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UNIT NEIGHBORHOODS IN TOPOLOGICAL RINGS

F. J. GARCÍA-PACHECO* AND PABLO PINIELLA

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Abstract. The concepts of open unit ball and closed unit ball in a real or complex normed space are naturally extended to the scope of topological rings with unity. We then define a type of open (closed) sets called open (closed) unit neighborhoods of 0. We show among other things that in $\mathbb{R}$ and $\mathbb{C}$ the only non-trivial open and closed unit neighborhoods of 0 are the open unit ball and the closed unit ball, respectively.

1. Introduction and notation

The concepts of balancedness and absorbance are fundamental in the theory of real or complex Banach spaces. For instance, these two concepts play a fundamental role in longstanding open problems such as the famous Separable Quotient Problem (see [1]). Also, it is a well-known fact that for every real or complex topological vector space it is possible to find a fundamental system of balanced and absorbing neighborhoods of zero which characterizes completely the vector topology ([2], p.146). These concepts are usually defined in the scope of real or complex topological vector spaces, but they can be extended naturally to the context of topological modules over absolute valued division rings using the unit ball as done in [3]. Thus, in order to define such concepts in the scope of topological modules we first need to define and study the unit neighborhoods of 0 in topological rings, which is the aim of this paper.

Throughout this manuscript all rings will be considered to be associative and unital, unless otherwise explicitly stated.

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* Corresponding author.
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1.1. **Notation.** We will start off by introducing the proper notation used throughout this manuscript.

- If X is a topological space and A is a subset of X, then $\mathrm{int}\,(A)$, $\mathrm{cl}\,(A)$, and $\mathrm{bd}\,(A)$ will denote the topological interior, the topological closure, and the topological boundary of A, respectively.
- Let R be an absolute semi-valued ring. We denote the open ball, the closed ball and the sphere of centre $x \in R$ and radius $r > 0$ as $\mathsf{U}_R\,(x, r)$, $\mathsf{B}_R\,(x, r)$ and $\mathsf{S}_R\,(x, r)$ respectively. When $x = 0$ and $r = 1$, we will simply write U_R, B_R and S_R respectively.

1.2. **Absolute-valued rings with unity.** Recall that if R is a ring, then an absolute semi-value on R is a map $|\cdot| : R \to \mathbb{R}$ satisfying the following conditions:

(1) $|r| \geq 0$ for all $r \in R$.
(2) $|0| = 0$.
(3) $|rs| = |r|\,|s|$ for all $r, s \in R$
(4) $|r + s| \leq |r| + |s|$ for all $r, s \in R$

Whenever $|x| = 0$ implies $x = 0$ we call $|\cdot|$ an absolute value. Basic properties of absolute semi-values follow:

- Either $|1| = 1$ or $|1| = 0$. The latter implies $|\cdot| = 0$.
- If $r \in \mathcal{U}\,(R)$ and $|1| = 1$, $|r| = |r^{-1}|^{-1}$. In particular, $|r^n| = |r|^n$ for all $n \in \mathbb{Z}$.
- Absolute-valued rings are integral domains.
- If $|r| = 1$ for all $r \in R \setminus \{0\}$, then $|\cdot|$ is called the discrete absolute value on R and it induces on R the discrete metric and hence the discrete topology.
- S_R is a submonoid of the multiplicative monoid R, and in case R is a division ring, then S_R is a normal subgroup of the multiplicative group $R \setminus \{0\}$.

2. Unit neighborhoods of 0

Before introducing the concept of unit neighbourhood of 0, we will recall a couple of basic definitions on monoid theory that are necessary for our purposes.

Remark 2.1. Recall that given a (multiplicative) monoid M and a non-empty subset A of M, then

- A is called symmetric when $A = A^{-1}$, where $A^{-1} = \{a^{-1} : a \in A\}$;
- A is called idempotent when $A^2 = A$, where $A^2 = \{ab : a, b \in A\}$.

2.1. **Notion and examples of unit neighborhoods of 0.** Recall that in a topological space X,

- a subset U is called regular open provided that $U = \mathrm{int}(\mathrm{cl}(U))$;
- a subset B is called regular closed provided that $B = \mathrm{cl}(\mathrm{int}(B))$;
- if U is open and B is closed, then $\mathrm{cl}(U)$ and $\mathrm{int}(B)$ are regular closed and regular open respectively.

