong generalized solutions with the help of Yosida operator approximation. Then, using previous results, we show that the range of the operators generated by the posed problem is dense.

1. INTRODUCTION

In this paper we study a Goursat problem generated by a second order functional equation with operator coefficients possessing variable domains in two dimensional time.

This work has been motivated by recent results on a similar family of problems in the one dimensional time, for this, we can cite the works by Lomovtsev [6]-[10]. Under certain conditions on the operator coefficients in a Hilbert space, the author show that the posed boundary value problems are well posed in the Hadamard sense. In the case where the operator coefficients have constant domains, various important results were proved under different assumptions (see [1, 4, 5, 11]). We

Date: Received: 3 May 2008; Accepted: 24 August 2008.

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2000 *Mathematics Subject Classification.* Primary 35B45; Secondary 35D05, 35L90, 35L10, 35L20.

Key words and phrases. Evolution equation, a priori estimate, strong generalized solution, hyperbolic equation, boundary value problem.

note that the case of non-local conditions, the one and second order equations where studied in [2, 3].

Our results extend those obtained in [1, 4, 11], a similar technique will be applied in this investigation to similar type of equations under different hypotheses. On the other hand the majority of the existing results are obtained in a one dimensional time while our paper deals with an two dimensional time problem bringing up some additional difficulties. When passing to the higher dimension, the problem changes completely, because for the second order hyperbolic equations, the characteristics on the plane are presented by two different sets.

Existence and uniqueness of the strong generalized solution are studied The proofs rely on a generalization of the well known energy integral method. First, we derive a priori estimates for the strong generalized solutions with the help of Yosida operator approximation. Then, using previous results, we show that the range of the operators generated by the posed problem is dense.

The plan of this paper is as follows. In the next section we expose the problem, present the main assumptions and define the function spaces. In section 3, we derive some a priori estimates, prove the uniqueness of the strong generalized solution if it exists and its continuous dependence with respect to data. The existence of the strong generalized solution is provided in section 4.

2. Assumptions and Function Spaces

We consider the following Goursat problem.

$$Lu = \frac{\partial^2 u}{\partial t_1 \partial t_2} + T(t)u = f(t) \tag{2.1}$$

$$u\mid_{t_2=0} = 0; u\mid_{t_1=0} = 0. \tag{2.2}$$

Where $t = (t_1, t_2) \in \Omega = \left]0, T_1\right[\times \left]0, T_2\right[$, a bounded domain in $\mathbb{R}^2$, u is the unknown function, f is the given function acting from Ω with values in a Hilbert space$H$, where the norm and the inner product are denoted respectively by $|\cdot|$ and $(\cdot,\cdot)$. We assume that $\{T(t), t \in \Omega\}$ is a family of linear unbounded and self-adjoint operators on H whose domains $D_t(T)$ are depending on t and dense in H.

Throughout the paper we shall make the following assumptions.

(a) The operators $T(t)$, for each $t = (t_1, t_2) \in \Omega$ are self-adjoint in H and strongly monotone, i.e., there exists a constant $c_1 > 0$ such that for almost all $t \in \Omega$ we have

$$|v(t)|_t^2 = (T(t)v(t), v(t)) \geq c_1 \left|v(t)\right|^2, \forall v(t) \in D_t(T).$$

(b) The inverses of the operators $T(t)$ exist for almost all $t \in \Omega$, are strongly differentiable with respect to t in H and

$$\frac{\partial T^{-1}(t)}{\partial t_1}, \frac{\partial T^{-1}(t)}{\partial t_2}, \frac{\partial^2 T^{-1}(t)}{\partial t_1 \partial t_2} \in \mathcal{B}(\Omega, \mathfrak{L}(H)$$

where $\mathfrak{L}(H)$ is the space of linear bounded operators from H into H equipped with the norm $\|T\|_{\mathfrak{L}(H)} = \sup_{\substack{u\in H\\ u\neq 0}} \dfrac{|Tu|}{|u|}$; $\mathcal{B}(\Omega, \mathfrak{L}(H)$ is a Banach space of bounded operators.

(c) There exists two positive constants α_i, $i = 1, 2$, such that for almost all $t \in \Omega$ we have

$$-Re\left(\frac{\partial T^{-1}(t)}{\partial t_i}u(t), u(t)\right) \leq \alpha_i \left(T^{-1}(t)u(t), u(t)\right), \forall u \in H, i = 1, 2.$$

Now we start the description of the function spaces.

