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# NONCOMMUTATIVE ORLICZ MODULAR SPACES ASSOCIATED WITH GROWTH FUNCTIONS

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Abstract. We study the noncommutative Orlicz modular spaces associated with growth functions. Some basic properties of such spaces, such as completeness and dominated convergence theorem, are present. Moreover, Young and Clarkson–McCarthy inequalities on these spaces proved.

## 1. Introduction

Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a normal faithful semi-finite trace $\tau$. Given $0 < p \leq \infty$ we denote by $L^p(\mathcal{M})$ the usual noncommutative $L^p$-spaces associated with $(\mathcal{M}, \tau)$. For the theory of noncommutative $L^p$-spaces, we refer the readers to [3, 4, 10, 12]. We denote the set of all $\tau$-measure operators by $L_0(\mathcal{M}, \tau)$, or simply $L_0(\mathcal{M})$. We consider the noncommutative Orlicz modular spaces $L^\Phi(\mathcal{M})$ associated with a growth functions $\Phi$ (see preliminaries for the definition). About the modular theory we refer the interested readers to [6, 9, 11, 13]. It is well-known that Orlicz spaces for convex functions are the generalization of $L_p$ spaces for $p \geq 1$. Many authors discuss Orlicz spaces for convex functions( see [6, 9, 11, 13] ). In this work we study noncommutative Orlicz spaces for the more general case of growth functions. These kinds of noncommutative Orlicz spaces are a generalization of noncommutative $L_p$ spaces for $p > 0$. In section 1, we show some results about the growth functions such

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as, mainly, characterizations of $\Delta_2$ and $\Delta_{\frac{1}{2}}$ conditions, and relationship between general growth functions and concave or convex growth functions.

T. Fack and H. Kosaki in [4] proved the dominated convergence theorems of $\tau$-measurable operators for noncommutative $L^p$-spaces. G. Sadeghi proved this theorem for noncommutative Orlicz spaces associated with convex functions in [13]. We extend this result to the noncommutative Orlicz spaces associated with growth functions. This will present in section 2. In section 3, we prove Young and Clarkson–McCarthy inequalities for noncommutative Orlicz spaces associated with growth functions.

## 2. Relationships of growth functions

**Definition 2.1.** We say a function $\Phi$ is a growth function, if $\Phi$ is a continuous and nondecreasing function from $[0,\infty)$ onto itself.

In this paper we always assume that for a growth function $\Phi$, $t\Phi^{'}(t)$ is also a growth function. There are many simple examples of growth functions $\Phi$ such that $t\Phi^{'}(t)$ are also growth functions. For example $\Phi(t) = t^p$, for every $p > 0$. There are also some growth functions $\Phi$ such that $t\Phi^{'}(t)$ are not growth functions, such as $\Phi(t) = \ln(1+t)$.

For a growth function $\Phi$, set $a = \sup\{t : \Phi(t) = 0\}$. Then $a < \infty$ and $\Phi(t) = 0$ for all $t \in [0,a]$. Hence we may assume that $\Phi(t) > 0$ for all $t > 0$ (otherwise replace $\Phi$ by $\Phi(a+\cdot)$). For a growth function $\Phi$, we have the following quantitative indices:

$$p_\Phi = \inf_{t>0} \frac{t\Phi^{'}(t)}{\Phi(t)}, \quad q_\Phi = \sup_{t>0} \frac{t\Phi^{'}(t)}{\Phi(t)}.$$

**Definition 2.2.** (i) A growth function $\Phi$ obeys $\Delta_2$-condition for all $t > 0$, often written as $\Phi \in \Delta_2$, if there is a constant $K > 1$ such that $\Phi(2t) \leq K\Phi(t)$ .

(ii) A growth function $\Phi$ said to satisfy the $\Delta_{\frac{1}{2}}$-condition for all $t > 0$ , denoted symbolically as $\Phi \in \Delta_{\frac{1}{2}}$, if there is a constant $0 < K < 1$ such that $\Phi(\frac{t}{2}) \leq K\Phi(t)$.

