Banach J. Math. Anal. 1 (2007), no. 1, 125–130

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

SUPERSTABILITY OF MULTIPLIERS AND RING DERIVATIONS ON BANACH ALGEBRAS

TAKESHI MIURA[1*], HIROKAZU OKA[2], GO HIRASAWA[3]
AND SIN-EI TAKAHASI[4]

This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by P. K. Sahoo

Abstract. In this paper, we will consider Hyers–Ulam–Rassias stability of multipliers and ring derivations between Banach algebras. As a corollary, we will prove superstability of ring derivations and multipliers. That is, approximate multipliers and approximate ring derivations are exact multipliers and ring derivations.

1. Introduction and results

It seems that the stability problem of functional equations had been first raised by S.M. Ulam (cf. [16, Chapter VI]). "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism? (An ε-automorphism of G means a transformation f of G into itself such that $\rho(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon$ for all $x, y \in G$.)"

D.H. Hyers [5, Theorem 1 and Corollary] gave an answer in the affirmative to the problem as follows.

Theorem 1.1. *Suppose that E_1 and E_2 are two real Banach spaces and $f\colon E_1 \to E_2$ is a mapping. If there exists $\varepsilon \geq 0$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

Date: Received: 28 May 2007; Accepted: 16 October 2007.
* Corresponding author.
2000 *Mathematics Subject Classification.* 46J10.
Key words and phrases. Hyers–Ulam–Rassias stability, multipliers, ring derivations.

for all $x, y \in E_1$, then the limit

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for each $x \in E_1$, and $T\colon E_1 \to E_2$ is the unique additive mapping such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for every $x \in E_1$. If, in addition, the mapping $\mathbb{R} \ni t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then T is linear.

This result is called the *Hyers–Ulam stability* of the *additive* Cauchy equation $g(x + y) = g(x) + g(y)$. Here we note that Hyers calls any solution of this equation a "linear" function or transformation. Hyers considered only *bounded* Cauchy difference $f(x+y) - f(x) - f(y)$. T. Aoki [1] introduced unbounded one and generalized a result [5, Theorem 1] of Hyers obtaining the stability of additive mapping. Th.M. Rassias [11], who independently introduced the unbounded Cauchy difference, was the first to prove the stability of the linear mapping between Banach spaces. The concept of the Hyers–Ulam–Rassias stability was originated from Rassias' paper [11] for the stability of the linear mapping and its importance in the proof of further results in functional equations. Th.M. Rassias [11] generalized Hyers' Theorem as follows:

Theorem 1.2. *Suppose E_1 and E_2 are two real Banach spaces and $f\colon E_1 \to E_2$ is a mapping. If there exist $\varepsilon \ge 0$ and $0 \le p < 1$ such that*

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

for every $x, y \in E_1$, then there is a unique additive mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \le \frac{2\varepsilon}{|2 - 2^p|}\,\|x\|^p$$

for every $x \in E_1$. If, in addition, the mapping $\mathbb{R} \ni t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then T is linear.

This result is what is called, the Hyers–Ulam–Rassias stability of the linear mapping. The result of Hyers is just the case of Rassias' Theorem when $p = 0$. Thus, the Theorem of Th.M. Rassias is a generalization to the case where 0 is less than or equal to p strictly less than 1. It should be mentioned that Rassias' Theorem for the stability of the linear mapping allows the Cauchy difference to be unbounded for all those values of p as well. During the 27th International Symposium on Functional Equations, Th.M. Rassias raised the problem whether a similar result holds for $1 \le p$. Z. Gajda [4, Theorem 2] proved that Theorem 1.2 is valid for $1 < p$. In the same paper [4, Example], he also gave an example that a similar result to the above does not hold for $p = 1$. Later, Th.M. Rassias and P. Šemrl [12, Theorem 2] gave another counter example for $p = 1$.

Note that if $p < 0$, then $\|0\|^p$ is obviously meaningless. However, if we assume that $\|0\|^p$ means ∞, then the proof given in [11] also works for $x \neq 0$. Moreover, with minor changes in the proof, we see that the result is also valid for $p < 0$. Thus, the Hyers–Ulam–Rassias stability of the additive Cauchy equation holds for all $p \in \mathbb{R} \setminus \{1\}$.

