Banach J. Math. Anal. 1 (2007), no. 1, 131-138

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

SOME BOUNDING INEQUALITIES FOR THE JACOBI AND RELATED FUNCTIONS

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This paper is dedicated to Professor Themistocles Michael Rassias.

Submitted by G. M. Milovanović

ABSTRACT. The main object of this paper is to present several bounding inequalities for the *classical* Jacobi function of the first kind. A number of closely-related inequalities for such other special functions as the *classical* Laguerre function are also considered.

1. INTRODUCTION

In the usual notation, the *classical* Jacobi function $P_\nu^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$ of the first kind is defined by (see, for example, [8, p. 433])

$$\begin{aligned} P_\nu^{(\alpha,\beta)}(z) &:= \sum_{k=0}^{\infty} \binom{\nu+\alpha}{\nu-k} \binom{\nu+\beta}{k} \left(\frac{z-1}{2}\right)^k \left(\frac{z+1}{2}\right)^{\nu-k} \\ &= \binom{\nu+\alpha}{\nu} {}_2F_1\left(-\nu, \alpha+\beta+\nu+1; \alpha+1; \frac{1-z}{2}\right) \qquad (\nu \in \mathbb{C}) \end{aligned} \tag{1.1}$$

Date: Received: 13 April 2007; Accepted: 27 October 2007.

2000 *Mathematics Subject Classification.* Primary 26A33, 33C45; Secondary 26D15, 33C15.

Key words and phrases. Jacobi functions, Jacobi polynomials, Gauss hypergeometric function, Riemann-Liouville fractional derivative, Laguerre function, Bessel function of the first kind, Krasikov's uniform bound for the Bessel function, confluent hypergeometric function, Landau's inequalities for the Bessel function, Love's inequalities, Luke's bound for the confluent hypergeometric function, Olenko's bound for the Bessel function.

in terms of the Gauss hypergeometric function ${}_2F_1$. Here, *and in what follows*, we make use of a generalized binomial coefficient given by

$$\binom{\kappa}{\mu} := \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\mu+1)\Gamma(\mu+1)} =: \binom{\kappa}{\kappa-\mu} \qquad (\kappa,\mu\in\mathbb{C}).$$

Together with the *classical* Jacobi function $Q_\nu^{(\alpha,\beta)}(z)$ $(\nu\in\mathbb{C})$ of the second kind, which possesses a hypergeometric representation given by (*cf.* [8, p. 449]; see also [13, p. 453, Problem 26])

$$\begin{aligned} Q_\nu^{(\alpha,\beta)}(z) &= 2^{\alpha+\beta+\nu}\, B(\alpha+\nu+1,\beta+\nu+1)\,(z-1)^{-\alpha-\nu-1}\,(z+1)^{-\beta} \\ &\cdot {}_2F_1\left(\nu+1,\alpha+\nu+1;\alpha+\beta+2\nu+2;\frac{2}{1-z}\right) \qquad (\nu\in\mathbb{C}), \end{aligned}$$

these *classical* Jacobi functions $P_\nu^{(\alpha,\beta)}(z)$ $(\nu\in\mathbb{C})$ and $Q_\nu^{(\alpha,\beta)}(z)$ $(\nu\in\mathbb{C})$ are known to satisfy the following differential equation:

$$(1-z^2)\frac{d^2w}{dz^2} + [\beta-\alpha-(\alpha+\beta+2)z]\frac{dw}{dz} + (\alpha+\beta+\nu+1)\nu w = 0$$
$$\left(w \equiv P_\nu^{(\alpha,\beta)}(z)\right),$$

$B(\alpha,\beta)$ being the familiar Beta function defined by

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.2}$$
$$\left(\min\{\Re(\alpha),\Re(\beta)\}>0\right).$$