By Proposition 1.4 in [1], we have that

$$\Phi \in \Delta_2 \Leftrightarrow q_\Phi < \infty, \quad \Phi \in \Delta_{\frac{1}{2}} \Leftrightarrow p_\Phi > 0. \tag{2.1}$$

If $\Phi$ satisfies the $\Delta_2$ and $\Delta_{\frac{1}{2}}$ conditions for all $t > 0$ , we denote symbolically as $\Phi \in \Delta_2 \cap \Delta_{\frac{1}{2}}$.

**Definition 2.3.** Growth functions $\Phi_1$, $\Phi_2$ are said to be equivalent, denoted by $\Phi_1 \sim \Phi_1$, if there exist positive constants $C_1, C_2, C_3, C_4$ such that

$$C_1\Phi_1(C_2t) \leq \Phi_2(t) \leq C_3\Phi_1(C_4t), \quad \forall\, t > 0.$$

In [1], we proved that if $\Phi_1 \sim \Phi_1$, then $\Phi_1 \in \Delta_2 \Leftrightarrow \Phi_2 \in \Delta_2$ and $\Phi_1 \in \Delta_{\frac{1}{2}} \Leftrightarrow \Phi_2 \in \Delta_{\frac{1}{2}}$ (see Proposition 1.4).

**Proposition 2.4.** *Let $\Phi$ be a growth function. Then the followings hold.*

(i) *For every* $p \in [0, p_\Phi]$, $t^{-p}\Phi(t)$ *is non-decreasing function and for every* $q \in [q_\Phi, \infty)$, $t^{-q}\Phi(t)$ *is non-increasing function.*
(ii) $\Phi \in \Delta_2$ *is equivalent to that for a constant* $C > 1$ *there is a constant* $K > 1$ *such that* $\Phi(Ct) \leq K\Phi(t)$ *for all* $t > 0$.
(iii) $\Phi \in \Delta_{\frac{1}{2}}$ *is equivalent to that for a constant* $0 < C < 1$ *there is a constant* $0 < K < 1$ *such that* $\Phi(Ct) \leq K\Phi(t)$ *for all* $t > 0$.

*Proof.* (i) For every $p \in (0, p_\Phi]$, $t^{-p}\Phi(t)$ is non-decreasing function. Indeed,

$$(t^{-p}\Phi(t))^{'} = t^{-p-1}(t\Phi^{'}(t) - p\Phi(t)) \geq t^{-p-1}(p_\Phi\Phi(t) - p\Phi(t)) \geq 0.$$

Similarly, we get $t^{-q}\Phi(t)$ is non-increasing for every $q \in [q_\Phi, \infty)$.

(ii) Let $\Phi \in \Delta_2$. If $C > 1$, then by (2.1) and (i),

$$\Phi(Ct) \leq C^{q_\Phi}\Phi(t), \quad \forall\, t > 0,$$

i.e., the desired result holds. Conversely, if there are constants $C > 1$ and $K > 1$ such that $\Phi(Ct) \leq K\Phi(t)$ for all $t > 0$, then there is an integer $n \geq 0$ such that $C^n < 2 \leq C^{n+1}$. It follows that $\Phi(2t) \leq \Phi(C^{n+1}t) \leq K^{n+1}\Phi(t)$ and $K^{n+1} > 1$, i.e., $\Phi \in \Delta_2$.

(iii) Let $\Phi \in \Delta_{\frac{1}{2}}$. If $0 < C < 1$, then by (2.1) and (i), we get

$$\Phi(Ct) \leq C^{p_\Phi}\Phi(t), \quad \forall t > 0.$$

This implies the desired result. Conversely, let $0 < C < 1$ and $0 < K < 1$ such that $\Phi(Ct) \leq K\Phi(t)$ for all $t > 0$. Then

$$\frac{1-C}{C}t\Phi'(t) \geq \int_{ct}^{t} \frac{s\Phi^{'}(s)}{s}ds = \Phi(t) - \Phi(ct) \geq (1-K)\Phi(t).$$

So $p_\Phi \geq \frac{C}{1-C}(1-K) > 0$. By (2.1), $\Phi \in \Delta_{\frac{1}{2}}$. □

We let $\Phi^{(\alpha)}(t) = \Phi(t^\alpha)$, where $\alpha$ is a positive real number. Then it is easy to see that $p_{\Phi^{(\alpha)}} = \alpha p_\Phi$, $q_{\Phi^{(\alpha)}} = \alpha q_\Phi$.