Definition 2.2. Let R be a topological ring.

- A regular open neighborhood U of 0 is said to be unit provided that U is symmetric for the addition, idempotent for the multiplication, and $1 \in \mathrm{cl}\,(U)$.
- A regular closed neighborhood B of 0 is said to be unit provided that $\mathrm{int}\,(B)$ is an open unit neighborhood of 0.

For simplification purposes we will refer to open (closed) unit neighborhoods of 0 as open (closed) units. From the definition above one can quickly infer that:

- R is always both an open and a closed unit, which will be called the trivial unit neighborhood of 0 or simply the trivial unit;
- if U is an open unit, then $\mathrm{cl}(U)$ is a closed unit;
- if B is a closed unit, then $\mathrm{int}(B)$ is an open unit.

Other (non-trivial) examples of open units and closed units follow.

Example 2.3. Let R be a topological ring.

- If R is endowed with the trivial topology, then R is the only open unit and the only closed unit.
- If R is endowed with the discrete topology, then every set containing 1 and being multiplicatively idempotent and additively symmetric is a clopen unit.
- Let $A \subset R$ be an arbitrary subset and $B = \bigcup_{n=1}^{\infty} (A \cup -A \cup \{-1, 1\})^n$. Notice that $(A \cup -A \cup \{-1, 1\})$ is the smallest additively symmetric subset containing A and 1. Also, see that powers of an additively symmetric are again additively symmetric. Finally, notice that the infinite union of powers of a given set is the smallest multiplicatively idempotent subset containing it. Thus, every open unit containing A must also contain B.
- If $\mathbb{K}$ denotes the real or the complex field, then $\mathsf{U}_{\mathbb{K}}$ and $\mathsf{B}_{\mathbb{K}}$ are an open and a closed unit neighborhood of 0, respectively.

The following example shows the existence of multiplicatively idempotent and additively symmetric open neighbourhoods of 0 which are not regular and contain the unity, justifying then the regularity in Definition 2.2.

Example 2.4. Let R be a topological ring such that $R \setminus \{-1, 1\}$ is dense in R. Consider $\mathbb{Z} \times R$ with the usual addition and the multiplication given by $(n_1, r_1)(n_2, r_2) = (n_1 n_2, n_1 r_2 + n_2 r_1)$ and endowed with the product topology. The reader may easily check that $\mathbb{Z} \times R$ with the given structure is a topological ring. Then $\{-1, 1\} \times R \cup \{0\} \times R \setminus \{-1, 1\}$ is a non-regular multiplicatively idempotent and additively symmetric open neighborhood of $(0, 0)$ that contains the unity of $\mathbb{Z} \times R$.

2.2. **Basic properties of open units and closed units.**

Remark 2.5. Recall that if R is a topological ring and A and B are non-empty subsets of R, then

- $\mathrm{cl}\,(-A) = -\mathrm{cl}\,(A)$.
- $\mathrm{int}(-A) = -\mathrm{int}(A)$.
- $\mathrm{cl}\,(\mathrm{cl}\,(A)\,\mathrm{cl}\,(B)) = \mathrm{cl}\,(AB)$.

- If $\text{int}(A)\text{int}(B)$ is open, then $\text{int}(A)\text{int}(B) \subseteq \text{int}(AB)$. This holds, for example, for division rings.

Proposition 2.6. *Let R be a topological ring. If B is a closed unit, then B is multiplicatively idempotent and additively symmetric.*

Proof. By bearing in mind Definition 2.2 we clearly have that $1 \in \text{cl}(\text{int}(B)) = B$. Therefore $B \subseteq B^2$. In accordance to Remark 2.5 and by keeping in mind Definition 2.2, we have that $B^2 = \text{cl}(\text{int}(B))^2 \subseteq \text{cl}(\text{int}(B)^2) = \text{cl}(\text{int}(B)) = B$. This shows that B is multiplicatively idempotent. The additive symmetricity of B is also a consequence of Remark 2.5. □

2.3. **Existence of non-trivial open units.** Every ring endowed with the trivial topology has only one open unit: itself. It is natural to wonder whether this is an exclusive property of the trivial topology. A negative answer is given through the next example.