On the domain $D_t(T)$ of the operator $T(t)$ we define an inner product $(,)_t$ by

$$(u, v)_t = (T(t)u, v), \forall u, v \in D_t(T),$$

that is called the energetic inner product. Moreover, it introduces a norm on $D_t(T)$ which is denoted by $|.|_t$ and is said to be the energetic norm, the space $(D_t(T), |.|_t)$ is called the energetic space and is a Hilbert space, denoted by H_T, that is obtained by completing $D_t(T)$ with respect to the energetic norm.

We suppose that the domain $D(L)$ of the operator L consisting on all functions $u \in L_2(\Omega, H)$ for that $u(t) \in D_t(T), \dfrac{\partial u}{\partial t_1}, \dfrac{\partial u}{\partial t_2}, \dfrac{\partial^2 u}{\partial t_1 \partial t_2}, T(t)u \in L_2(\Omega, H)$ and satisfying conditions (2.2).

We denote by E the completion of $D(L)$ according to the norm

$$\begin{aligned} \|u\|_1^2 \quad = \quad & \sup_{(s_1,s_2)\in\overline{\Omega}} \left\{ \int_0^{s_1} \varkappa(t_1, s_2) \left(\left| \frac{\partial u(t_1, s_2)}{\partial t_1} \right|^2 + |u(t_1, s_2)|_t^2 \right) dt_1 + \right. \\ & \left. \int_0^{s_2} \varkappa(s_1, t_2) \left(\left| \frac{\partial u(s_1, t_2)}{\partial t_2} \right|^2 + |u(s_1, t_2)|_t^2 \right) dt_2 \right\}, \end{aligned}$$

where $\varkappa(t_1, t_2) = (T_1 - t_1)(T_2 - t_2)$.

As the space of right hand side f of equation (2.1), we choose the space F that is the completion of $L_2(D, H)$ according to the norm

$$\|f\|_2 = \sup_{v\in E_0} \left(\left| \iint_\Omega \varkappa(t_1, t_2) (f, \Lambda v) \, dt_1 dt_2 \right| / \|v\|_0 \right)$$

where $\Lambda u = \dfrac{\partial u}{\partial t_1} + \dfrac{\partial u}{\partial t_2}$ and E_0 is the set of all functions $v \in L_2(\Omega, H)$ for that $\dfrac{\partial v}{\partial t_1}, \dfrac{\partial v}{\partial t_2}, \dfrac{\partial^2 v}{\partial t_1 \partial t_2} \in L_2(\Omega, H)$ and satisfying conditions (2.2), equipped with the norm

$$\|v\|_0^2 = \sup_{(s_1,s_2)\in\overline{\Omega}} \left\{ \int_0^{s_1} \varkappa(t_1,t_2) \left(\left| \frac{\partial v(t_1,s_2)}{\partial t_1} \right|^2 + c_1 \left| v(t_1,s_2) \right|^2 \right) dt_1 + \int_0^{s_2} \varkappa(t_1,t_2) \left(\left| \frac{\partial v(s_1,t_2)}{\partial t_2} \right|^2 + c_1 \left| v(s_1,t_2) \right|^2 \right) dt_2 \right\}.$$

Lemma 2.1. *Assume that the hypothesis (a) and (b) hold, then $D(L)$ is dense in $L_2(\Omega, H)$.*

Proof. The proof is the same as in [2]. □

3. A priori estimates

Theorem 3.1. *Assume that the hypothesis (a)-(c) hold, then for every $u \in D(L)$ the following a priori estimate holds*

$$\|u\|_1 \leq C \|Lu\|_2 , \tag{3.1}$$

where C is a positive constant independent on t and on u.

Proof. We introduce the Yosida approximation operators $T_\varepsilon^{-1}(t) = (I + \varepsilon T(t))^{-1}$, $\varepsilon \geq 0$. They commute with $T(t)$ with range $D_t(T)$ and have the following properties (see [9]).

$$\text{(Y.A)} \begin{cases} \dfrac{\partial T_\varepsilon^{-1}(t)}{\partial t_i} = \varepsilon T(t) T_\varepsilon^{-1}(t) \frac{\partial T^{-1}(t)}{\partial t_i} T(t) T_\varepsilon^{-1}(t), i = 1,2 \\ \varepsilon T(t) T_\varepsilon^{-1}(t) = I - T_\varepsilon^{-1}(t) \\ \dfrac{\partial (T(t) T_\varepsilon^{-1}(t))}{\partial t_i} = \dfrac{-1}{\varepsilon} \dfrac{\partial T_\varepsilon^{-1}(t)}{\partial t_i} \\ \|T_\varepsilon^{-1}(t)\|_{\mathfrak{L}(H)} \leq 1 \\ \|\varepsilon T(t) T_\varepsilon^{-1}(t) u\| = \|u - T_\varepsilon^{-1}(t) u\| \to 0; \varepsilon \to 0, \forall u \in H \end{cases}$$

