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WEAK APPROXIMATION PROPERTIES OF SUBSPACES

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Abstract. The paper is concerned with weak approximation properties which are weaker than the classical approximation property. For $\lambda \geq 1$, we prove that a Banach space X has the λ-bounded weak approximation property (λ-BWAP) if and only if every locally 1-complemented subspace of X has the λ-BWAP, and that if X has the λ-BWAP and Z is a locally μ-complemented subspace of X, then Z has the $(2\mu+4)\mu\lambda$-BWAP. It also follows that X has the weak approximation property (WAP) if and only if every locally complemented subspace of X has the WAP.

1. Introduction and Main results

A Banach space X is said to have the *approximation property* (AP) if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank and continuous linear map (operator) S on X such that $\sup_{x \in K} \|Sx - x\| \leq \varepsilon$, briefly, $id_X \in \overline{\mathcal{F}(X)}^{\tau_c}$, where id_X is the identity map on X, $\mathcal{F}(X)$ is the space of all finite rank operators on X and τ_c is the topology of uniformly compact convergence on the space $\mathcal{L}(X)$ of all operators on X. For $\lambda \geq 1$, if $id_X \in \overline{\{S \in \mathcal{F}(X) : \|S\| \leq \lambda\}}^{\tau_c}$, then we say that X has the λ-*bounded approximation property* (λ-BAP). Choi and the first author [1, 7] introduced and studied weaker forms of the AP. A Banach space X is said to have the *weak approximation property* (WAP) if $\mathcal{K}(X) \subset \overline{\mathcal{F}(X)}^{\tau_c}$, where $\mathcal{K}(X)$ is the space of all compact operators on X. For $\lambda \geq 1$,

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we say that X has the λ-*bounded weak approximation property* (λ-BWAP) if for every $T \in \mathcal{K}(X)$, we have $T \in \overline{\{S \in \mathcal{F}(X) : \|S\| \le \lambda\|T\|\}}^{\tau_c}$.

For $\mu \ge 1$, a closed subspace Z of a Banach space X is called *locally μ-complemented* in X if for every finite-dimensional subspace E of X and every $\varepsilon > 0$, there exists an operator $T : E \to Z$ with $\|T\| \le \mu + \varepsilon$ such that $Tx = x$ for all $x \in E \cap Z$. It is well known that the AP and the BAP are inherited by locally complemented subspaces (cf. [3, Theorem 2.4]). The first author [7, Theorem 1.4] obtained the analogue for the WAP and the BWAP under an assumption. In this paper, we have:

Theorem 1.1. *If X has the λ-BWAP and Z is a locally μ-complemented subspace of X, then Z has the $(2\mu + 4)\mu\lambda$-BWAP.*

For a closed subspace Z of a Banach space X, a map $\Phi : Z^* \to X^*$ is called an *extension operator* if $(\Phi z^*)(z) = z^*(z)$ for every $z^* \in Z^*$ and $z \in Z$. A closed subspace Z of X is called an *ideal* if there exists an extension operator Φ from Z^* to X^* with $\|\Phi\| = 1$. The operator Φ is called a *Hahn–Banach extension operator*. It is well known that Z is an ideal in X if and only if Z is locally 1-complemented in X (cf. Lemma 2.2), and that X has the λ-BAP (resp. AP) if and only if every ideal in X has the λ-BAP (resp. AP) (cf. [8, Proposition 4.3 and Theorem 2.2] and [3, Theorem 2.4]). In this paper, we have:

Theorem 1.2. *Let $\lambda \ge 1$. The following statements are equivalent.*
(a) *X has the λ-BWAP.*
(b) *Every ideal in X has the λ-BWAP.*
(c) *If for every separable closed subspace Y of X, there exists a separable closed subspace Z of X with $Y \subset Z$ such that Z has the λ-BWAP.*

Theorem 1.3. *The following statements are equivalent.*
(a) *X has the WAP.*
(b) *Every locally complemented subspace of X has the WAP.*
(c) *Every ideal in X has the WAP.*
(d) *If for every separable closed subspace Y of X, there exists a separable closed subspace Z of X with $Y \subset Z$ such that Z has the WAP.*

2. Proof of Theorem 1.1

Lemma 2.1. [5, Lemma 3.2] *If Z is a locally μ-complemented subspace of X, then for every closed subspace Y of X containing Z with $\dim Y/Z < \infty$, there exists a projection P from Y onto Z with $\|P\| \le 2\mu + 4$.*

The following lemma is a simple extension of [5, Theorem 3.4 and Theorem 3.5] (cf. [2]).

