



REMARKS ON LIPSCHITZIAN MAPPINGS AND SOME FIXED POINT THEOREMS

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ABSTRACT. Let X, Y be the normed spaces, $C \subset X$ a convex set, and $T : C \rightarrow Y$ a continuous mapping. Some weak conditions implying the Lipschitz continuity of T are presented. Applications to the fixed point theory and theory of composition operators are presented.

1. INTRODUCTION

Lipschitzian mappings play important role in the fixed-point theory and its applications to nonlinear functional equation (cf. for instance J. Dugundij and A. Granas [2], also D.H. Hyers, G. Isac, Th.M. Rassias [4]). The contractions and nonexpansive mappings are the typical examples. It turns out however that sometime the Lipschitz condition forces a map to be affine. For instance, if the substitution (or Nemytskii) operator $T : Lip[0, 1] \rightarrow Lip[0, 1]$, generated by a function $h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$, given by the formula

$$T(\varphi)(x) := h(x, \varphi(x)), \quad \varphi \in Lip[0, 1], \quad (x \in [0, 1]),$$

is globally Lipschitzian with respect to the norm of the Banach space $Lip[0, 1]$, i.e. if, for some nonnegative real L ,

$$\|T(\varphi_1) - T(\varphi_2)\|_{Lip} \leq L \|\varphi_1 - \varphi_2\|_{Lip}, \quad \varphi_1, \varphi_2 \in Lip[0, 1],$$

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then there exist $a, b \in Lip[0, 1]$ such that

$$h(x, y) = a(x)y + b(x), \quad x \in [0, 1], y \in \mathbb{R},$$

and, consequently, the operator $T - b$ is linear (cf. [5]). Similar facts hold true for the substitution operator in some other function Banach spaces Hölder spaces, BV spaces, C^n (cf. [1], Chapters 6, and 7). In these cases the Banach contraction principle as well as its generalizations (including the Boyd-Wong theorem) which implicitly assume the Lipschitz continuity of the mapping, are not directly applicable for the relevant nonlinear problems.

In this context an open question arises whether the contractivity condition in some fixed point theorems can be modified in a way allowing to solve the above mentioned problems.

In this note we show that even a very weak substitute of the Lipschitz continuity of a map implies its Lipschitz continuity. As a by-product, we obtain purely formal generalizations of some fixed point theorems.

Let X, Y be the normed spaces, $C \subset X$ a convex, set and $T : C \rightarrow Y$ a continuous mapping. In the first section we show that the existence of a real $c \geq 0$ and a sequence of positive real numbers (t_n) , $\lim_{n \rightarrow \infty} t_n = 0$, such that for all $n \in \mathbb{N}$, $x, y \in C$,

$$\|x - y\| = t_n \implies \|T(x) - T(y)\| \leq ct_n,$$

implies the Lipschitz continuity of T with the constant c (cf. Theorem 1). This result improves a result in [6] where the uniform continuity of T is assumed). With the aid of some properties of subadditive functions, the above condition is replaced by a weaker one (Theorem 2).

In section 2, applying these results for a selfmapping T of a nonempty bounded and closed subset C of a uniformly convex Banach space, we present a (formal) generalization of the Browder-Goehde-Kirk theorem (Theorem 3) and its counterpart which guarantees the uniqueness of the fixed point. Instead of the nonexpansivity of T , the existence of a sequence of positive real numbers (t_n) , $\lim_{n \rightarrow \infty} t_n = 0$, such that

$$\liminf_{n \rightarrow \infty} \frac{\sup \{\|T(x) - T(y)\| : \|x - y\| = t_n, x, y \in C\}}{t_n} \leq 1$$

is required. As a corollary we obtain the following result. *Let X be a uniformly convex Banach space and let $C \subset X$ be a nonempty bounded closed and convex set. If $T : C \rightarrow C$ and*

$$\limsup_{\|x-y\| \rightarrow 0} \frac{\|T(x) - T(y)\|}{\|x - y\|} \leq 1,$$

then there exists a fixed-point of T in C . If, moreover, this inequality is strict, then the fixed-point is unique.