D.G. Bourgin [3] proved *superstability* of ring homomorphisms. Suppose that A and B are Banach algebras and suppose that B is with unit. If $f\colon A \to B$ is a surjective mapping such that

$$\|f(a+b) - f(a) - f(b)\| \leq \varepsilon \qquad (a, b \in A)$$

$$\|f(ab) - f(a)f(b)\| \leq \delta \qquad (a, b \in A)$$

for some $\varepsilon \geq 0$ and $\delta \geq 0$, then f is a ring homomorphism, that is,

$$f(a+b) = f(a) + f(b) \quad \text{and} \quad f(ab) = f(a)f(b)$$

for all $a, b \in A$. J.A. Baker [2] proved that if f is a mapping from a semigroup S into $\mathbb{C}$, the complex number field, satisfying $|f(a+b) - f(a)f(b)| \leq \varepsilon$ $(a, b \in S)$ for some $\varepsilon \geq 0$, then either $|f(a)| \leq (1 + \sqrt{1+4\varepsilon})/2$ for all $a \in S$ or f is multiplicative, that is, $f(a+b) = f(a)f(b)$ for all $a, b \in S$.

Let B be a Banach algebra. We say that a mapping $T\colon B \to B$ is a *multiplier* if $aT(b) = T(a)b$ for each $a, b \in B$. A mapping $T\colon B \to B$ is a left (right) multiplier if $T(ab) = T(a)b$ (resp. $T(ab) = aT(b)$) for each $a, b \in B$. If an additive mapping $T\colon B \to B$ satisfies

$$T(ab) = aT(b) + T(a)b \qquad (a, b \in B),$$

then T is said to be a *ring derivation*. The first, third and fourth authors [7, 8] considered perturbation of multipliers and ring derivations, and they proved stability results in the sense of Hyers–Ulam–Rassias. Under a mild assumption, they also proved superstability of multipliers and ring derivations. That is, if f is an approximate multiplier (ring derivation), then f is an exact multiplier (resp. ring derivation).

In this paper, we will consider multipliers, left (right) multipliers and ring derivations on Banach algebra B. To unify these mappings, we consider the mapping Φ_f defined by

$$\Phi_f(a, b) = \alpha f(ab) + \beta a f(b) + \gamma f(a) b \qquad (a, b \in B), \tag{1.1}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$. Recall that left (right) annihilator algebra of B, denoted by $A_l(B)$ (resp. $A_r(B)$), is the set $\{a \in B : aB = 0\}$ (resp. $\{a \in B : Ba = 0\}$).

Theorem 1.3. *Let $\varepsilon \geq 0$, $p \geq 0$, $p \neq 1$ and $\alpha, \beta, \gamma \in \mathbb{C}$. Let B be a Banach algebra with $A_l(B) = \{0\}$ and $f\colon B \to B$ a mapping satisfying*

$$\|f(a+b) - f(a) - f(b)\| \leq \varepsilon(\|a\|^p + \|b\|^p) \qquad (a, b \in \qquad (a, b \in B), \tag{1.2}$$

$$\|\Phi_f(a, b)\| \leq \varepsilon \|a\|^p \, \|b\|^p \qquad (a, b \in B). \tag{1.3}$$

where $\Phi_f(a, b)$ is defined by (1.1). *If $\gamma \neq 0$, then $\Phi_f(a, b) = 0$ for every $a, b \in B$.*

Corollary 1.4. *Let $\varepsilon \geq 0$ and $p \geq 0$, $p \neq 1$. Let B be a Banach algebra with $A_l(B) = \{0\}$, and $f\colon B \to B$ a mapping satisfying* (1.2).

(a) *If $\|f(ab) - f(a)b\| \leq \varepsilon \|a\|^p \, \|b\|^p$ for each $a, b \in B$, then f is a left multiplier.*
(b) *If $\|af(b) - f(a)b\| \leq \varepsilon \|a\|^p \, \|b\|^p$ for each $a, b \in B$, then f is a multiplier.*
(c) *If $\|f(ab) - af(b) - f(a)b\| \leq \varepsilon \|a\|^p \, \|b\|^p$ for each $a, b \in B$, then f is a ring derivation.*