**Theorem 2.5.** *Let* $\Phi$ *be a growth function and* $\Phi \in \Delta_{\frac{1}{2}} \cap \Delta_2$. *Set* $\Phi_0(t) = t\Phi^{'}(t)$.

(i) *If* $q_{\Phi_0}\alpha \leq 1$, *then* $\Phi^{(\alpha)}$ *is a concave growth function.*
(ii) *If* $p_{\Phi_0}\alpha \geq 1$, *then* $\Phi^{(\alpha)}$ *is a convex growth function.*

*Proof.* (i) Set $a = \sup\{t : \ \Phi'(t) = 0\}$. Then $a < \infty$ and $\Phi(t) = 0$ for all $t \in (0, a]$. Hence we may assume that $\Phi'(t) >$ for all $t > 0$ (otherwise replace $\Phi$ by $\Phi(a+\cdot)$).

Since $\Phi_0(t)$ is nondecreasing function on $[0, \infty)$, $\Phi^{''}(t)$ exists for all $t > 0$ except a countable set of points in which we take $\Phi^{''}(t)$ as the derivative from the right. Computing we get that

$$(\Phi^{(\alpha)}(t))^{'} = \alpha t^{\alpha-1}\Phi^{'}(t^\alpha)$$

and

$$\begin{aligned}(\Phi^{(\alpha)}(t))^{''} &= \alpha(\alpha-1)t^{\alpha-2}\Phi^{'}(t^\alpha) + \alpha^2 t^{2(\alpha-1)}\Phi^{''}(t^\alpha) \\ &= \alpha^2 t^{\alpha-2}\Big(t^\alpha\Phi^{''}(t^\alpha) + (1-\frac{1}{\alpha})\Phi^{'}(t^\alpha)\Big) \\ &= \alpha^2 t^{\alpha-2}\Phi^{'}(t^\alpha)\Big(\frac{t^\alpha\Phi^{''}(t^\alpha)}{\Phi^{'}(t^\alpha)} + 1 - \frac{1}{\alpha}\Big).\end{aligned}$$

Since $p_\Phi \Phi(t) \leq t\Phi'(t) \leq q_\Phi \Phi(t)$, we get $\Phi \sim \Phi_0$. By Proposition 1.4 of [1], $0 < p_{\Phi_0} \leq q_{\Phi_0} < \infty$. Hence

$$p_{\Phi_0} \leq \frac{t(\Phi_0(t))'}{\Phi_0(t)} = \frac{t(t\Phi'(t))'}{t\Phi'(t)} \leq q_{\Phi_0}, \quad \forall\, t > 0,$$

so

$$p_{\Phi_0} - 1 \leq \frac{t\Phi''(t)}{\Phi'(t)} \leq q_{\Phi_0} - 1, \quad \forall\, t > 0.$$

It follows that

$$p_{\Phi_0} - 1 \leq \frac{t^\alpha \Phi''(t^\alpha)}{\Phi'(t^\alpha)} \leq q_{\Phi_0} - 1, \quad \forall\, t > 0. \tag{2.2}$$

Since $q_{\Phi_0}\alpha \leq 1$, by (2.2) we have that

$$(\Phi^{(\alpha)}(t))'' \leq \alpha^2 t^\alpha \Phi'(t^\alpha)(q_\Phi - \frac{1}{\alpha}) \leq 0, \quad \forall\, t > 0,$$

which implies that $\Phi^{(\alpha)}$ is a concave growth function.

(ii) It is proved in [1]. □

## 3. Noncommutative Orlicz spaces

Let $\mathcal{M}$ be a semi-finite von Neumann algebra equipped with a normal faithful semi-finite trace $\tau$. Let $L_0(\mathcal{M})$ denote the topological $*$-algebra of measurable operators with respect to $(\mathcal{M}, \tau)$. The topology of $L_0(\mathcal{M})$ is determined by the convergence in measure.