Now, for the *Riemann–Liouville fractional derivative operator* $\mathcal{D}_z^\mu$ of (real or complex) order μ defined by (*cf.* [1, Vol. II, p. 181 *et seq.*]; see also [3])

$$\mathcal{D}_z^\mu\{f(z)\} := \begin{cases} \dfrac{1}{\Gamma(-\mu)}\displaystyle\int_0^z (z-t)^{-\mu-1}\, f(t)\,dt & \left(\Re(\mu)<0\right) \\[2ex] \dfrac{d^m}{dz^m}\left\{\mathcal{D}_z^{\mu-m}\{f(z)\}\right\} & \left(m-1\leqq \Re(\mu)<m\ \ (m\in\mathbb{N})\right), \end{cases}$$

it is known that

$$\mathcal{D}_z^\mu\left\{z^\lambda\right\} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}\, z^{\lambda-\mu} \qquad \left(\Re(\lambda)>-1\right) \tag{1.3}$$

and that

$$\mathcal{D}_z^\mu\{f(z)\cdot g(z)\} = \sum_{j=0}^\infty \binom{\mu}{j}\mathcal{D}_z^{\mu-j}\{f(z)\}\cdot D_z^j\{g(z)\} \qquad (\mu\in\mathbb{C}), \tag{1.4}$$

which, in the special case when

$$\mu = m \qquad (m\in\mathbb{N}_0 := \mathbb{N}\cup\{0\};\ \mathbb{N} := \{1,2,3,\cdots\}),$$

yields the familiar Leibniz rule of calculus, D_z^j being the *ordinary* derivative operator of order $j\in\mathbb{N}_0$ with respect to z.

By *correctly* applying these last properties (1.3) and (1.4), it is fairly straightforward to observe that the first-kind Jacobi function $P_\nu^{(\alpha,\beta)}(z) \quad (\nu \in \mathbb{C})$ would satisfy the following Rodrigues formula:

$$\begin{aligned} P_\nu^{(\alpha,\beta)}(z) & \frac{(-2)^{-\nu}}{\Gamma(\nu+1)} \, (1-z)^{-\alpha} \, (1+z)^{-\beta} \\ & \cdot \mathcal{D}_z^\nu \left\{ (1-z)^{\alpha+\nu} \, (1+z)^{\beta+\nu} \right\} \qquad (\nu \in \mathbb{C}), \end{aligned} \tag{1.5}$$

only in the case of the *classical* Jacobi polynomials $P_n^{(\alpha,\beta)}(z) \quad (n \in \mathbb{N}_0)$, that is, *only* when

$$\nu = n \qquad (n \in \mathbb{N}_0).$$

The obviously erroneous formula (1.5) (*with* $\nu \in \mathbb{C}$) was interpreted as the definition of the so-called *fractional Jacobi function* in a recent seemingly invalid rederivation of some of the familiar properties of the well-known (rather *classical*) Jacobi function $P_\nu^{(\alpha,\beta)}(z) \quad (\nu \in \mathbb{C})$ by Gogovcheva and Boyadjiev (cf. [2, p. 433, Definition 2]; see also another similar work by Mirevski *et al.* [9]).

In our present investigation, we aim at deriving several bounding inequalities for the Jacobi function $|P_\nu^{(\alpha,\beta)}(z)| \quad (\nu \in \mathbb{C})$, which is defined by (1) above. Our method is based largely upon some results derived in a recent work by Pogány and Srivastava [11].

2. A set of lemmas and other preliminaries

For the classical Laguerre function $L_\nu^{(\mu)}(z) \quad (\nu \in \mathbb{C})$ defined, in terms of the confluent hypergeometric function ${}_1F_1$, by

$$\begin{aligned} L_\nu^{(\mu)}(z) &:= \sum_{k=0}^{\infty} \binom{\mu+\nu}{\nu-k} \frac{(-z)^k}{k!} \\ &= \binom{\mu+\nu}{\nu} \, {}_1F_1\big(-\nu; \mu+1; \, z\big) \qquad (\nu \in \mathbb{C}), \end{aligned} \tag{2.1}$$

a bounding inequality (asserted by Lemma 2.1 below) was proven by Eric Russell Love (1912–2001) [6] by making use of the following well-known integral representation:

$$L_\nu^{(\mu)}(x) = \frac{e^x \, x^{-\frac{\mu}{2}}}{\Gamma\big(\nu+1\big)} \int_0^\infty e^{-t} \, t^{\nu+\frac{\mu}{2}} J_\mu\big(2\sqrt{xt}\big) \, dt \tag{2.2}$$