Example 2.7. Take $\mathbb{R}$ and consider the finite-measure-complement topology

$$\mathcal{T} = \left\{ U \subset \mathbb{R} : \begin{array}{c} U \text{ is open in the usual topology} \\ \text{and } \mathbb{R} \setminus U \text{ has finite measure} \end{array} \right\}$$

It is easy to see that $\mathcal{T}$ is a ring topology. Now, suppose U is an open unit. Since $\mathbb{R} \setminus U$ has finite measure, there exists $x \in U$ such that $x > 1$. Since U is a neighborhood of 0 in the usual topology, it contains a ball $\mathsf{B}_{\mathbb{R}}(0, r)$ for $r > 0$. Since U is multiplicatively idempotent, then $\mathbb{R} = \bigcup_{n \in \mathbb{N}} x^n \mathsf{B}_{\mathbb{R}}(0, r) \subset U$, so U is trivial.

3. Unit neighborhoods of 0 in absolute valued rings

Let R be an absolute semi-valued ring. Notice that if U is a neighborhood of 0, then we can consider the strictly positive (possibly infinite) number

$$\tau(U) := \sup \{\varepsilon > 0 : \mathsf{U}_R(0, \varepsilon) \subseteq U\}.$$

It is fairly obvious that $\mathsf{U}_R(0, \tau(U)) \subseteq U$. We will refer to $\tau(U)$ as the unit radius of U and will be relying on it throughout most of this section.

3.1. Topological properties of open and closed units inherited from the absolute value.

Remark 3.1. Let R be an absolute semi-valued ring. Let $a, x \in R$ and $t > 0$. Then:

(1) If $|a| \neq 0$, then $a\mathsf{U}_R(x, t) \subseteq \mathsf{U}_R(ax, |a|t)$.
(2) If $|1| = 1$ and $a \in \mathcal{U}(R)$, then $a\mathsf{U}_R(x, t) = \mathsf{U}_R(ax, |a|t)$.

What follows next is a series of technical lemmas which are helpful towards accomplishing our purposes.

Lemma 3.2. *Let R be an absolute semi-valued ring. Let U be an open neighborhood of 0. If $U^2 \subseteq U$, then either $U = R$ or $U \cap \mathcal{U}(R) \subseteq \mathsf{B}_R$.*

Proof. Assume that there exists $b \in U \cap \mathcal{U}(R)$ with $|b| > 1$. Then Remark 3.1 allows us to conclude that

$$U \supseteq \bigcup_{n\in\mathbb{N}} b^n U \supseteq \bigcup_{n\in\mathbb{N}} b^n \mathsf{U}_R\left(0, \tau(U)\right) = \bigcup_{n\in\mathbb{N}} \mathsf{U}_R\left(0, |b|^n \tau(U)\right) = R.$$

□

Lemma 3.3. *Let R be an absolute semi-valued ring with $|1| = 1$. Let U be an open neighborhood of 0. If D is a connected component of U with non-empty interior such that $d(D, 0) > 0$, then for every sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathsf{U}_R \cap \mathcal{U}(R)$ converging to 1, there exists $n_0 \in \mathbb{N}$ such that $a_{n_0} D \cap D \neq \varnothing$, $a_{n_0} D \cap (R \setminus D) \neq \varnothing$, and thus $a_{n_0} D \cap bd(D) \neq \varnothing$.*

Proof. Fix $x \in \operatorname{int}(D)$ and $0 < t < |x|$ such that $\mathsf{U}_R(x, t) \subseteq D$ (notice that $|x| \geq d(D, 0) > 0$). Now choose $n_0 \in \mathbb{N}$ so that $|1 - a_{n_0}| < \frac{t}{|x|}$. With this choice of n_0 we have that $a_{n_0} x \in \mathsf{U}_R(x, t)$ so $a_{n_0} D \cap D \neq \varnothing$. Also observe that $|a_{n_0}| > 0$ since otherwise we would have the contradiction that

$$1 = |1| = ||1| - |a_{n_0}|| \leq |1 - a_{n_0}| < \frac{t}{|x|} < 1.$$

Now take

$$0 < \varepsilon < \frac{(1 - |a_{n_0}|)\, d(D, 0)}{|a_{n_0}|}$$

and $y \in U$ such that

$$|y| < d(D, 0) + \varepsilon.$$

With this choice of ε and y, we have that $a_{n_0} y \notin D$. Indeed, if $a_{n_0} y \in D$, then

$$d(D, 0) \leq |a_{n_0} y| = |a_{n_0}|\, |y| < |a_{n_0}|\, d(D, 0) + |a_{n_0}|\, \varepsilon < d(D, 0).$$

