D'ALEMBERT'S FUNCTIONAL EQUATION ON COMPACT GROUPS

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by M. Abel

ABSTRACT. At the Fourty-forth International Symposium on Functional Equations in 2006, Louisville, Kentucky, USA, T.M.K. Davison presented a talk under the title "D'Alembert's functional equation on compact groups" in which he offered a solution method for this equation on any compact group. In this paper we offer another approach based on some very recent results on spectral synthesis, which method seems to be useful in the case of other equations.

1. INTRODUCTION AND PRELIMINARIES

D'Alembert's functional equation has been studied in several papers by several authors under different assumptions. The equation has the form

$$f(x\,y) + f(x\,y^{-1}) = 2\,f(x)\,f(y)\,, \tag{1.1}$$

where the unknown function f is supposed to be defined on a group G having complex values: $f : G \to \mathbb{C}$ satisfies (1.1) for all x, y in G. For the history of d'Alembert's equation and for diverse results concerning it see e.g. [1, 2, 4, 5].

At the Fourty-forth International Symposium on Functional Equations in 2006, Louisville, Kentucky, USA, T. M. K. Davison presented some new results concerning (1.1). He supposes that G is a compact group and he introduces the notion of a basic d'Alembert function: a continuous solution $f : G \to \mathbb{C}$ of (1.1) for which $f(xy) = f(x)$ for all x in G implies that $y = e$, the identity of the

group G. He then shows that every d'Alembert function factors through a basic
d'Alembert function. He also proves that the only compact groups that support a
basic d'Alembert function are isomorphic to compact subgroups of $SU2(\mathbb{C})$. Each
subgroup (compact or not) of $SU2(\mathbb{C})$ supports a basic d'Alembert function.

In this note we apply a completely different approach to solve (1.1) on compact
groups. This approach is based on spectral synthesis over compact groups.

1.1. **Spectral analysis and spectral synthesis over compact groups.** Spec-
tral synthesis deals with the description of translation invariant function spaces
over locally compact groups. For the most relevant results and developments with
applications see [6]. However, the study of spectral synthesis in [6] is restricted
mainly to discrete Abelian groups. In [7] we sketched a possible approach of spec-
tral analysis and spectral synthesis problems over not necessarily commutative
groups. As a main result we proved that spectral analysis and spectral synthesis
in this extended sense holds over compact groups. Here we shortly summarize
the main ideas and results of [7].

If G is a compact group, then $\mathcal{C}(G)$ denotes the Banach space of all continuous
complex valued functions defined on G equipped with the point-wise operations
and with the topology of uniform convergence.

For each y in G the symbol τ_y denotes the *right translation operator by y* which
is defined on each f in $\mathcal{C}(G)$ by the formula

$$\tau_y f(x) = f(xy) \,,$$

whenever x is in G. A linear subspace V of $\mathcal{C}(G)$ is called *right invariant*, if $\tau_y f$
belongs to V, whenever f is in V. A right invariant closed linear subspace of
$\mathcal{C}(G)$ is called a *right variety*. If this is different from $\{0\}$ and $\mathcal{C}(G)$, then we call
it a *proper right variety*. Moreover, if V is a right variety in $\mathcal{C}(G)$, then a subset
of V which is a right variety is called a *proper right subvariety*, if it is different
from $\{0\}$ and from V. We can analogously define the concepts of *left translation
operator, left invariant subspace, left variety* and *proper left variety*, etc. A set
which is a right, or left variety, is called a *one-sided variety*. A right variety,
which is also a left variety is called a *two-sided variety*, or simply a *variety*.

A nonzero right (left, or two-sided) variety in $\mathcal{C}(G)$ is called *reducible*, if it
has a proper subvariety, otherwise it is called *irreducible*. A right (left, or two-
sided) variety in $\mathcal{C}(G)$ is called *decomposable*, if it is the direct sum of two proper
subvarieties, otherwise it is called *indecomposable*.

In general, if G is a locally compact (not necessarily compact) Abelian group,
then the building blocks of spectral analysis and spectral synthesis are the ex-
ponential monomials. A continuous homomorphism of G into the multiplica-
tive group of nonzero complex numbers is called an *exponential*, and a con-
tinuous homomorphism of G into the additive group of complex numbers is
called an *additive function*. A complex valued function on G having the form
$x \mapsto P\big(a_1(x), a_2(x), \ldots, a_n(x)\big)$ is called a *polynomial*, if $P : \mathbb{C}^n \to \mathbb{C}$ is a complex

polynomial and $a_1, a_2, \ldots, a_n : G \to \mathbb{C}$ are additive functions. Hence polynomials are the elements of the function algebra generated by the constants and the additive functions.