$$\big(x \geqq 0; \; \Re(\mu+\nu) > -1\big)$$

involving the first-kind Bessel function $J_\nu(z)$ of order ν, defined by

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \, \Gamma(\nu+n+1)} \, \left(\frac{x}{2}\right)^{\nu+2n} \tag{2.3}$$

$$\big(z \in \mathbb{C} \setminus (-\infty, 0); \; \nu \in \mathbb{C}\big).$$

Lemma 2.1. *The following bounding inequality holds true for the Laguerre function* $L_\nu^{(\mu)}(x)$*:*

$$\left|L_\nu^{(\mu)}(x)\right| \leqq \frac{\Gamma\big(\Re(\mu+\nu)+1\big)\,\Gamma\left(\Re(\mu)+\frac{1}{2}\right)}{\left|\Gamma\big(\nu+1\big)\right|\Gamma\big(\Re(\mu)+1\big)\,\left|\Gamma\left(\mu+\frac{1}{2}\right)\right|}\,e^x \tag{2.4}$$

$$\left(x>0;\ \Re(\mu)>-\frac{1}{2};\ \Re(\mu+\nu)>-1\right)$$

Recently, Pogány and Srivastava [11] applied some inequalities due to Yudell Leo Luke (1918–1983) [7], Landau [5], Olenko [10] and Krasikov [4] with a view to presenting several remarkable improvements over Love's inequality (2.2). We recall here the bounding inequalities of Pogány and Srivastava [11] in the form of the following lemmas.

Lemma 2.2. *The following inequality holds true for the Laguerre function* $L_\nu^{(\mu)}(x)$*:*

$$\left|L_\nu^{(\mu)}(x)\right| \leqq \frac{(\mu+\nu)e^x}{\nu(\mu-|\nu|)B(\mu,\nu)\,(1+x)}$$

$$\big(x\geqq 0;\ \mu>|\nu|;\ \nu>-1\big),$$

where $B(\alpha,\beta)$ *denotes the familiar Beta function defined already by* (1.2)*.*

Lemma 2.3. *The following bounding inequality holds true for the Laguerre function* $L_\nu^{(\mu)}(x)$*:*

$$\left|L_\nu^{(\mu)}(x)\right| \leqq m_\nu^\mu(x)\,\frac{e^x\,x^{-\frac{\mu}{2}}}{\Gamma(\nu+1)}$$

$$\left(x>0;\ \mu>0;\ \nu>-1;\ \mu+2\nu>-\frac{3}{2}\right),$$

where

$$m_\nu^\mu(x) := \min_{x,\mu,\nu}\left\{\frac{b_L\Gamma\left(\frac{\mu}{2}+\nu+1\right)}{\mu^{\frac{1}{3}}},\ \frac{c_L\Gamma\left(\frac{\mu}{2}+\nu+\frac{5}{6}\right)}{\sqrt[3]{2}\,x^{\frac{1}{6}}},\ \frac{d_O\Gamma\left(\frac{\mu}{2}+\nu+\frac{3}{4}\right)}{\sqrt{2}\,x^{\frac{1}{4}}}\right\},$$

and the coefficients b_L, c_L *and* d_O *are given, respectively, by*

$$b_L := \sqrt[3]{2}\sup_{x\in\mathbb{R}^+}\{\mathrm{Ai}(x)\}, \tag{2.5}$$

$$c_L := \sup_{x\in\mathbb{R}^+}\left\{x^{\frac{1}{3}}J_0(x)\right\} \tag{2.6}$$

and

$$d_O := b_L\sqrt{\mu^{\frac{1}{3}}+\frac{\alpha_1}{\mu^{\frac{1}{3}}}+\frac{3\alpha_1^2}{10\mu}} \qquad (\mu>0) \tag{2.7}$$

in terms of the Bessel function $J_\nu(z)$ *defined by* (2.3) *and the familiar Airy function* $\mathrm{Ai}(x)$ *defined by*

$$Ai(x) := \frac{\pi}{3}\sqrt{\frac{x}{3}}\left[J_{-\frac{1}{3}}\left(2\left(\frac{x}{3}\right)^{\frac{3}{2}}\right) + J_{\frac{1}{3}}\left(2\left(\frac{x}{3}\right)^{\frac{3}{2}}\right)\right].$$