As a consequence, $a_{n_0} y \notin D$ and thus $a_{n_0} D \cap (R \setminus D) \neq \varnothing$. Finally, the connectedness of $a_{n_0} D$ implies that $a_{n_0} D \cap \operatorname{bd}(D) \neq \varnothing$ (notice that $r \mapsto a_{n_0} r$ is a homeomorphism on R since a_{n_0} is invertible). □

Lemma 3.4. *Let R be an absolute semi-valued ring such that the open balls are connected. Let U be an open neighborhood of 0. If D is a connected component of U, then D is open.*

Proof. Let $x \in D$. There exists $t > 0$ such that $\mathsf{U}_R(x, t) \subseteq U$. Let E denote the connected component of U containing $\mathsf{U}_R(x, t)$. Then $D \cap E \neq \varnothing$ therefore $\mathsf{U}_R(x, t) \subseteq E = D$. □

Now we are ready to state and prove the main theorems in this subsection.

Theorem 3.5. *Let R be an absolute semi-valued ring such that $|1| = 1$, the open balls are connected and $\operatorname{int}(\mathsf{B}_R) = \mathsf{U}_R$. Consider a non-trivial open unit U of R. If $U \cap \mathcal{U}(R)$ is dense in U, then $U \subseteq \mathsf{U}_R$ and U is connected.*

Proof. In the first place, it suffices to keep in mind Lemma 3.2 to conclude that $U \cap \mathcal{U}(R) \subseteq \mathsf{B}_R$. The denseness of $U \cap \mathcal{U}(R)$ in U implies that $U \subseteq \mathsf{B}_R$ and thus $U \subseteq \operatorname{int}(\mathsf{B}_R) = \mathsf{U}_R$. In the second and final place, assume to the contrary that U is not connected and consider C_0 to be the connected component of 0. By hypothesis, $\mathsf{U}_R(0,\tau(U)) \subseteq C_0$. By assumption there must exist another connected component D verifying that $D \cap C_0 = \varnothing$, which implies that $d(D,0) > 0$. In accordance to Lemma 3.4, we have that D is open. Now, Lemma 3.3 applies to ensure the existence of $a \in U \cap \mathcal{U}(R)$ such that $aD \cap D \neq \varnothing$ and $aD \cap (R \setminus D) \neq \varnothing$. Now observe that $aD \subseteq U$ and aD is connected (because a is invertible). Therefore, aD is a connected component of U such that $aD \cap D \neq \varnothing$, which implies that $aD = D$. This contradicts the fact that $aD \cap (R \setminus D) \neq \varnothing$. As a consequence, U must be connected. □

Using this theorem we can fully characterize the open and closed unit neighborhoods of zero in $\mathbb{R}$ and $\mathbb{C}$. We will need the next lemma.

Lemma 3.6. *Let R be a topological ring. If U is an open unit and $z \notin U$, then $z\left(U \cap \mathcal{U}(R)\right)^{-1} \cap U = \emptyset$.*

Proof. Suppose $z\left(U \cap \mathcal{U}(R)\right)^{-1} \cap U \neq \varnothing$. There exist $u \in U \cap \mathcal{U}(R)$ and $v \in U$ such that $zu^{-1} = v$. But then $z = vu \in U^2 = U$, contradiction. □

Theorem 3.7. *If $\mathbb{K}$ denotes the real or the complex field, then $\mathsf{U}_{\mathbb{K}}$ and $\mathsf{B}_{\mathbb{K}}$ are the only non-trivial open and closed unit neighborhoods of 0, respectively.*

Proof. Let U be an open unit neighborhood of 0 in $\mathbb{K}$. By Theorem 3.5, $U \subset \mathsf{U}_{\mathbb{K}}$ and U is connected. Since U is an open connected subset of $\mathbb{K}$, it is also path-connected. Path-connectedness and $\pm 1 \in \mathrm{cl}(U)$ implies the real case.