2. Proofs of results

Lemma 2.1. *Let $\varepsilon \geq 0$ and $p \geq 0$, $p \neq 1$. Let B be a Banach algebra and $f\colon B \to B$ a mapping satisfying (1.2) and (1.3). Then there exists a unique additive mapping $T\colon B \to B$ such that*

$$\|f(a) - T(a)\| \leq \frac{2\varepsilon}{|2-2^p|}\,\|a\|^p \qquad (a \in B) \tag{2.1}$$

and that

$$\Phi_T(a,b) = 0 \qquad (a,b \in B). \tag{2.2}$$

Proof. Suppose that $p \neq 1$. By (1.2), it follows from Theorem 1.2 (see [1, 4, 5, 11]) that there exists a unique additive mapping $T\colon B \to B$ such that (2.1) holds for every $a \in B$. So, we need to prove (2.2). Set $s = (1-p)/|1-p|$. Then $s = \pm 1$. Take $a, b \in B$ arbitrarily. Since T is additive, we see that $T(a) = n^{-s}T(n^s a)$ for each $n \in \mathbb{N}$, the set of all natural numbers. Now it follows from (2.1) that

$$\begin{aligned}\|n^{-s}f(n^s a) - T(a)\| &= n^{-s}\,\|f(n^s a) - T(n^s a)\| \\ &\leq n^{-s}\frac{2\varepsilon}{|2-2^p|}\,\|n^s a\|^p = n^{s(p-1)}\,\frac{2\varepsilon}{|2-2^p|}\,\|a\|^p\end{aligned}$$

for all $n \in \mathbb{N}$. Since $s(p-1) < 0$, we have

$$\|n^{-s}f(n^s a) - T(a)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.3}$$

A quite similar argument to the above shows that

$$\|n^{-2s}f(n^{2s}ab) - T(ab)\| \leq n^{2s(p-1)}\,\frac{2\varepsilon}{|2-2^p|}\,\|ab\|^p$$

for all $n \in \mathbb{N}$, and hence

$$\|n^{-2s}f(n^{2s}ab) - T(ab)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.4}$$

By (1.3), we get, for each $n \in \mathbb{N}$,

$$\|\Phi_f(n^s a, n^s b)\| \leq \varepsilon\|n^s a\|^p\,\|n^s b\|^p = n^{2sp}\varepsilon\|a\|^p\,\|b\|^p$$

Since $s(p-1) < 0$, we have

$$n^{-2s}\|\Phi_f(n^s a, n^s b)\| \leq n^{2s(p-1)}\varepsilon\|a\|^p\|b\|^p \to 0 \quad \text{as} \quad n \to \infty. \tag{2.5}$$

Now we are ready to prove $\Phi_T(a,b) = 0$. By the triangle inequality, we have

$$\begin{aligned}\|\Phi_T(a,b)\| &= \|\alpha T(ab) + \beta a T(b) + \gamma T(a) b\| \\ &\leq |\alpha|\,\|T(ab) - n^{-2s}f(n^{2s}ab)\| \\ &\quad + \|\alpha n^{-2s}f(n^{2s}ab) + \beta a T(b) + \gamma T(a) b\|.\end{aligned} \tag{2.6}$$

By (2.4), the first term of the right hand side tends to 0 as $n \to \infty$. Another application of the triangle inequality to the second term of the right hand side shows

$$\begin{aligned}\|\alpha n^{-2s} f(n^{2s}ab) + \beta aT(b) + \gamma T(a)b\| \\ \le \|\alpha n^{-2s} f(n^{2s}ab) + \beta n^{-s} af(n^s b) + \gamma f(n^s a)n^{-s}b\| \\ + \|\beta aT(b) - \beta n^{-s} af(n^s b)\| + \|\gamma T(a)b - \gamma f(n^s a)n^{-s}b\| \\ \le n^{-2s}\|\Phi_f(n^s a, n^s b)\| + |\beta|\,\|a\|\,\|T(b) - n^{-s} f(n^s b)\| \\ + |\gamma|\,\|T(a) - n^{-s} f(n^s a)\|\,\|b\|.\end{aligned}$$

By (2.3) and (2.5), we thus obtain

$$\|\alpha n^{-2s} f(n^{2s}ab) + \beta aT(b) + \gamma T(a)b\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, by (2.6), we have $\|\Phi_T(a, b)\| = 0$, and so $\Phi_T(a, b) = 0$. □