A function which is a product of a polynomial and an exponential is called an *exponential monomial*. Therefore the general form of exponential monomials is

$$\varphi(x) = p(x)m(x),$$

where $m : G \to \mathbb{C}$ is an exponential and $p : G \to \mathbb{C}$ is a polynomial.

Nevertheless, if G is compact Abelian, then each exponential function is a character and each polynomial is a constant, hence exponential monomials are constant multiples of characters and exponential polynomials are trigonometric polynomials. It turns out, however, that trigonometric polynomials are not the proper function class which can be used to build up varieties in the noncommutative case. In what follows we adopt these concepts for the noncommutative case. In [7] we showed that these new concepts coincide with the old ones on commutative groups.

Let G be a compact group and H a finite dimensional Hilbert space with the scalar product $\langle\,,\,\rangle$. By a *representation* of G on H we mean a continuous homomorphic mapping F of G into the algebra $\mathcal{B}(H)$ of bounded linear operators on H. The representation $F : G \to \mathcal{B}(H)$ is called *irreducible*, if H has no proper nonzero subspace which is invariant under all operators $F(x)$ with x in G. Otherwise the representation is called *reducible*. Given a representation of this type $F : G \to \mathcal{B}(H)$ a nonzero proper closed subspace of H is called a *reducing subspace* for F, if the subspace and its orthogonal complement are invariant subspaces of the representation F. If a representation has a reducing subspace, then it is called *decomposable*, otherwise it is called *indecomposable*. An irreducible representation is obviously indecomposable.

The following definitions seem to be reasonable. Let G be a locally compact group and let V be a variety in $\mathcal{C}(G)$. We say that *spectral analysis* holds in V, if V contains a nonzero exponential monomial. We say that *spectral synthesis* holds in V, if all exponential monomials in V span a dense subvariety in V. If V is a nonzero variety, then clearly spectral synthesis in V implies spectral analysis in V. We say that *spectral analysis*, respectively *spectral synthesis* holds on G, if spectral analysis, respectively spectral synthesis holds in every proper variety in $\mathcal{C}(G)$. Similarly, we can define the concepts of spectral analysis and spectral synthesis for one-sided varieties, and also one-sided spectral analysis and one-sided spectral synthesis for locally compact groups. Also, the locally compactness of the group is not necessary to formulate the above concepts, but in this paper we consider this setting only.

Let G be a locally compact group with identity e and let V be a nonzero finite dimensional right variety in $\mathcal{C}(G)$. Suppose that the complex valued continuous functions $f_1, f_2, \ldots, f_n : G \to \mathbb{C}$ form a basis of V. Then there exist functions

$c_{i,j} : G \to \mathbb{C}$ $(i,j = 1,2,\ldots,n)$ such that the system of functional equations

$$f_j(xy) = \sum_{i=1}^{n} c_{i,j}(y)f_i(x) \qquad (1.2)$$

holds for each x,y in G and for $j = 1,2,\ldots,n$. We denote by $C(y)$ the matrix $\big(c_{i,j}(y)\big)_{i,j=1,2,\ldots,n}$, for each y in G. Clearly, $C(e)$ is the identity matrix.

Replacing y by yz in (1.2) we obtain the equations

$$f_j(xyz) = \sum_{k=1}^{n} c_{k,j}(yz)f_k(x) \qquad (1.3)$$

holds for each x,y,z in G and for $j = 1,2,\ldots,n$. On the other hand, if we put xy for x and z for y in (1.2), then we have

$$f_j(xyz) = \sum_{i=1}^{n} c_{i,j}(z)f_i(xy) = \sum_{i=1}^{n}\sum_{k=1}^{n} c_{i,j}(z)c_{k,i}(y)f_k(x) \qquad (1.4)$$

holds for each x,y,z in G and for $j = 1,2,\ldots,n$. Comparing (1.3) and (1.4), and using the linear independence of the function $f_1, f_2, \ldots, f_n$ we have that

$$C(yz) = C(y)C(z) \qquad (1.5)$$

holds for each y,z in G. Again, by the linear independence of the functions $f_1, f_2, \ldots, f_n$, there are elements x_k in G such that the matrix $\big(f_j(x_k)\big)_{j,k=1,2,\ldots,n}$ is regular. Substituting x_k for x in (1.2) with j is fixed and $k = 1,2,\ldots,n$ we get a system of linear equations for the unknowns $c_{i,j}(y)$ with $i = 1,2,\ldots,n$ from which these unknowns can be expressed as linear combinations of some left translates of f_j. In particular, the functions $c_{i,j}$ are continuous functions. Therefore in this case $C : G \to \mathcal{B}(\mathbb{C}^n)$ is a representation of G on the finite dimensional Hilbert space $\mathbb{C}^n$ equipped with the standard inner product. This will be called a *representation of G corresponding to the right variety V*, using the given basis. It is clear that if C, D are two representations of G corresponding to the same nonzero finite dimensional variety, using different bases, then there exists a regular matrix S such that the equality $D = S^{-1}DS$ holds. Further, it follows that if the right variety V is also left invariant, then the functions $c_{i,j}$ for $i,j = 1,2,\ldots,n$ belong to V, too, moreover, by (1.2), they span V. The following theorem is a special case of a result in [7].