2. SOME RESULTS ON LIPSCHITZIAN MAPPINGS

Theorem 2.1. *Let X, Y be normed spaces and $C \subset X$ a convex set. Suppose $T : C \rightarrow Y$ is continuous. If there exist a real $c \geq 0$ and a sequence of positive*

real numbers (t_n) , $\lim_{n \rightarrow \infty} t_n = 0$, such that

$$\|x - y\| = t_n \implies \|T(x) - T(y)\| \leq ct_n \quad (2.1)$$

for all $n \in \mathbb{N}$, $x, y \in C$, then

$$\|T(x) - T(y)\| \leq c\|x - y\|, \quad x, y \in C.$$

Proof. Put

$$A := \{t \geq 0 : \|x - y\| = t \implies \|T(x) - T(y)\| < ct_n, (x, y \in C)\}.$$

If $t \geq \text{diam } C$ and $t < \infty$ then, of course, $t \in A$. Moreover, by assumption,

$$t_n \in A, \quad n \in \mathbb{N}. \quad (2.2)$$

Let $x, y \in C$ be such that, for some $k, n \in \mathbb{N}$,

$$\|x - y\| = kt_n.$$

Taking

$$z_j := x + \frac{j}{k}(y - x), \quad j = 0, 1, \dots, k,$$

we have

$$z_0 = x, \quad z_k = y; \quad \|z_j - z_{j-1}\| = t_n, \quad j = 1, \dots, k,$$

and, making use of (2.2),

$$\begin{aligned} \|T(x) - T(y)\| &= \left\| \sum_{j=1}^k T(z_j) - T(z_{j-1}) \right\| \leq \sum_{j=1}^k \|T(z_j) - T(z_{j-1})\| \\ &\leq c \sum_{j=1}^k \|z_j - z_{j-1}\| = ckt_n. \end{aligned}$$

This proves that $kt_n \in A$ for all $k, n \in \mathbb{N}$. Since the set $\{kt_n : k, n \in \mathbb{N}\}$ is dense in $[0, +\infty)$, we infer that so is A .

Now take arbitrary $x, y \in C$, $x \neq y$, and put

$$t = \|x - y\|.$$

By the density of A there is a sequence (s_n) of real numbers such that

$$s_n \in A, \quad 0 < s_n < t \quad \text{for all } n \in \mathbb{N}; \quad \lim_{n \rightarrow \infty} s_n = t.$$

Put

$$x_n := \frac{s_n}{t}x + \left(1 - \frac{s_n}{t}\right)y, \quad n \in \mathbb{N}.$$

Of course we have

$$x_n \in C \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad \lim_{n \rightarrow \infty} x_n = x.$$

Since $s_n \in A$ we have

$$\|T(x_n) - T(y)\| \leq c\|x_n - y\|, \quad n \in \mathbb{N}.$$

From the assumed continuity of T , letting $n \rightarrow \infty$, we hence get

$$\|T(x) - T(y)\| \leq c\|x - y\|,$$

which was to be shown. \square

The following example shows that the assumption of the continuity of the mapping T cannot be omitted.

Example 2.2. Let $X = Y = C = \mathbb{R}$, and let $T : C \rightarrow \mathbb{R}$ be defined by

$$T(x) := \begin{cases} x + 1 & \text{for } x \in \mathbb{Q} \\ x + 2 & \text{for } x \notin \mathbb{Q} \end{cases},$$

where \mathbb{Q} denotes the set of rational numbers. Then, for all $x, y \in C$,

$$|x - y| \in \mathbb{Q} \implies |T(x) - T(y)| = |x - y|.$$

In particular, with every sequence of positive rational numbers (t_n) , such that $\lim_{n \rightarrow \infty} t_n = 0$, the mapping T satisfies condition (2.1).

Now we can prove the main result of this section.

Theorem 2.3. *Let X, Y be the normed spaces, $C \subset X$ a convex set, and $T : C \rightarrow Y$ a continuous map. If there exists a sequence of positive real numbers (t_n) , $\lim_{n \rightarrow \infty} t_n = 0$, such that*

$$c_0 := \liminf_{n \rightarrow \infty} \frac{\sup \{\|T(x) - T(y)\| : \|x - y\| = t_n, x, y \in C\}}{t_n} < \infty,$$

then

$$\|T(x) - T(y)\| \leq c_0 \|x - y\|, \quad x, y \in C.$$

Proof. Replacing, if necessary, the sequence (t_n) by a subsequence, we can assume that

$$c_0 = \lim_{n \rightarrow \infty} \frac{\sup \{\|T(x) - T(y)\| : \|x - y\| = t_n, x, y \in C\}}{t_n} < \infty$$

and that, for some $c \geq c_0$,

$$\frac{\sup \{\|T(x) - T(y)\| : \|x - y\| = t_n, x, y \in C\}}{t_n} \leq c, \quad n \in \mathbb{N}.$$

It follows that condition (2.1) is satisfied. Applying Theorem 1 we obtain

$$\|T(x) - T(y)\| \leq c \|x - y\|, \quad x, y \in C. \quad (2.3)$$

Put

$$P := \{\|x - y\| : x, y \in C\}.$$

The convexity of C implies that either $P = [0, b]$ for some $b < +\infty$ or $P = [0, b)$ for some $b \leq +\infty$. Define the function $f : [0, +\infty) \rightarrow [0, +\infty]$ by

$$f(t) := \begin{cases} \sup \{\|T(x) - T(y)\| : \|x - y\| = t; x, y \in C\} & \text{for } t \in P \\ 0 & \text{for } t \notin P \end{cases}.$$

Of course we have $f(0) = 0$. From (2.3) we infer that f is finite, i.e. $f : [0, +\infty) \rightarrow [0, +\infty)$, and that f is right-continuous at 0.