Proof of Theorem 1.3. By Lemma 2.1, there exists a unique additive mapping T satisfying (2.1) and (2.2). Suppose $A_l(B) = \{0\}$. Take $a, x \in B$ and $n \in \mathbb{N}$ arbitrarily. Set $s = (1 - p)/|1 - p|$. By (1.3), we have

$$\|\alpha f(an^s x) + \beta af(n^s x) + \gamma f(a)n^s x\| \le \varepsilon\|a\|^p\,\|n^s x\|^p,$$

and so

$$\begin{aligned}\|\alpha n^{-s} f(n^s(ax)) + \beta an^{-s} f(n^s x) + \gamma f(a)x\| \\ \le \varepsilon n^{-s}\|a\|^p\,\|n^s x\|^p = \varepsilon n^{s(p-1)}\|a\|^p\,\|x\|^p.\end{aligned}$$

Since $s(p - 1) < 0$, it follows from (2.3) that

$$\alpha T(ax) + \beta aT(x) + \gamma f(a)x = 0. \tag{2.7}$$

Here we notice, by (2.2), that

$$\alpha T(ax) + \beta aT(x) + \gamma T(a)x = 0. \tag{2.8}$$

Subtracting (2.8) from (2.7), we obtain $\gamma\{f(a) - T(a)\}x = 0$. Since $\gamma \ne 0$, we have

$$\{f(a) - T(a)\}B = 0.$$

Since $A_l(B) = \{0\}$, we thus conclude that $f(a) = T(a)$ for each $a \in B$. By (2.2), we have $\Phi_f(a, b) = \Phi_T(a, b) = 0$ for each $a, b \in B$. □

Proof of Corollary 1.4. This is a direct corollary to Theorem 1.3.

(a) Take $\alpha = 1$, $\beta = 0$, $\gamma = -1$.
(b) Take $\alpha = 0$, $\beta = 1$, $\gamma = -1$.
(c) Take $\alpha = 1$, $\beta = \gamma = -1$.

By Theorem 1.3, each approximate mapping is an exact one. This completes the proof. □

Remark 2.2. Let B be a Banach algebra without order. The first, third and fourth authors [7, Theorem 1.1] proved that if $f\colon B \to B$ satisfies $\|af(b) - f(a)b\| \le \varepsilon\|a\|^p\,\|b\|^p$ $(a, b \in B)$, then f is a multiplier.

References

1. T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.
2. J.A. Baker, *The stability of the cosine equation*, Proc. Amer. Math. Soc. **80** (1980), 411–416.
3. D.G. Bourgin, *Approximately isometric and multiplicative transforms on continuous function rings*, Duke Math. J. **16** (1949), 385–397.
4. Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci., **14** (1991), 431-434.
5. D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222-224.
6. D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser Boston, Basel, Berlin, 1998.
7. T. Miura, G.Hirasawa and S.-E. Takahasi, *Stability of multipliers on Banach algebras*, Int. J. Math. Math. Sci., **45** (2004), 2377–2381.
8. T. Miura, G. Hirasawa and S.-E. Takahasi, *A perturbation of ring derivations on Banach algebras*, J. Math. Anal. Appl., **319** (2006), 522–530.
9. C.-G. Park and Th.M. Rassias, *On a generalized Trif's mapping in Banach modules over a C*-algebra*, J. Korean Math. Soc. **43** (2006), 323–356.
10. A. Prastaro and Th.M. Rassias, *Ulam stability in geometry of PDE's*, Nonlinear Funct. Anal. Appl. **8** (2003), 259–278.
11. Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.
12. Th.M. Rassias and P. Šemrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989-993.
13. Th.M. Rassias, *The problem of S.M.Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.
14. Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.
15. Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
16. S.M. Ulam, *A collection of mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York-London, 1960.

[1,4] Department of Applied Mathematics and Physics, Graduate School of Science and Engineering, Yamagata University, Yonezawa 992-8510, Japan.
E-mail address: miura@yz.yamagata-u.ac.jp and sin-ei@emperor.yz.yamagata-u.ac.jp

[2,3] Faculty of Engineering, Ibaraki University, Hitachi 316-8511, Japan.
E-mail address: oka@mx.ibaraki.ac.jp and gou@mx.ibaraki.ac.jp