**Definition 3.1.** Let $\Phi$ be a growth function. We define the corresponding noncommutative Orlicz space on $(\mathcal{M}, \tau)$ by

$$L^\Phi(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \tau(\Phi(|\lambda x|)) < \infty \; for\ some\ \lambda > 0\}.$$

Let a growth function $\Phi$ satisfies the $\Delta_2$-condition. By Proposition 2.4, we know that $\tau(\Phi(|\lambda x|)) < \infty$ for some $\lambda > 0$ are equivalent to $\tau(\Phi(|x|)) < \infty$. So $L^\Phi(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \tau(\Phi(|x|)) < \infty\}$ for $\Phi \in \Delta_2$.

**Definition 3.2.** Let $X$ be an arbitrary vector space. A functional $\rho : X \to [0, \infty]$ is called a modular, if for arbitrary $x, y \in X$,

(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scalar $\alpha$ with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$.

We now define a functional $\rho_\Phi$ on $L_0(\mathcal{M})$ by

$$\rho_\Phi(x) = \tau(\Phi(|x|)), \quad \forall\, x \in L_0(\mathcal{M}).$$

It follows from corollary 2.8 of [4] that $\rho_\Phi(x) = \tau(\Phi(|x|)) = \int_0^\infty \Phi(\mu_t(x))dt.$

**Theorem 3.3.** *Let $\Phi$ be a growth function and $\Phi \in \Delta_{\frac{1}{2}} \cap \Delta_2$. Then $\rho_\Phi$ is a modular on $L_0(\mathcal{M})$.*

*Proof.* It is clear that $\rho_\Phi$ satisfies the (i) and (ii) of Definition 3.2. Let $x, y$ be $\tau$-measurable operators, $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. By Lemma 4.3 of [4]( also see [7]), there exist partial isometries $u, v \in \mathcal{M}$ such that

$$|\alpha x + \beta y| \leq \alpha u|x|u^* + \beta v|y|v^*.$$

Using Theorem 2.5 we find a natural number $n$ such that $\Phi^{(\frac{1}{n})}$ is concave growth function. By Proposition 4.6 of [4],

$$\begin{aligned}
\rho_\Phi(\alpha x + \beta y) &= \tau(\Phi(|\alpha x + \beta y|)) = \int_0^\infty \Phi(\mu_t(\alpha x + \beta y))dt \\
&\leq \int_0^\infty \Phi(\mu_t(\alpha u|x|u^* + \beta v|y|v^*))dt \\
&= \int_0^\infty \Phi^{(\frac{1}{n})}((\mu_t(\alpha u|x|u^* + \beta v|y|v^*))^n)dt \\
&= \int_0^\infty \Phi^{(\frac{1}{n})}(\mu_t((\alpha u|x|u^* + \beta v|y|v^*)^n))dt \\
&\leq \int_0^\infty \Phi^{(\frac{1}{n})}(\mu_t(\alpha(u|x|u^*)^n + \beta(v|y|v^*)^n))dt \\
&\leq \int_0^\infty \Phi^{(\frac{1}{n})}(\mu_t(\alpha(u|x|u^*)^n))dt + \int_0^\infty \Phi^{(\frac{1}{n})}(\mu_t(\beta(v|y|v^*)^n))dt \\
&\leq \int_0^\infty \Phi^{(\frac{1}{n})}(\alpha\mu_t(|x|^n))dt + \int_0^\infty \Phi^{(\frac{1}{n})}(\beta\mu_t(|y|^n))dt \\
&= \int_0^\infty \Phi(\alpha^{\frac{1}{n}}\mu_t(x))dt + \int_0^\infty \Phi(\beta^{\frac{1}{n}}\mu_t(y))dt \\
&\leq \int_0^\infty \Phi(\mu_t(x))dt + \int_0^\infty \Phi(\mu_t(y))dt \\
&= \tau(\Phi(|x|)) + \tau(\Phi(|y|)) \\
&= \rho_\Phi(x) + \rho_\Phi(y).
\end{aligned}$$

Thus $\rho_\Phi$ is a modular on $L_0(\mathcal{M})$. □

**Theorem 3.4.** *Let $\Phi$ be a growth function and $\Phi \in \Delta_{\frac{1}{2}} \cap \Delta_2$. Then noncommutative Orlicz space $L^\Phi(\mathcal{M})$ is $\rho_\Phi$-complete.*