Take $s, t \geq 0$. If $s + t \notin P$ then, obviously,

$$0 = f(s + t) \leq f(s) + f(t).$$

Suppose that $s + t \in P$. Let $x, y \in C$ be such that

$$\|x - y\| = s + t.$$

By the convexity of C ,

$$z := \frac{t}{s+t}x + \frac{s}{s+t}y \in C.$$

Moreover we have

$$\|x - z\| = s, \quad \|z - y\| = t.$$

By the triangle inequality,

$$\|T(x) - T(y)\| \leq \|T(x) - T(z)\| + \|T(z) - T(y)\|,$$

whence, by the definition of f ,

$$\|T(x) - T(y)\| \leq f(s) + f(t).$$

Taking the supremum of the left hand side over all $x, y \in C$ such that $\|x - y\| = s + t$, we hence get

$$f(s + t) \leq f(s) + f(t),$$

which shows that f is subadditive in $[0, +\infty)$.

According to well-known properties of subadditive functions (cf. [3] where the measurability of f is assumed), the limit

$$\lim_{t \rightarrow 0} \frac{f(t)}{t} \text{ exists}$$

and

$$\lim_{t \rightarrow 0} \frac{f(t)}{t} = \sup \left\{ \frac{f(t)}{t} : t > 0 \right\}.$$

Since, by the definition of the function f and (2.3),

$$c_0 = \lim_{n \rightarrow \infty} \frac{f(t_n)}{t_n},$$

we hence infer that

$$c_0 = \sup \left\{ \frac{f(t)}{t} : t > 0 \right\},$$

whence

$$f(t) \leq c_0 t, \quad t \geq 0.$$

This inequality and the definition of f imply that

$$\|T(x) - T(y)\| \leq c_0 \|x - y\|, \quad x, y \in C,$$

which completes the proof. \square

Theorem 2.4. *Let X, Y be the normed spaces, $C \subset X$ a convex set and $T : C \rightarrow Y$. If there exists a function $\gamma : [0, \infty) \rightarrow [0, \infty)$ such that*

$$c_0 := \limsup_{t \rightarrow 0} \frac{\gamma(t)}{t} < \infty,$$

and

$$\|T(x) - T(y)\| \leq \gamma(\|x - y\|), \quad x, y \in C,$$

then

$$\|T(x) - T(y)\| \leq c_0 \|x - y\|, \quad x, y \in C.$$

Proof. The continuity of T follows from the inequality $\|T(x) - T(y)\| \leq \gamma(\|x - y\|)$ for all $x, y \in C$. Taking a sequence of positive real numbers (t_n) , $\lim_{n \rightarrow \infty} t_n = 0$, such that

$$c_0 = \limsup_{n \rightarrow \infty} \frac{\gamma(t_n)}{t_n}$$

we obtain

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{\sup \{\|T(x) - T(y)\| : \|x - y\| = t_n, x, y \in C\}}{t_n} \\ &= \limsup_{n \rightarrow \infty} \frac{\gamma(t_n)}{t_n} = c_0, \end{aligned}$$

and the result follows from Theorem 2. \square

Note the following obvious

Corollary 2.5. *Let X, Y be the normed spaces, $C \subset X$ a convex set, $T : C \rightarrow Y$ and $\gamma : [0, \infty) \rightarrow [0, \infty)$. Suppose that*

$$\|T(x) - T(y)\| \leq \gamma(\|x - y\|), \quad x, y \in C.$$

If one of the following conditions holds true:

(1)

$$\limsup_{t \rightarrow 0} \frac{\gamma(t)}{t} < \infty,$$

(2) T is continuous and

$$c_0 := \liminf_{t \rightarrow 0} \frac{\gamma(t)}{t} < \infty,$$

then

$$\|T(x) - T(y)\| \leq c_0 \|x - y\|, \quad x, y \in C.$$

3. SOME FIXED POINT THEOREMS

The following result is a formal generalization of Browder-Goehde-Kirk fixed point theorem (cf. Dugundji-Granas [2, p. 34]).

Theorem 3.1. *Let X be a uniformly convex Banach space and let $C \subset X$ be a nonempty bounded closed and convex set. Suppose that $T : C \rightarrow C$ is a continuous map. If there exists a sequence of positive real numbers (t_n) , $\lim_{n \rightarrow \infty} t_n = 0$, such that*

$$\liminf_{n \rightarrow \infty} \frac{\sup \{\|T(x) - T(y)\| : \|x - y\| = t_n, x, y \in C\}}{t_n} \leq 1,$$

then T has a fixed point in C .