For $\mathbb{K} = \mathbb{C}$ suppose $\tau(U) < 1$. The compactness of $\mathsf{S}_{\mathbb{C}}(0,\tau(U))$ and the maximality of $\tau(U)$ allow that $\mathsf{S}_{\mathbb{C}}(0,\tau(U)) \nsubseteq U$. Then there exists some $z \notin U$ with $|z| = \tau(U)$ and $z, -z \in \mathrm{cl}(U)$. Since U is path-connected we can find a continuous path $f : [-1,1] \to \mathbb{C}$ such that $f(-1) = -z$, $f(0) = 1$, $f(1) = z$, and verifying that $f(x) \in U$ for $x \notin \{-1,0,1\}$. It is clear that we can take $f(x) \neq 0$ for every $x \in [-1,1]$. We can define $F : [-1,1] \to \mathbb{C}$ given by $F(x) = zf(x)^{-1}$, which is still continuous. Notice that $F(-1) = -1$, $F(0) = z$, and $F(1) = 1$. If $x \in (-1,0) \cup (0,1)$, then $F(x) = zf(x)^{-1} \in zU^{-1}$ and by Lemma 3.6, $F(x) \notin U$. Then $F(-1,1)$ is a continuous open path connecting -1 and 1, thus cutting the open unit ball in two disjoint open components by Jordan's Curve Theorem; but $F(-1,1) \cap U = \varnothing$ and $U \subset \mathsf{U}_{\mathbb{C}}$, so U cannot be path-connected, contradiction. Thus $\tau(U) = 1$ and $U = \mathsf{U}_{\mathbb{C}}$.

Finally, $U = \mathsf{U}_{\mathbb{K}}$ implies that the only closed unit is $\mathsf{B}_{\mathbb{K}}$ by its own definition. □

3.2. **Unit balls vs unit neighborhoods of** 0. Notice that not every open unit ball in an absolute-valued ring is an open unit. For example, consider a ring endowed with the discrete absolute value. But there are more non-discrete examples.

Example 3.8. Take the unitalized ring $R = \mathbb{Z} \times \mathbb{R}$. It is easy to prove that

$$d\left((n,x),(m,y)\right) = |n-m| + \frac{|x-y|}{1+|x-y|}$$

is a metric which induces the product topology on R. Then

$$\mathsf{U}_R = \{(n,x) \in R : d((n,x),(0,0)) < 1\} = \{0\} \times \mathbb{R}$$

but $\mathsf{U}_R^2 = 0 \neq \mathsf{U}_R$. Also

$$\mathsf{B}_R = \{(n,x) \in R : d((n,x),(0,0)) \leq 1\} = \mathsf{U}_R \cup \{\pm 1\} \times \{0\},$$

but $\mathrm{cl}(\mathrm{int}(\mathsf{B}_R)) = \mathsf{U}_R \neq \mathsf{B}_R$. Thus, neither U_R is an open unit nor B_R is a closed unit.

This poses a natural question: *which properties must be imposed to an absolute-valued ring to ensure its open and closed unit balls are open and closed units, respectively?* First, we give a sufficient condition for ensuring the regularity of U_R.

Proposition 3.9. *Let R be an absolute semi-valued ring such that $|1| = 1$. If $\mathsf{U}_R \cap \mathcal{U}(R)$ is dense in U_R, then U_R is a regular open set.*

Proof. First off, notice that $\mathsf{U}_R \subset \mathrm{int}(\mathrm{cl}(\mathsf{U}_R))$ is obvious. Suppose the inclusion is strict and take $x \in \mathrm{int}(\mathrm{cl}(\mathsf{U}_R)) \setminus \mathsf{U}_R \subseteq \mathsf{S}_R$ since $\mathrm{cl}(\mathsf{U}_R) \subseteq \mathsf{B}_R$. There exists $r > 0$ such that $\mathsf{U}_R(x,r) \subset \mathrm{cl}(\mathsf{U}_R) = \mathrm{cl}\,(\mathsf{U}_R \cap \mathcal{U}(R))$. Since R is first-countable (because of its metric structure) then there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathsf{U}_R \cap \mathcal{U}(R)$ converging also to x. Finally, consider $y_n = v_n^{-1}x^2$, which is out of B_R (since $|y_n| > 1$) and converges to x. Thus, some $y_n \in \mathsf{U}_R(x,r) \subset \mathrm{cl}(\mathsf{U}_R) \subseteq \mathsf{B}_R$, which is a contradiction and, by reductio ad absurdum, implies $\mathsf{U}_R = \mathrm{int}(\mathrm{cl}(\mathsf{U}_R))$. □

Lemma 3.10. *Let R be an absolute semi-valued ring such that $|1| = 1$.*

1. *$\mathsf{U}_R \cap \mathcal{U}(R) \neq \varnothing$ if and only if $\mathcal{U}(R) \nsubseteq \mathsf{S}_R$.*
2. *If $(u_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{U}(R)$ converging to $u \in \mathcal{U}(R)$, then $(u_n^{-1})_{n\in\mathbb{N}}$ converges to u^{-1}.*
3. *$\mathcal{U}(R)$ is a topological multiplicative group.*

Assume, in addition, that the absolute semi-value is complete.