*Proof.* Let $\{x_n\}$ be a $\rho_\Phi$-Cauchy sequence in $L^\Phi(\mathcal{M})$. Then

$$\rho_\Phi(x_n - x_m) \to 0 \ (n, m \to \infty)$$

From this we get that for all $t > 0$,

$$\mu_t(x_n - x_m) \to 0 \ (n, m \to \infty). \tag{3.1}$$

Indeed, if $\mu_t(x_n - x_m) \nrightarrow 0 \ (n, m \to \infty)$, then there are $\varepsilon_0 > 0$, $t_0 > 0$ and for any $k \in \mathbb{N}$ there exit $n_k, m_k > k$ such that

$$\mu_{t_0}(x_{n_k} - x_{m_k}) \geq \varepsilon_0.$$

By the right continuity of $\mu_t$, there exists $\delta > 0$ such that

$$\mu_t(x_{n_k} - x_{m+k}) \geq \frac{\varepsilon_0}{2}, \quad \forall\, t \in (t_0, t_0 + \delta).$$

By the definition of a growth function $\Phi$, we have

$$\begin{aligned} \rho_\Phi(x_{n_k} - x_{m_k}) &= \tau(\Phi(x_{n_k} - x_{m_k})) \\ &= \int_0^\infty \Phi(\mu_t(x_{n_k} - x_{m_k}))dt \\ &\geq \int_{t_0}^{t_0+\delta} \Phi(\mu_t(x_{n_k} - x_{m_k}))dt \\ &\geq \Phi(\frac{\varepsilon_0}{2})\delta > 0. \end{aligned}$$

That is contraction. Thus (3.1) holds.
By Lemma 3.1 of [4], $\{x_n\}$ is a cauchy sequence in $L_0(\mathcal{M})$. Since $L_0(\mathcal{M})$ is a complete, there exist a $\tau$-measurable operator $x \in L_0(\mathcal{M})$ such that $\{x_n\}$ convergence to $x$ in measure topology. By the Lemma 3.4 of [4] and continuity of $\Phi$, we obtain

$$\begin{aligned} \rho_\Phi(x) &= \tau(\Phi(x)) = \int_0^\infty \Phi(\mu_t(x))dt \\ &\leq \int_0^\infty \lim_{n\to\infty} \inf \Phi(\mu_t(x_n))dt \\ &\leq \lim_{n\to\infty} \inf \int_0^\infty \Phi(\mu_t(x_n))dt \\ &= \lim_{n\to\infty} \inf \rho_\Phi(x_n). \end{aligned}$$

Similarly,

$$\rho_\Phi(x - x_n) \leq \lim_{m\to\infty} \inf \rho_\Phi(x_m - x_n), \quad \forall\, n \in \mathbb{N}. \tag{3.2}$$

On the other hand, since $\Phi \in \Delta_2$,

$$\begin{aligned} \rho_\Phi(x) = \rho_\Phi(x - x_n + x_n) &\leq \rho_\Phi(2(x - x_n)) + \rho_\Phi(2x_n)) \\ &\leq K(\rho_\Phi(x - x_n) + \rho_\Phi(x_n)). \end{aligned}$$

This implies $\rho_\Phi(x) < \infty$, i.e. $x \in L^\Phi(\mathcal{M})$. By (3.2), it follows that the sequence $\{x_n\}$ is $\rho_\Phi$-convergent to $x$. □

Next, we prove the dominated convergence theorems for $\tau$-measurable operators with respect to the modular $\rho_\Phi$.

**Theorem 3.5.** *Let $\Phi$ be a growth function and $\Phi \in \Delta_{\frac{1}{2}} \cap \Delta_2$. Let $\{x_n\}$ be a sequence of $\tau$-measurable operators converging to $x$ in the measure topology. Assume that there exist $\tau$-measurable operators $y_n$, $n = 1, 2, \cdots$ and $y$ in $L^\Phi(\mathcal{M})$, satisfying the following conditions:*

(i) $|x_n| \leq |y_n|$,
(ii) $\rho_\Phi(y) = \lim_{n\to\infty} \rho_\Phi(y_n)$,
(iii) *the sequence $\{y_n\}$ converges to $y$ in the measure topology.*