Proof. By Theorem 2 the mapping T is nonexpansive. Now the result is a consequence of the Browder-Goehde-Kirk theorem. \square

Now we prove the following

Theorem 3.2. *Let X be a uniformly convex Banach space and let $C \subset X$ be a nonempty bounded closed and convex set. Suppose that $T : C \rightarrow C$ is a continuous map. If there exists a sequence of positive real numbers (t_n) , $\lim_{n \rightarrow \infty} t_n = 0$, such that, for all $n \in \mathbb{N}$,*

$$\sup \{ \|T(x) - T(y)\| : \|x - y\| = t_n, x, y \in C \} < t_n \quad (3.1)$$

then T has a unique fixed point in C .

Proof. The existence of a fixed point follows from Theorem 3. By Theorem 1 we have

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad x, y \in C. \quad (3.2)$$

Take arbitrary $x, y \in C$ such that $\|x - y\| > 0$. Since $\lim_{n \rightarrow \infty} t_n = 0$, there is a $k \in \mathbb{N}$ such that $t_k < \|x - y\|$. By the convexity of C ,

$$z := \left(1 - \frac{t_k}{\|x - y\|}\right)x + \frac{t_k}{\|x - y\|}y \in C,$$

and

$$\|x - z\| = t_k, \quad \|z - y\| = \|x - y\| - t_k,$$

whence, by (3.1),

$$\|T(x) - T(z)\| < \|x - z\| = t_k \quad (3.3)$$

Now, making use of (3.2) and (3.3), we obtain

$$\begin{aligned} \|T(x) - T(y)\| &\leq \|T(x) - T(z)\| + \|T(z) - T(y)\| \\ &< t_k + (\|x - y\| - t_k) = \|x - y\|. \end{aligned}$$

Thus we have shown that

$$\|T(x) - T(y)\| < \|x - y\|, \quad x, y \in C, x \neq y,$$

which implies the uniqueness of the fixed-point. This completes the proof. \square

As an immediate consequence of Theorems 4 and 5 note the following

Corollary 3.3. *Let X be a uniformly convex Banach space and let $C \subset X$ be a nonempty bounded closed and convex set. If $T : C \rightarrow C$ and*

$$\limsup_{\|x-y\| \rightarrow 0} \frac{\|T(x) - T(y)\|}{\|x - y\|} \leq 1,$$

then there exists a fixed-point of T in C . If, moreover, this inequality is strict, then the fixed-point is unique.

From Corollary 1 and the Browder-Goehde-Kirk fixed point theorem we get

Corollary 3.4. *Let X be a uniformly convex Banach space, $C \subset X$ a nonempty convex and closed set, T a selfmap of C and $\gamma : [0, \infty) \rightarrow [0, \infty)$. Suppose that*

$$\|T(x) - T(y)\| \leq \gamma(\|x - y\|), \quad x, y \in C.$$

If one of the following conditions holds true:

$$(1) \quad \limsup_{t \rightarrow 0} \frac{\gamma(t)}{t} \leq 1,$$

(2) T is continuous and

$$c_0 := \liminf_{t \rightarrow 0} \frac{\gamma(t)}{t} \leq 1$$

then T has a fixed point in C . If moreover

$$\|T(x) - T(y)\| \neq \|x - y\| \quad \text{for all } x, y \in C, x \neq y,$$

the fixed point is unique.

Remark 3.5. The arguments used in the proofs of Theorems 1 and 2 allow to prove their counterparts in metrically convex spaces.

4. REMARK ON GLOBALLY LIPSCHITZIAN SUBSTITUTION OPERATORS

Applying Theorem 2 and the main result of [5] we obtain the following

Corollary 4.1. *Let a function $h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$. Suppose that the substitution operator T defined by*

$$T(\varphi)(x) := h(x, \varphi(x)), \quad (x \in [0, 1]),$$

maps continuously the Banach space $Lip[0, 1]$ into itself. If there exists a sequence of positive real numbers (t_n) , $\lim_{n \rightarrow \infty} t_n = 0$, such that, for all $n \in \mathbb{N}$,

$$\liminf_{n \rightarrow \infty} \frac{\sup \left\{ \|T(\varphi) - T(\psi)\|_{lip} : \|\varphi - \psi\|_{lip} = t_n, \varphi, \psi \in Lip[0, 1] \right\}}{t_n} < \infty,$$

then there are $a, b \in Lip[0, 1]$ such that

$$h(x, y) = a(x)y + b(x), \quad x \in [0, 1], y \in \mathbb{R}.$$

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