4. *If $s \in \mathsf{U}_R$, then $1 - s$ is invertible and $(1-s)^{-1} = \sum_{n=0}^{\infty} s^n$.*
5. *$\mathcal{U}(R)$ is open.*

These facts are proved on almost every manual concerning topological groups and topological rings, such as [3]. We are in the right position to state and prove the main results in this subsection.

Proposition 3.11. *Let R be an absolute semi-valued ring such that $|1| = 1$.*

1. *If there exists an element $x \in \mathsf{U}_R$ with maximum absolute value, then either $\mathsf{U}_R = \{z \in R : |z| = 0\}$ or U_R is not idempotent.*
2. *If U_R is an open unit, then $\mathsf{U}_R \neq \{z \in R : |z| = 0\}$ and consequently $\sup_{z \in \mathsf{U}_R} |z|$ is not attained.*

Proof.

1. Assume that $\mathsf{U}_R \neq \{z \in R : |z| = 0\}$ and U_R is idempotent. Note that in this case $|x| > 0$. By hypothesis, there are $r, s \in \mathsf{U}_R$ such that $x = rs$ so $|x| = |rs| = |r||s| < |r| \leq |x|$, which is a contradiction.
2. Assume that U_R is an open unit such that $\mathsf{U}_R = \{z \in R : |z| = 0\}$. Notice that $\mathsf{U}_R\left(1, \frac{1}{2}\right) \cap \{z \in R : |z| = 0\} = \varnothing$, which contradicts the fact that $1 \in \mathrm{cl}\left(\mathsf{U}_R\right)$. The fact that $\sup_{z \in \mathsf{U}_R} |z|$ is not attained is a direct consequence of the previous item.

□

Definition 3.12. Let R be an absolute semi-valued ring. We will say that $e \in \mathsf{B}_R$ is

- a weak extreme point of B_R provided that the condition $2e = r + s$ with $r, s \in \mathsf{B}_R$ implies that $r = s$;
- an extreme point of B_R provided that the condition $2e = r + s$ with $r, s \in \mathsf{B}_R$ implies that $r = s = e$.

We will denote the set of (weak) extreme points by (w-)ext(B_R).

Proposition 3.13. *Let R be an absolute semi-valued ring.*

(1) $\mathrm{ext}(\mathsf{B}_R) \subseteq w\text{-}\mathrm{ext}(\mathsf{B}_R) \subseteq \mathsf{B}_R$.
(2) *If* $\mathrm{char}(R) = 2$ *and* $R \neq 0$, *then* $\mathrm{ext}(\mathsf{B}_R) = \varnothing$ *and* $w\text{-}\mathrm{ext}(\mathsf{B}_R) = \mathsf{B}_R$.
(3) *If* $\mathrm{char}(R) \neq 2$ *and R is an integral domain, then* $\mathrm{ext}(\mathsf{B}_R) = w\text{-}\mathrm{ext}(\mathsf{B}_R)$.

Proof.

(1) It is fairly obvious that $\mathrm{ext}(\mathsf{B}_R) \subseteq w\text{-}\mathrm{ext}(\mathsf{B}_R) \subseteq \mathsf{B}_R$.
(2) Let $e \in \mathsf{B}_R$ and write $2e = r + s$ with $r, s \in \mathsf{B}_R$. Then $2e = 0$ so $r = s$ since $r + r = 0$. This shows that w-ext(B_R) $= \mathsf{B}_R$. In order to prove that $\mathrm{ext}(\mathsf{B}_R) = \varnothing$ we choose $e \neq r \in \mathsf{B}_R$ and notice that $2e = 0 = r + r$ (observe that since $R \neq 0$ we may take $e = 1$ and $r = 0$).
(3) Let $e \in w\text{-}\mathrm{ext}(\mathsf{B}_R)$ and assume that $r, s \in \mathsf{B}_R$ are so that $2e = r + s$. By hypothesis we have that $r = s$ so $2e = 2r$ and thus $2(e - r) = 0$. Again by hypothesis we deduce that $e = r$.