*Then $x_n$ and $x$ are in $L^\Phi(\mathcal{M})$ and*

$$\lim_{n\to\infty} \rho_\Phi(x_n - x) = 0.$$

*Proof.* We use the method in the proof of Theorem 3.11 of [13]. Since $\Phi \in \Delta_2$,

$$L^\Phi(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \tau(\Phi(|x|)) < \infty\}.$$

From the condition (i) and Lemma 2.5 of [4] we know that $\mu_t(x_n) \le \mu_t(y_n)$. Therefore for the growth function $\Phi$,

$$\int_0^\infty \Phi(\mu_t(x_n))dt \le \int_0^\infty \Phi(\mu_t(y_n))dt.$$

Whence

$$\rho_\Phi(x_n) \le \rho_\Phi(y_n).$$

This implies that $x_n \in L^\Phi(\mathcal{M})$, since $y_n \in L^\Phi(\mathcal{M})$. From the proof of Theorem 3.4, we know that

$$\rho_\Phi(x) \le \lim_{n\to\infty} \inf \rho_\Phi(x_n) \le \lim_{n\to\infty} \inf \rho_\Phi(y_n) \le \rho_\Phi(y).$$

This inequality ensures that $x \in L^\Phi(\mathcal{M})$.

Since $\Phi$ is a growth function and satisfies the $\Delta_2$-condition, by the Theorem 2.5, there is a natural number $n_0$ such that $\Phi^{(\frac{1}{n_0})}$ concave growth function. Hence

$$\begin{aligned}
\Phi(\alpha+\beta) &= \Phi^{(\frac{1}{n_0})}((\alpha+\beta)^{n_0}) \le \Phi^{(\frac{1}{n_0})}(2^{n_0-1}\alpha^{n_0} + 2^{n_0-1}\beta^{n_0}) \\
&\le \Phi^{(\frac{1}{n_0})}(2^{n_0-1}\alpha^{n_0}) + \Phi^{(\frac{1}{n_0})}(2^{n_0-1}\beta^{n_0}) \\
&\le \Phi(2\alpha) + \Phi(2\beta) \\
&\le K(\Phi(\alpha) + \Phi(\beta)), \quad \forall\, \alpha, \beta \ge 0.
\end{aligned}$$

Applying this we obtain that

$$\begin{aligned}
\Phi(\mu_t(x_n - x)) &\le \Phi(\mu_{\frac{t}{2}}(x_n) + \mu_{\frac{t}{2}}(x)) \\
&\le \Phi(\mu_{\frac{t}{2}}(y_n) + \mu_{\frac{t}{2}}(x)) \\
&\le K(\Phi(\mu_{\frac{t}{2}}(y_n)) + \Phi(\mu_{\frac{t}{2}}(x))).
\end{aligned}$$

On the other hand, it follows from Lemma 3.1 and 3.4 of [4] that

$$\lim_{n\to\infty} \mu_t(x_n - x) = 0 \quad \textit{and} \quad \mu_{\frac{t}{2}}(y) \le \lim_{n\to\infty} \inf \mu_{\frac{t}{2}}(y_n).$$

Hence

$$\begin{aligned}
&\lim_{n\to\infty} \inf\{K[\Phi(\mu_{\frac{t}{2}}(y_n)) + \Phi(\mu_{\frac{t}{2}}(x))] - \Phi(\mu_t(x_n - x))\} \\
&\ge K[\Phi(\mu_{\frac{t}{2}}(y)) + \Phi(\mu_{\frac{t}{2}}(x))].
\end{aligned}$$

By usual Fatou Lemma, we have that

$$\begin{aligned}&\int_0^\infty K[\Phi(\mu_{\frac{t}{2}}(y))+\Phi(\mu_{\frac{t}{2}}(x))]dt\\ &\le \int_0^\infty \lim_{n\to\infty}\inf\{K[\Phi(\mu_{\frac{t}{2}}(y_n))+\Phi(\mu_{\frac{t}{2}}(x))]-\Phi(\mu_t(x_n-x))\}dt\\ &\le \lim_{n\to\infty}\inf\int_0^\infty\{K[\Phi(\mu_{\frac{t}{2}}(y_n))+\Phi(\mu_{\frac{t}{2}}(x))]-\Phi(\mu_t(x_n-x))\}dt.\end{aligned}$$