We Consider now the scalar product

$$e^{c(s_1+s_2-t_1-t_2)} \varkappa(t_1,t_2) (Lu, T_\varepsilon^{-1}(t) \Lambda u)$$

then by integrating this expression by parts over the domain $\Omega_s =]0, s_1[\times]0, s_2[\subset \Omega$, we get the identity

$$\int_0^{s_1}\left[e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)\left(\left|T_\varepsilon^{-\frac{1}{2}}(t)\frac{\partial u}{\partial t_1}\right|^2+\left|T_\varepsilon^{-\frac{1}{2}}(t)T^{\frac{1}{2}}(t)u(t)\right|^2\right)\right]_{t_2=s_2}dt_1$$

$$+\int_0^{s_2}\left[e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)\left(\left|T_\varepsilon^{-\frac{1}{2}}(t)\frac{\partial u}{\partial t_2}\right|^2+\left|T_\varepsilon^{-\frac{1}{2}}(t)T^{\frac{1}{2}}(t)u(t)\right|^2\right)\right]_{t_1=s_1}dt_2$$

$$\begin{aligned}
= \;& 2\mathrm{Re}\iint_{\Omega_s} e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)(Lu,T_\varepsilon^{-1}(t)\Lambda u)dt_1dt_2 \qquad (3.2)\\
& -2c\iint_{\Omega_s} e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)\left|T_\varepsilon^{-\frac{1}{2}}(t)u(t)\right|_t^2dt_1dt_2\\
& -c\iint_{\Omega_s} e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)\left(\left|T_\varepsilon^{-\frac{1}{2}}(t)\frac{\partial u}{\partial t_2}\right|^2+\left|T_\varepsilon^{-\frac{1}{2}}(t)\frac{\partial u}{\partial t_1}\right|^2\right)dt_1dt_2\\
& -\iint_{\Omega_s} e^{c(s_1+s_2-t_1-t_2)}\varkappa_1(t_1)\left(\left|T_\varepsilon^{-\frac{1}{2}}(t)\frac{\partial u}{\partial t_1}\right|^2+\left|T_\varepsilon^{-\frac{1}{2}}(t)u(t)\right|_t^2\right)dt_1dt_2\\
& -\iint_{\Omega_s} e^{c(s_1+s_2-t_1-t_2)}\varkappa_2(t_2)\left(\left|T_\varepsilon^{-\frac{1}{2}}(t)\frac{\partial u}{\partial t_2}\right|^2+\left|T_\varepsilon^{-\frac{1}{2}}(t)u(t)\right|_t^2\right)dt_1dt_2\\
& +\mathrm{Re}\iint_{\Omega_s} e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)\Big[\Big(\frac{\partial u}{\partial t_1},\frac{\partial T_\varepsilon^{-1}(t)}{\partial t_2}\frac{\partial u}{\partial t_1}\Big)+\Big(\frac{\partial u}{\partial t_2},\frac{\partial T_\varepsilon^{-1}(t)}{\partial t_1}\frac{\partial u}{\partial t_2}\Big)\\
& \qquad +\Big(\frac{\partial (T(t)T_\varepsilon^{-1}(t))}{\partial t_1}u,u\Big)+\Big(\frac{\partial (T(t)T_\varepsilon^{-1}(t))}{\partial t_2}u,u\Big)\Big]dt_1dt_2
\end{aligned}$$

where $\varkappa_1(t_1,t_2)=(T_1-t_1)$ and $\varkappa_2(t_1,t_2)=(T_2-t_2)$.