□

Definition 3.14. Let R be an absolute semi-valued ring. The absolute stabilizer of $r \in R$ is defined as

$$\mathsf{E}_r := \{s \in R : |r + s| = |r - s| = |r|\}.$$

The reader may find easy to realize that $\mathsf{E}_0 = \{s \in R : |s| = 0\}$ is a closed two-sided ideal of R.

Lemma 3.15. *Let R be an absolute semi-valued ring.*

(1) $\mathsf{E}_0 \subseteq \mathsf{E}_r$ *for every* $r \in R$.
(2) *If* $\mathrm{char}(R) \neq 2$ *and R is an integral domain, then* $\mathsf{E}_1 \neq \{0\}$ *implies that* $1 \notin \mathrm{ext}\,(\mathsf{B}_R)$.

Proof.

(1) Let $s \in \mathsf{E}_R$. Simply notice that

$$|r| = ||r| - |s|| \leq |r - s| \leq |r| + |s| = |r|$$

and

$$|r| = ||r| - |-s|| \leq |r + s| \leq |r| + |s| = |r|.$$

(2) Let $s \in \mathsf{E}_1 \setminus \{0\}$. Then $2 = (1 - s) + (1 + s)$ with $|1 - s| = |1 + s| = |1|$. Suppose that $1 - s = 1 + s$, then $2s = 0$ and, by hypothesis, $s = 0$, contradiction. Then $1 - s \neq 1 + s$, meaning that $1 \notin \mathrm{ext}\,(\mathsf{B}_R)$.

□

Theorem 3.16. *Let R be an absolute semi-valued ring such that $|1| = 1$. Suppose that R is an integral domain and of characteristic different from 2. Assume that:*

- *1 is an extreme point of B_R.*
- *R is complete.*
- *$\mathsf{U}_R \cap \mathcal{U}(R) \neq \varnothing$.*

Then $1 \in cl(\mathsf{U}_R)$.

Proof. By the third hypothesis there exists $r \in \mathcal{U}(R)$ with $0 < |r| < 1$. Notice that the sequence $(r^n)_{n\in\mathbb{N}}$ converges to 0, so $(1 - r^n)_{n\in\mathbb{N}}$ converges to 1, and $1 - r^n$ is invertible for all $n \in \mathbb{N}$ (see Lemma 3.10 above). We will distinguish now several cases:

- There exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $|1 - r^{n_k}| > 1$ for all $k \in \mathbb{N}$. In this case, we apply Lemma 3.10 to assure that $\left((1 - r^{n_k})^{-1}\right)_{k\in\mathbb{N}} \subseteq \mathsf{U}_R$ converges to 1, so $1 \in \mathrm{cl}(\mathsf{U}_R)$.
- There exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $|1 - r^{n_k}| < 1$ for all $k \in \mathbb{N}$. In this case, we immediately conclude that $1 \in \mathrm{cl}(\mathsf{U}_R)$.
- There exists $n_0 \in \mathbb{N}$ such that $|1 - r^n| = 1$ for all $n \geq n_0$. We will show that $|1 + r^n| \neq 1$ for all $n \geq n_0$. Indeed, suppose to the contrary that there exists $n_1 \geq n_0$ with $|1 + r^{n_1}| = 1$. Then $r^{n_1} \in \mathsf{E}_1 \setminus \{0\}$, which implies in virtue of (2) of Lemma 3.15 that $1 \notin \mathrm{ext}(\mathsf{B}_R)$. This contradicts our hypotheses, therefore $|1 + r^n| \neq 1$ for all $n \geq n_0$. Finally, we only need to take the sequence
$$x_n = \begin{cases} 1 + r^n & \text{if } |1 + r^n| < 1 \\ (1 + r^n)^{-1} & \text{if } |1 + r^n| > 1 \end{cases}$$
for $n \geq n_0$.

□

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Department of Mathematics, University of Cadiz, Puerto Real, 11510, Spain.
E-mail address: garcia.pacheco@uca.es
E-mail address: piniella5050@gmail.com