From the assumption (ii) and $x,\ y\in L^\Phi(\mathcal{M})$, we get

$$-\lim_{n\to\infty}\sup\int_0^\infty \Phi(\mu_t(x_n-x))dt\ge 0.$$

That is $\lim_{n\to\infty}\rho_\Phi(x_n-x)=0$. □

**Corollary 3.6.** *Let $\Phi$ be a growth function and $\Phi\in\Delta_{\frac{1}{2}}\cap\Delta_2$. Let $\{x_n\}$ be a sequence of $\tau$-measurable operators converging to $x$ in the measure topology. If there exists an operator $y\in L^\Phi(\mathcal{M})$ such that $|x_n|\le y$ for $n=1,2,\cdots$, then*

$$\lim_{n\to\infty}\rho_\Phi(x_n)=\rho_\Phi(x).$$

## 4. The Clarkson–McCarthy inequalities

In this section we extend Young and Clarkson–McCarthy inequalities on noncommutative Orlicz modular space associated with growth functions.

The classical Young inequality for two nonnegative real numbers $a$, $b$, is

$$ab\le\frac{a^p}{p}+\frac{b^q}{q},$$

where $p,\ q>1$ are such that $p^{-1}+q^{-1}=1$. We have the following result.

**Theorem 4.1.** *Let $\Phi$ be a growth function and $\Phi\in\Delta_{\frac{1}{2}}\cap\Delta_2$. Let $p,q\ge 1$ such that $p^{-1}+q^{-1}=1$. If $x\in L^{\Phi^{(p)}}(\mathcal{M})$ and $y\in L^{\Phi^{(q)}}(\mathcal{M})$, then*

$$\rho_\Phi(xy)\le\frac{1}{p}\rho_{\Phi^{(p)}}(x)+\frac{1}{q}\rho_{\Phi^{(q)}}(y).$$

*Consequently, $xy\in L^\Phi(\mathcal{M})$.*

*Proof.* By the classical Young inequality,

$$\mu_t(a)\mu_t(b)\le\frac{\mu_t(a)^p}{p}+\frac{\mu_t(b)^q}{q},\quad \forall a,b\in L_0(\mathcal{M}).\tag{4.1}$$

By Theorem 2.5, there is a $n\in\mathbb{N}$ such that $\Phi^{(n)}$ is a convex growth function. Hence $\Phi(e^t)=\Phi^{(n)}(e^{\frac{t}{n}})$ is also convex function. Then, by (iii) of Theorem 4.2 in

[4] and (4.1), we get

$$\begin{aligned}
\rho_\Phi(xy) &= \int_0^\infty \Phi(\mu_t(xy))dt \\
&\leq \int_0^\infty \Phi(\mu_t(x)\mu_t(y))dt \\
&= \int_0^\infty \Phi^{(n)}(\mu_t(|x|^{\frac{1}{n}})\mu_t(|y|^{\frac{1}{n}}))dt \\
&\leq \int_0^\infty \Phi^{(n)}(\frac{\mu_t(|x|^{\frac{1}{n}})^p}{p} + \frac{\mu_t(|y|^{\frac{1}{n}})^q}{q})dt \\
&\leq \frac{1}{p}\int_0^\infty \Phi^{(n)}(\mu_t(x)^{\frac{p}{n}})dt + \frac{1}{q}\int_0^\infty \Phi^{(n)}(\mu_t(y)^{\frac{q}{n}})dt \\
&= \frac{1}{p}\tau(\Phi^{(p)}(|x|)) + \frac{1}{q}\tau(\Phi^{(q)}(|y|)) \\
&= \frac{1}{p}\rho_{\Phi^{(p)}}(x) + \frac{1}{q}\rho_{\Phi^{(q)}}(y).
\end{aligned}$$

This is the desired result. □

The following is a generalization of Clarkson–McCarthy inequalities (see [2, 5, 8]).