Lemma 2.4. *The following bounding inequality holds true for the Laguerre function* $L_\nu^{(\mu)}(x)$ *when* $x>0$, $\mu>0$ *and* $r\in(0,2)$*:*

$$\begin{aligned}\left|L_\nu^{(\mu)}(x)\right| \leqq \frac{\sqrt{\Gamma\left(\mu+2\nu+\frac{1}{2}\right)e^x\, x^{-\frac{\mu}{2}}}}{\Gamma(\nu+1)\,(2-r)^{\frac{\mu}{2}+\nu+\frac{1}{4}}} &\cdot \left(\frac{d_O^2}{2\sqrt{x}}\left[1-\exp\left(-\frac{r}{16x}\left[\lambda+(\lambda+1)^{\frac{2}{3}}\right]\right)\right]\right. \\ &\left.+\frac{4\mathcal{K}_\mu}{\pi\, r^{\frac{3}{2}}}\,\Gamma\left(\frac{3}{2},\frac{r}{16x}\left[\lambda+(\lambda+1)^{\frac{2}{3}}\right]\right)\right)^{\frac{1}{2}},\end{aligned} \tag{2.8}$$

where

$$\lambda := (2\mu+1)(2\mu+3) \qquad \textit{and} \qquad \mathcal{K}_\mu := \left[(2\mu+1)(2\mu+3)+1\right]^{\frac{2}{3}} - 2(2\mu+1),$$

$\Gamma(z,\kappa)$ *being the incomplete Gamma function of the second kind, defined by*

$$\Gamma(z,\kappa) := \int_\kappa^\infty t^{z-1}\, e^{-t}\, dt \qquad \big(\Re(z)>0;\ \kappa\in\mathbb{C}\big). \tag{2.9}$$

3. Bounding inequalities for the Jacobi function

First of all, in light the hypergeometric representations in (1.1) and (2.1), we find from the Eulerian integral [cf. Equation (2.6)]:

$$\Gamma(z,0)\equiv\Gamma(z) := \int_0^\infty t^{z-1}\, e^{-t}\, dt \qquad \big(\Re(z)>0\big)$$

that

$$\Gamma(\alpha+\beta+\nu+1)P_\nu^{(\alpha,\beta)}(z) = \int_0^\infty t^{\alpha+\beta+\nu}\, e^{-t}\, L_\nu^{(\alpha)}\left(\frac{1}{2}(1-z)t\right)dt \tag{3.1}$$

$$\big(\Re(\alpha+\beta+\nu)>-1\big),$$

which, in the special case when

$$\nu = n \qquad (n\in\mathbb{N}_0),$$

happens to be a well-known result (cf., e.g., [13, p. 94, Problem 24]).

By writing (3.1) in the following (*relatively simpler*) form:

$$\left|P_\nu^{(\alpha,\beta)}(1-2x)\right| \leqq \frac{1}{\Gamma(\alpha+\beta+\nu+1)}\int_0^\infty t^{\alpha+\beta+\nu}\, e^{-t}\, \left|L_\nu^{(\alpha)}(xt)\right| dt$$

$$(\alpha,\beta,\nu\in\mathbb{R};\ \alpha+\beta+\nu>-1),$$

and then appealing to the corresponding version of Love's inequality (2.4), we obtain Theorem 3.1 below.

Theorem 3.1. *The following bounding inequality holds true for the classical Jacobi function*:

$$\left|P_\nu^{(\alpha,\beta)}(1-2x)\right| \leqq \binom{\nu+\alpha}{\nu}(1-x)^{-\alpha-\beta-\nu-1}$$

$$(0<x<1;\ \alpha>-1;\ \nu+\alpha>-1;\ \alpha+\beta+\nu>-1)\,.$$

In a similar manner, we can apply Lemmas 2.2 and 2.3 with a view to deducing the results asserted by Theorems 3.2 and 3.3 below. In particular, in our proof of Theorem 3.2, we make use *also* of the following known result [1, p. 137, Entry 4.3(7)]:

$$\int_0^\infty t^\sigma\ (t+\kappa)^{-1}\ e^{-st}dt = \kappa^\sigma\ e^{\kappa s}\ \Gamma(\sigma+1)\Gamma(-\sigma,\kappa s)$$

$$\big(\Re(s)>0;\ \Re(\sigma)>-1;\ |\arg(\kappa)|<\pi\big),$$

where $\Gamma(z,\kappa)$ is the incomplete Gamma function of the second kind defined by (2.9). The details involved are being left as an exercise for the interested reader.