To estimate the last integral in the right hand side of (3.2), we use the $\delta-$Cauchy inequality with $\delta=1$, we apply properties (Y.A) then the conditions (b), finally by passing to the limit when ε tends to 0 in the both sides of the resultant relation it yields

$$\begin{aligned}
&\int_0^{s_1}\left[e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)\Big(\left|\frac{\partial u}{\partial t_1}\right|^2+|u|_t^2\Big)\right]_{t_2=s_2}dt_1+\\
&\int_0^{s_2}\left[e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)\Big(\left|\frac{\partial u}{\partial t_2}\right|^2+|u|_t^2\Big)\right]_{t_1=s_1}dt_2 \qquad (3.3)\\
\leq\;& +2\mathrm{Re}\iint_{\Omega_s} e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)(Lu,\Lambda u)dt_1dt_2+\\
&(-2c+\alpha_1+\alpha_2)\iint_{\Omega_s} e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)\,|u|_t^2\,dt_1dt_2\\
&+(-c+1/2)\iint_{\Omega_s} e^{c(s_1+s_2-t_1-t_2)}\varkappa(t_1,t_2)\left[\left|\frac{\partial u}{\partial t_1}\right|^2+\left|\frac{\partial u}{\partial t_2}\right|^2\right]dt_1dt_2
\end{aligned}$$

it is easy to see that if $c \geq \max(1/2, \frac{\alpha_1+\alpha_2}{2}) = \alpha$, the two last terms in the right hand side of (3.3) are negative then from elementary inequalities it yields

$$\int_0^{s_1} \varkappa(t_1, s_2) \left(\left| \frac{\partial u(t_1, s_2)}{\partial t_1 \left\| u \right\|_1} \right|^2 + \left| u(t_1, s_2) \right|_t^2 \right)_{t_2 = s_2} dt_1 +$$
$$\int_0^{s_2} \varkappa(s_1, t_2) \left(\left| \frac{\partial u(s_1, t_2)}{\partial t_2} \right|^2 + \left| u(s_1, t_2) \right|_t^2 \right)_{t_1 = s_1} dt_2$$

$$\leq 2e^{\alpha(T_1+T_2)} \left| \iint_\Omega \varkappa(t_1, t_2)(Lu, \Lambda u) dt_1 dt_2 \right| \tag{3.4}$$

Replacing the left hand side of (3.4) by its upper bound with respect to $(s_1, s_2) \in \overline{\Omega}$, dividing the both sides by $\left\| u \right\|_1$ then taking into account condition (a)we get that $\left\| u \right\|_1 \geq \left\| u \right\|_0, \forall u \in D(L)$ and consequently

$$\left\| u \right\|_1 \leq 2e^{\alpha(T_1+T_2)} \left\| Lu \right\|_2$$

This achieves the proof of the Theorem. □

We prove in a standard manner that the operator L is closable. Let $\overline{L}$ be its closure with domain of definition $D(\overline{L}) = \overline{D(L)}$.

Definition 3.2. The solution of the equation

$$\overline{L}u = f$$

is called strong generalized solution of the posed problem.

Since the functions $u \in D(\overline{L})$ are the limits of the sequences $u_n \in D(L)$, by passing to the limit we extend inequality (3.1) to the strong generalized solutions

$$\left\| u \right\|_1 \leq 2e^{\alpha(T_1+T_2)} \left\| \overline{L}u \right\|_2 ; \forall u \in D(\overline{L}) \tag{3.5}$$

Remark 3.3. From (3.5), we deduce the uniqueness of the strong generalized solution, when it exists, its continuous dependence on the data f and the closure of the range $R(\overline{L})$ of $\overline{L}$ in F,

$$R(\overline{L}) = \overline{R(L)}; (\overline{L})^{-1} = \overline{L^{-1}}.$$

Consequently, to prove the existence of the strong generalized solution for arbitrary $f \in F$, it suffices to prove that $R(L)$ is dense in F, that is $\overline{R(L)} = F$.

4. Solvability of the problem

Theorem 4.1. *Assume that the conditions of Theorem 3.1 are fulfilled, then for any $f \in F$, there exist one and only one strong generalized solution $u \in D(\overline{L})$ such that $u = (\overline{L})^{-1}(f)$ for the problem (2.1)-(2.2) satisfying the inequality (3.5).*

Proof. From a corollary of Hahn–Banach Theorem, it suffices to prove that if

$$\iint_{\Omega} \varkappa(t_1,t_2)\,(Lu,\Lambda v)\,dt_1dt_2 = 0, \forall u \in D(L) \tag{4.1}$$

for $v \in E_0$, then $v = 0$.

Denote $\mathcal{C}(\Omega,H)$ the set of all functions $v \in C^{\infty}$ such that $v\mid_{t_2=0}=0; v\mid_{t_1=0}=0$, since $\mathcal{C}(\Omega,H)$ is dense in E_0, we will prove that if (4.1) is true for $v \in \mathcal{C}(\Omega,H)$ then $v=0$.