**Theorem 4.2.** *Let $\Phi$ be a growth function and $\Phi \in \Delta_{\frac{1}{2}} \cap \Delta_2$. Set $\Phi_0(t) = t\Phi'(t)$.*

(i) *If $p_{\Phi_0} \geq 2$, then for $x,\ y \in L^\Phi(\mathcal{M})$, we have*

$$\rho_\Phi(x+y) + \rho_\Phi(x-y) \geq 2(\rho_\Phi(x) + \rho_\Phi(y)).$$

*Consequently,*

$$\rho_\Phi(x+y) + \rho_\Phi(x-y) \leq 2^{-1}(\rho_\Phi(2x) + \rho_\Phi(2y)).$$

(ii) *If $q_{\Phi_0} \leq 2$, then for $x,\ y \in L^\Phi(\mathcal{M})$, we have*

$$\rho_\Phi(x+y) + \rho_\Phi(x-y) \geq 2^{-1}(\rho_\Phi(2x) + \rho_\Phi(2y)).$$

*Consequently,*

$$\rho_\Phi(x+y) + \rho_\Phi(x-y) \leq 2(\rho_\Phi(x) + \rho_\Phi(y)).$$

*Proof.* (i) By the proof of Theorem 2.5, we know that $\Phi^{(\frac{1}{2})}$ is a convex growth function. We know that for a convex function,

$$\Phi^{(\frac{1}{2})}(\alpha) + \Phi^{(\frac{1}{2})}(\beta) \geq 2\Phi^{(\frac{1}{2})}(\frac{\alpha+\beta}{2}). \tag{4.2}$$

Using (4.2), (iii) of Theorem 4.4 and (ii) of Proposition 4.6 in [4] we obtain that

$$\begin{aligned}
\rho_\Phi(x+y)+\rho_\Phi(x-y) &= \int_0^\infty \Phi(\mu_t(x+y))dt + \int_0^\infty \Phi(\mu_t(x-y))dt \\
&= \int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(x+y)^2)dt + \int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(x-y)^2)dt \\
&\geq 2\int_0^\infty \Phi^{(\frac{1}{2})}(\frac{\mu_t(x+y)^2+\mu_t(x-y)^2}{2})dt \\
&\geq 2\int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(\frac{|x+y|^2+|x-y|^2}{2}))dt \\
&= 2\int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(\frac{(x+y)^*(x+y)+(x-y)^*(x-y)}{2}))dt \\
&= 2\int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(|x|^2+|y|^2))dt \\
&= 2\tau(\Phi^{(\frac{1}{2})}(|x|^2+|y|^2)) \\
&\geq 2(\tau(\Phi^{(\frac{1}{2})}(|x|^2))+2\tau(\Phi^{(\frac{1}{2})}(|y|^2))) \\
&= 2(\rho_\Phi(x)+\rho_\Phi(y)).
\end{aligned}$$

Replaced $x$ and $y$ by $x+y$ and $x-y$, then we obtain

$$\rho_\Phi(x+y)+\rho_\Phi(x-y)\leq 2^{-1}(\rho_\Phi(2x)+\rho_\Phi(2y)).$$

(ii) By Theorem 2.5, we know that $\Phi^{(\frac{1}{2})}$ is a concave growth function. Using (i) of Theorem 4.6 in [4] we get

$$\begin{aligned}
\rho_\Phi(x+y)+\rho_\Phi(x-y) &= \int_0^\infty \Phi(\mu_t(x+y))dt + \int_0^\infty \Phi(\mu_t(x-y))dt \\
&= \int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(x+y)^2)dt + \int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(x-y)^2)dt \\
&\geq \int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(|x+y|^2+|x-y|^2))dt \\
&= \int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(2|x|^2+2|y|^2))dt \\
&= \int_0^\infty \Phi^{(\frac{1}{2})}(\mu_t(\frac{|2x|^2+|2y|^2}{2}))dt \\
&\geq 2^{-1}[\tau(\Phi^{(\frac{1}{2})}(|2x|^2))+\tau(\Phi^{(\frac{1}{2})}(|2y|^2))] \\
&= 2^{-1}(\rho_\Phi(2x)+\rho_\Phi(2y)).
\end{aligned}$$

Replace $x$ and $y$ by $\frac{x+y}{2}$ and $\frac{x-y}{2}$, we get that

$$\tau(\Phi(|x+y|))+\tau(\Phi(|x-y|))\leq 2(\tau(\Phi(|x|))+\tau(\Phi(|y|))).$$

□

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