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ON CONVERGENCE OF GREEDY APPROXIMATIONS FOR THE TRIGONOMETRIC SYSTEM

SERGEI V. KONYAGIN[1]

This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by M.S. Moslehian

ABSTRACT. In this note we discuss the convergence of greedy approximants for trigonometric Fourier expansion in $L_p(\mathbb{T})$, $1 \le p < 2$.

1. INTRODUCTION

We study in this paper the following nonlinear method of summation of trigonometric Fourier series. Consider a periodic function $f \in L_p(\mathbb{T})$, $1 \le p \le \infty$, $(L_\infty(\mathbb{T}) = C(\mathbb{T}))$, defined on the torus $\mathbb{T}$. Let a number $m \in \mathbb{N}$ be given and Λ_m be a set of $k \in \mathbb{Z}$ with the properties:

$$\min_{k\in\Lambda_m} |\hat{f}(k)| \ge \max_{k\notin\Lambda_m} |\hat{f}(k)|, \quad |\Lambda_m| = m,$$

where

$$\hat{f}(k) := (2\pi)^{-1} \int_{\mathbb{T}} f(x) e^{-ikx} dx$$

is a Fourier coefficient of f. We define

$$G_m(f) := S_{\Lambda_m}(f) := \sum_{k\in\Lambda_m} \hat{f}(k) e^{ikx}$$

and call it a m-th greedy approximant of f with regard to the trigonometric system $\mathcal{T} := \{e^{ikx}\}_{k\in\mathbb{Z}}$. Clearly, a m-th greedy approximant may not be unique. In this paper we do not impose any extra restrictions on Λ_m.

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It has been proved in [1] for $p < 2$ and in [5] for $p \neq 2$ that there exists a $f \in L_p(\mathbb{T})$ such that $\{G_m(f)\}$ does not converge in L_p. It was remarked in [6] that the method from [5] gives a little more: 1) There exists a continuous function f such that $\{G_m(f)\}$ does not converge in $L_p(\mathbb{T})$ for any $p > 2$; 2) There exists a function f that belongs to any $L_p(\mathbb{T})$, $p < 2$, such that $\{G_m(f)\}$ does not converge in measure. Thus the above negative results show that the condition $f \in L_p(\mathbb{T})$, $p \neq 2$, does not guarantee convergence of $\{G_m(f)\}$ in the L_p-norm. The main goal of this paper is to discuss additional (to $f \in L_p$) conditions on f to guarantee that $\|f - G_m(f)\|_p \to 0$ as $m \to \infty$. Some results in this direction have already been obtained in [2].

For a mapping $\alpha : W \to W$ we denote α_k its k-fold iteration: $\alpha_k := \alpha \circ \alpha_{k-1}$. In [3] we studied quantitative versions of Cauchy's convergence criterion for greedy approximants and proved the following theorems.

Theorem 1.1. *Let $\alpha : \mathbb{N} \to \mathbb{N}$ be strictly increasing. Then the following conditions are equivalent:*

(a) *for some $k \in \mathbb{N}$ and for any sufficiently large $m \in \mathbb{N}$ we have $\alpha_k(m) > e^m$;*
(b) *if $f \in C(\mathbb{T})$ and*

$$\left\|G_{\alpha(m)}(f) - G_m(f)\right\|_\infty \to 0 \quad (m \to \infty)$$

then

$$\|f - G_m(f)\|_\infty \to 0 \quad (m \to \infty).$$

Theorem 1.2. *Let $p = 2q$, $q \in \mathbb{N}$, be an even integer, $\delta > 0$. Assume that $f \in L_p(\mathbb{T})$ and there exists a sequence of positive integers $M(m) > m^{1+\delta}$ such that*

$$\|G_{M(m)}(f) - G_m(f)\|_p \to 0 \quad \text{as} \quad m \to \infty.$$

Then we have

$$\|f - G_m(f)\|_p \to 0 \quad \text{as} \quad m \to \infty.$$

Theorem 1.3. *For any $p \in (2, \infty)$ there exists a function $f \in L_p(\mathbb{T})$ with divergent in the $L_p(\mathbb{T})$ sequence $\{G_m(f)\}$ of greedy approximations with the following property. For any sequence $\{M(m)\}$ such that $m \leq M(m) \leq m^{1+o(1)}$ we have*

$$\|G_{M(m)}(f) - G_m(f)\|_p \to 0 \quad (m \to 0).$$

The proofs of Theorems 1.1 and 1.2 give also "sequential" versions of those results.