Replacing the function u in the relation (4.1) by $T_\varepsilon^{-1}(t)h$, where $h \in E_0$, then we get

$$\begin{aligned} & 2\mathrm{Re}\int\int_{\Omega}\varkappa(t_1,t_2)(T_\varepsilon^{-1}(t)\frac{\partial^2 h}{\partial t_1\partial t_2},\Lambda v)dt_1dt_2 \\ = \; & -2\mathrm{Re}\int\int_{\Omega}\varkappa(t_1,t_2)(\frac{\partial^2 T_\varepsilon^{-1}(t)}{\partial t_1\partial t_2}h,\Lambda v)dt_1dt_2 \\ & -2\mathrm{Re}\int\int_{\Omega}\varkappa(t_1,t_2)(\frac{\partial T_\varepsilon^{-1}(t)}{\partial t_2}\frac{\partial h}{\partial t_1}+\frac{\partial T_\varepsilon^{-1}(t)}{\partial t_1}\frac{\partial h}{\partial t_2},\Lambda v)dt_1dt_2 \\ & -2\mathrm{Re}\int\int_{\Omega}\varkappa(t_1,t_2)(T(t)T_\varepsilon^{-1}(t)h,\Lambda v)dt_1dt_2. \end{aligned} \tag{4.2}$$

From properties (Y.A) and condition (b), we see that $\dfrac{\partial^2 T_\varepsilon^{-1}(t)}{\partial t_1\partial t_2}$, $\dfrac{\partial T_\varepsilon^{-1}(t)}{\partial t_i}$ and $T(t)T_\varepsilon^{-1}(t)$ are bounded in H. Using Cauchy-Schwartz inequality and Poincaré –Friedrichs inequality, we get that the absolute value of the right hand side of (4.2) is estimate by

$$\gamma\left(\left\|\frac{\partial h}{\partial t_2}\right\|^2_{L_2(\Omega,H)}+\left\|\frac{\partial h}{\partial t_2}\right\|^2_{L_2(\Omega,H)}\right)^{1/2}\left\|\varkappa(t_1,t_2)T_\varepsilon^{-1}(t)\Lambda v\right\|_{L_2(\Omega,H)}$$

where γ is a positive constant. From this we conclude that the linear functional that maps

$$\left(\frac{\partial h}{\partial t_1},\frac{\partial h}{\partial t_2}\right)\to\int\int_{\Omega}\varkappa(t_1,t_2)\left(\frac{\partial^2 h}{\partial t_1\partial t_2},T_\varepsilon^{-1}(t)\Lambda v\right)dt_1dt_2$$

is bounded in $L_2(\Omega,H)\times L_2(\Omega,H)$, consequently the derivatives of the function $\varkappa(t_1,t_2)T_\varepsilon^{-1}(t)\dfrac{\partial v}{\partial t_i}$ are in $L_2(\Omega,H)$ and vanishes on the line $t_i=T_i$, $i=1,2$.

Let w be a solution of the following problem

$$\begin{cases} e^{c(T_1+T_2-t_1-t_2)}t_1t_2\Lambda w=\Lambda v \\ w(0,t_2)=w(t_1,0)=0 \end{cases}$$

We integrate (4.2) by parts, then we set $h = w$ to get

$$
\begin{aligned}
& \int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa_1(t_1,t_2)t_1t_2(\left|T_{\varepsilon}^{\frac{-1}{2}}(t)\frac{\partial w}{\partial t_1}\right|^2 + \left|T_{\varepsilon}^{-\frac{1}{2}}(t)u(t)\right|_t^2)dt_1dt_2 + \\
& \int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa_2(t_1,t_2)t_1t_2(\left|T_{\varepsilon}^{\frac{-1}{2}}(t)\frac{\partial w}{\partial t_2}\right|^2 + \left|T_{\varepsilon}^{-\frac{1}{2}}(t)u(t)\right|_t^2)dt_1dt_2 \\
= \quad & \int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_2(\left|T_{\varepsilon}^{\frac{-1}{2}}(t)\frac{\partial w}{\partial t_1}\right|^2 + \left|T_{\varepsilon}^{-\frac{1}{2}}(t)u(t)\right|_t^2)dt_1dt_2 + \\
& \int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_2(\left|T_{\varepsilon}^{\frac{-1}{2}}(t)\frac{\partial w}{\partial t_2}\right|^2 + \left|T_{\varepsilon}^{-\frac{1}{2}}(t)u(t)\right|_t^2)dt_1dt_2 \\
& -c\int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_1t_2\left(\left|T_{\varepsilon}^{\frac{-1}{2}}(t)\frac{\partial w}{\partial t_1}\right|^2 + \left|T_{\varepsilon}^{\frac{-1}{2}}(t)\frac{\partial w}{\partial t_2}\right|^2\right)dt_1dt_2 \\
& -2c\int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_1t_2\left|T_{\varepsilon}^{\frac{-1}{2}}(t)w\right|_t^2 dt_1dt_2
\end{aligned}
$$