Theorem 3.2. *The following bounding inequality holds true for the classical Jacobi function*:

$$\begin{aligned}\left|P_\nu^{(\alpha,\beta)}(1-2x)\right| \leqq \frac{\alpha+\nu}{\nu(\alpha-|\nu|)B(\alpha,\nu)}\ & x^{-\alpha-\beta-\nu-1}\ e^{(1-x)/x}\\ &\cdot\Gamma\left(-\alpha-\beta-\nu,\frac{1-x}{x}\right)\end{aligned}$$

$$(0<x<1;\ \alpha>|\nu|;\ \nu>-1;\ \alpha+\beta+\nu>-1)\,.$$

Theorem 3.3. *The following bounding inequality holds true for the classical Jacobi function*:

$$\left|P_\nu^{(\alpha,\beta)}(1-2x)\right| \leqq \mathcal{M}_\nu^{\alpha,\beta}(x)\ \frac{x^{-\frac{\alpha}{2}}\ (1-x)^{-\frac{\alpha}{2}-\beta-\nu-1}}{\Gamma(\nu+1)\Gamma(\alpha+\beta+\nu+1)}$$

$$\bigg(0<x<1;\ \alpha>0;\ \alpha+2\nu>-\frac{3}{2};\qquad \alpha+2(\beta+\nu)>-\frac{3}{2};\ \min\{\nu,\alpha+\beta+\nu\}>-1\bigg),$$

where

$$\mathcal{M}_{\nu}^{\alpha,\beta}(x) := \min_{x,\alpha,\beta,\nu} \left\{ \frac{b_L}{\alpha^{\frac{1}{3}}}\, \Gamma\left(\frac{\alpha}{2}+\nu+1\right)\Gamma\left(\frac{\alpha}{2}+\beta+\nu+1\right), \right.$$
$$\frac{c_L}{\sqrt[3]{2}}\, \Gamma\left(\frac{\alpha}{2}+\nu+\frac{5}{6}\right)\Gamma\left(\frac{\alpha}{2}+\beta+\nu+\frac{5}{6}\right)\left(\frac{x}{1-x}\right)^{-\frac{1}{6}},$$
$$\left. \frac{d_O}{\sqrt{2}}\, \Gamma\left(\frac{\alpha}{2}+\nu+\frac{3}{4}\right)\Gamma\left(\frac{\alpha}{2}+\beta+\nu+\frac{3}{4}\right)\left(\frac{x}{1-x}\right)^{-\frac{1}{4}} \right\},$$

the coefficients b_L, c_L *and* d_O *being given by* (2.5), (2.6) *and* (2.7), *respectively.*

4. Concluding remarks and observations

In this concluding section, we present several brief remarks and observations concerning the methodology and techniques which are used here and elsewhere for finding bounding inequalities for a considerably large variety of special functions and polynomials.

Remark 4.1. Our method of proof of Theorems 3.1, 3.2 and 3.3 above, which is based heavily upon the integral representation (3.1) for the Jacobi function, does not seem to apply easily to the bounding inequality (2.8) asserted by Lemma 2.4.

Remark 4.2. The matrix methods (described and applied, among others, by Rassias and Srivastava [12]) require the use of a three-term recurrence relation which is satisfied by a fairly large family of special functions including (for example) such classical orthogonal polynomials as the Jacobi polynomials and their many relatives. Consequently, in the absence of an appropriate three-term recurrence relation, it does not seem to be possible to apply these matrix methods to the classical Jacobi function $P_{\nu}^{(\alpha,\beta)}(z)$ $(\nu \in \mathbb{C})$, the classical Laguerre function $L_{\nu}^{(\mu)}(z)$ $(\nu \in \mathbb{C})$, and so on.

Remark 4.3. The bounding inequalities for the Jacobi function, which are presented in the preceding section, are consequences of several potentially useful and reasonably sharp inequalities for the classical Laguerre functions (see Section 2).

Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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