Theorem 1.1. *Let G be a compact group and let V be a nonzero finite dimensional variety in $\mathcal{C}(G)$. Then any representation of G corresponding to V is indecomposable if and only if V is indecomposable. Further, any representation of G corresponding to V is irreducible if and only if V is irreducible.*

The next theorem is a special case of one of the main results in [7].

Theorem 1.2. *Let G be a compact group. Then spectral synthesis holds in every finite dimensional variety in $\mathcal{C}(G)$. Moreover, every nonzero finite dimensional variety in $\mathcal{C}(G)$ has an irreducible subvariety.*

The following theorem is taken from [7].

Theorem 1.3. *Let G be a locally compact group and let V be a variety in $\mathcal{C}(G)$. Spectral analysis holds in V if and only if V has a nonzero finite dimensional subvariety. Spectral synthesis holds in V if and only if V is the sum of its finite dimensional subvarieties.*

In this paper we shall need the following basic result of [7].

Theorem 1.4. *Spectral synthesis holds on compact groups.*

Proof. By Theorem 1.3 it is enough to prove that any variety in $\mathcal{C}(G)$ is the sum of finite dimensional subvarieties.

The fundamental theorem of almost periodic functions (see e.g. [3], p.47.) states that on any group any variety consisting of almost periodic functions is the sum of finite dimensional irreducible subvarieties. However, if G is a compact group, then every continuous complex valued function on G is almost periodic. The proof is complete. $\square$

2. D'ALEMBERT'S FUNCTIONAL EQUATION ON COMPACT GROUPS

Let G be a compact group and suppose that the continuous function $f : G \to \mathbb{C}$ satisfies the functional equation (1.1) of d'Alembert:

$$f(x\,y) + f(x\,y^{-1}) = 2\,f(x)\,f(y)$$

for each x, y in G. Using the spectral synthesis result Theorem 1.4 we can prove the following theorem.

Theorem 2.1. *Let G be a compact group and suppose that the continuous function $f :\to \mathbb{C}$ satisfies the functional equation (1.1). Then there exists a finite dimensional complex Hilbert–space, a unitary representation $M : G \to \mathcal{B}(H)$, an element x_0 in G and elements ξ, η in H such that*

$$f(y) = \frac{1}{2\,\langle M(x_0)\xi, \xi\rangle}[\langle M(y)\xi, \eta\rangle + \langle M(y^{-1})\xi, \eta\rangle]$$

for each y in G.

Proof. By the substitution $x = y = e$ (e is the identity of the group) it follows from (1.1) that $f(e) = 1$ or $f(e) = 0$. In the latter case f is identically zero by the substitution $y = e$ in (1.1) and we shall exclude this possibility. Hence $f(e) = 1$ and substituting $x = e$ in (1.1) we get that f is an even function. Interchanging x and y in (1.1) it follows that $f(xy) = f(yx)$ holds for each x, y in G. This means that the left translation invariant closed subspace generated by f is also right translation invariant, that is, it is translation invariant. By spectral synthesis there exists a finite dimensional Hilbert–space H, an indecomposable representation $M : G \to \mathcal{B}(H)$ and an element ξ in H such that the non-identically zero function function

$$x \mapsto \langle M(x)\xi, \xi\rangle$$

belongs to the translation invariant closed subspace generated by f, hence it satisfies (1.1) on G for any fixed y in G. (Here, as above $\langle . \rangle$ denotes the inner

product in H.) This means

$$\langle M(xy)\xi, \xi \rangle + \langle M(xy^{-1})\xi, \xi \rangle = \langle M(x)\xi, \xi \rangle f(y)$$

holds for each x, y in G. Putting $x = e$ and using the fact that $x \mapsto \langle M(x)\xi, \xi \rangle$ is not identically zero, hence there exists an x_0 in G such that $\langle M(x_0)\xi, \xi \rangle \neq 0$ and it follows

$$f(y) = \frac{1}{2\langle M(x_0)\xi, \xi \rangle}[\langle M(x_0\,y)\xi, \xi \rangle + \langle M(x_0\,y^{-1})\xi, \xi \rangle]$$

for each y in G. Using the notation $M^{-1}(x_0)\xi = \eta$ it follows

$$f(y) = \frac{1}{2\langle M(x_0)\xi, \xi \rangle}[\langle M(y)\xi, \eta \rangle + \langle M(y^{-1})\xi, \eta \rangle]$$

for each y in G and the proof is complete. $\qquad \square$

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