$$
\begin{aligned}
& -2\mathrm{Re}\int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_1t_2(\frac{\partial^2 T_{\varepsilon}^{-1}(t)}{\partial t_1\partial t_2}w, \Lambda w)dt_1dt_2 \\
& -2\mathrm{Re}\int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_1t_2(\frac{\partial T_{\varepsilon}^{-1}(t)}{\partial t_1}\frac{\partial w}{\partial t_2} + \\
& \frac{\partial T_{\varepsilon}^{-1}(t)}{\partial t_2}\frac{\partial w}{\partial t_1}, \Lambda w)dt_1dt_2 + \mathrm{Re}\int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_1t_2 \times \\
& \left((\frac{\partial T_{\varepsilon}^{-1}(t)}{\partial t_1}\frac{\partial w}{\partial t_2}, \frac{\partial w}{\partial t_1}) + (\frac{\partial T_{\varepsilon}^{-1}(t)}{\partial t_2}\frac{\partial w}{\partial t_1}, \frac{\partial w}{\partial t_2})\right)dt_1dt_2 \\
& +\mathrm{Re}\int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2) \times \\
& \left((\frac{\partial(T(t)T_{\varepsilon}^{-1}(t))}{\partial t_2}w, w) + (\frac{\partial(T(t)T_{\varepsilon}^{-1}(t))}{\partial t_1}w, w)\right)dt_1dt_2 \qquad (4.3)
\end{aligned}
$$

Using the Cauchy-Schwartz inequality and the fact that $\left\|\frac{\partial^2 T_{\varepsilon}^{-1}(t)}{\partial t_1\partial t_2}\right\|$ and $\left\|\frac{\partial T_{\varepsilon}^{-1}(t)}{\partial t_i}\right\|$ tend to zero when ε tends to 0 then the fifth, sixth and seventh integrals in the right hand side of (4.3) can be estimated by zero. Applying the properties (Y.A) and condition (c) to the last integral, then regrouping similar terms, it yields

$$
\begin{aligned}
& \int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa_1(t_1,t_2)t_1t_2(\left|\frac{\partial w}{\partial t_1}\right|^2 + |w(t)|_t^2)dt_1dt_2 + \qquad (4.4) \\
& \int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa_2(t_1,t_2)t_1t_2(\left|\frac{\partial w}{\partial t_2}\right|^2 + |w(t)|_t^2)dt_1dt_2
\end{aligned}
$$

$$\leq \int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_2\left(-ct_1+1\right)\left|\frac{\partial w}{\partial t_1}\right|^2 dt_1dt_2$$
$$\int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_1\left(-ct_2+1\right)\left|\frac{\partial w}{\partial t_2}\right|^2 dt_1dt_2$$
$$(-2c+\alpha_2+\alpha_1)\int\int_{\Omega} e^{c(T_1+T_2-t_1-t_2)}\varkappa(t_1,t_2)t_1t_2\left|w\right|_t^2 dt_1dt_2+$$

for $c \geq \max\left\{1/T_1, 1/T_2, (\alpha_1+\alpha_2)/2\right\}$, the right hand side of (4.4) is negative, then $\left|\frac{\partial w}{\partial t_1}\right| = \left|\frac{\partial w}{\partial t_2}\right| = |w|_t^2 = 0$, and so $v = 0$, indeed, for $v \in \mathcal{C}(\Omega, H)$, the operator Λ is invertible and if $\Lambda v = g$ then

$$v = \begin{cases} \int_{t_1-t_2}^{t_1} g(s_1, s_1-t_1+t_2)ds_{1,}t_1 \geq t_2 \\ \int_{t_2-t_1}^{t_2} g(s_2+t_1-t_2, s_2)ds_{2,}t_1 < t_2 \end{cases}$$

□

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