Theorem 1.4. *Let $\{m_j\}_{j \in \mathbb{N}}$ be a strictly increasing sequence of positive integers. Then the following conditions are equivalent:*

(a) *for some $k \in \mathbb{N}$ and for all $j \in \mathbb{N}$ we have $m_{j+k} > e^{m_j}$;*
(b) *if $f \in C(\mathbb{T})$ and*

$$\left\|G_{m_{j+1}}(f) - G_{m_j}(f)\right\|_\infty \to 0 \quad (j \to \infty)$$

then

$$\left\|f - G_{m_j}(f)\right\|_\infty \to 0 \quad (j \to \infty).$$

Theorem 1.5. *Let $p = 2q$, $q \in \mathbb{N}$, be an even integer, $\delta > 0$. Assume that $f \in L_p(\mathbb{T})$ and there exists a sequence of positive integer $\{m_j\}_{j\in\mathbb{N}}$ such that $m_{j+1} > m_j^{1+\delta}$ for all j and*

$$\left\|G_{m_{j+1}}(f) - G_{m_j}(f)\right\|_p \to 0 \quad (j \to \infty)$$

Then we have

$$\left\|f - G_{m_j}(f)\right\|_p \to 0 \quad (j \to \infty).$$

2. Results

In this note we announce some results for the spaces $L_p(\mathbb{T})$, $1 \le p < 2$.

Theorem 2.1. *Let $\alpha : \mathbb{N} \to \mathbb{N}$ be strictly increasing such that for some $k \in \mathbb{N}$ and for all $m \in \mathbb{N}$ we have $\alpha_k(m) > e^m$. Assume that $1 \le p < 2$, $f \in L_p(\mathbb{T})$, and*

$$\left\|G_{\alpha(m)}(f) - G_m(f)\right\|_p \to 0 \quad (m \to \infty).$$

Then

$$\|f - G_m(f)\|_p \to 0 \quad (m \to \infty).$$

Theorem 2.2. *Let $1 \le p < 2$. Assume that $f \in L_p(\mathbb{T})$ and there exist a sequence of positive integer $\{m_j\}_{j\in\mathbb{N}}$ and a positive integer k such that $m_{j+k} > e^{m_j}$ for all j and*

$$\left\|G_{m_{j+1}}(f) - G_{m_j}(f)\right\|_p \to 0 \quad (j \to \infty)$$

Then we have

$$\left\|f - G_{m_j}(f)\right\|_p \to 0 \quad (j \to \infty).$$

We can partially reverse Theorem 2.2 for $p = 1$.

Theorem 2.3. *Let $\delta > 0$, $\{m_j\}_{j\in\mathbb{N}}$ be a sequence of positive integers such that $\log m_{j+1} > (\log m_j)^{2+\delta}$ for all j and for any k the inequality $m_{j+k} < e^{m_j}$ holds for some j. Then there exists a function $f \in L_1(\mathbb{T})$ such that*

$$\left\|G_{m_{j+1}}(f) - G_{m_j}(f)\right\|_1 \to 0 \quad (j \to \infty)$$

but

$$\sup_j \|G_{m_j}(f)\|_1 = \infty.$$

Probably, the condition $\log m_{j+1} > (\log m_j)^{2+\delta}$ is not essential. However, we expect that Theorem 2.2 for $p > 1$ and Theorem 2.1 are not sharp.

The proofs of Theorems 2.1 and 2.2 follow the technique of [3]. The proof of Theorem 2.3 is based on [4].

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[1] Department of Mechanics and Mathematics, Moscow State University, Moscow, 119992, Russia.

E-mail address: konyagin@ok.ru