Lemma 2.2. *Let Z be a closed subspace of X and let $\mu \ge 1$. The following statements are equivalent.*
(a) *For every Banach space Y and every $T \in \mathcal{K}(Z, Y)$, there exists a $\widetilde{T} \in \mathcal{K}(X, Y)$ with $\|\widetilde{T}\| \le \mu\|T\|$ such that $\widetilde{T}x = Tx$ for all $x \in Z$.*
(b) *Z is locally μ-complemented in X.*

Proof of Theorem 1.1. Let $T \in \mathcal{K}(Z)$. Let K be a compact subset of Z and let $\varepsilon > 0$ be given. Since Z is locally μ-complemented in X, by Lemma 2.2 there exists a $\widetilde{T} \in \mathcal{K}(X, Z)$ with $\|\widetilde{T}\| \le \mu\|T\|$, which is an extension of T. Since X has the λ-BWAP, there exists an $S_0 \in \mathcal{F}(X)$ with $\|S_0\| \le \lambda\|i\widetilde{T}\|$, where $i : Z \to X$ is the inclusion map, such that

$$\sup_{z\in K} \|S_0 z - iTz\| \le \frac{\varepsilon}{2\mu + 4}.$$

Put $Y := Z + S_0(X)$. Then $\dim Y/Z < \infty$. By Lemma 2.1 there exists a projection P from Y onto Z with $\|P\| \le 2\mu + 4$. Let $S := PS_0|_Z$. Then $S \in \mathcal{F}(Z)$ and for every $z \in K$

$$\|Sz - Tz\| = \|PS_0 z - PTz\| \le \varepsilon$$

and we also have

$$\|S\| \le (2\mu + 4)\lambda\|\widetilde{T}\| \le (2\mu + 4)\mu\lambda\|T\|,$$

hence Z has the $(2\mu + 4)\mu\lambda$-BWAP.

□

3. Proofs of Theorems 1.2 and 1.3

We denote by $\|\cdot\|_\pi$ and $\|\cdot\|_{\mathcal{I}}$, respectively, the projective tensor norm and the integral ideal norm. The following lemmas are needed to prove Theorem 1.2.

Lemma 3.1. [6, Theorem 2.5] *Let $T \in \mathcal{L}(X)$ and let $\lambda \ge 1$. The following statements are equivalent.*

(a) $T \in \overline{\{S \in \mathcal{F}(X) : \|S\| \le \lambda\|T\|\}}^{\tau_c}$.
(b) $\|TS\|_\pi \le \lambda\|T\|\|S\|_{\mathcal{I}}$ *for every* $S \in \mathcal{F}(X)$.
(c) $\|TS\|_\pi \le \lambda\|T\|\|S\|_{\mathcal{I}}$ *for every Banach space* Y *and every* $S \in \mathcal{F}(Y, X)$.

Lemma 3.2. *Let $\mathcal{A}(X)$ be a subset of $\mathcal{L}(X)$. Let $T \in \mathcal{L}(X)$ and let $\lambda \ge 1$. Then $T \in \overline{\{S \in \mathcal{A}(X) : \|S\| \le \lambda\|T\|\}}^{\tau_c}$ if and only if for every finite-dimensional subspace F of X and every $\varepsilon > 0$, there exists an $S \in \mathcal{A}(X)$ with $\|S\| \le \lambda\|T\|$ such that $\|Sx - Tx\| \le \varepsilon\|x\|$ for every $x \in F$.*

Proof of Theorem 1.2. (a)⇒(b) Let Z be an ideal in X. Let $T \in \mathcal{K}(Z)$. We use Lemma 3.1(b) to prove that $T \in \overline{\{S \in \mathcal{F}(Z) : \|S\| \le \lambda\|T\|\}}^{\tau_c}$. Let $j : Z \to X$ be the inclusion map and let $\Phi : Z^* \to X^*$ be a Hahn–Banach extension operator. For a Banach space B, we denote by i_B the canonical isometry from B to B^{**}.

Now, since the operator $T^* : Z^* \to Z^*$ is weak* to weak continuous, T^{**} maps from Z^{**} into $i_Z(Z)$. Thus the operator $i_Z^{-1}T^{**}\Phi^* i_X : X \to Z$ is well defined and we claim that

$$i_Z^{-1}T^{**}\Phi^* i_X j = T.$$

Indeed, for every $z \in Z$ and $z^* \in Z^*$,

$$\Phi^* i_X j(z)(z^*) = i_X j(z)(\Phi(z^*)) = z^*(z) = i_Z(z)(z^*).$$

Then for every $z \in Z$, we have

$$i_Z^{-1}T^{**}\Phi^* i_X j(z) = i_Z^{-1}T^{**} i_Z(z) = Tz.$$

Now, let $S \in \mathcal{F}(Z)$. Since Z is an ideal in X, $Z^*\hat{\otimes}_\pi Z$ is a closed subspace of $Z^*\hat{\otimes}_\pi X$ (cf. [10, Theorem 3.4]). Thus $\|TS\|_\pi = \|ji_Z^{-1}T^{**}\Phi^* i_X jS\|_\pi$. Since X has the λ-BWAP, by Lemma 3.1(c) we have

$$\begin{aligned}\|TS\|_\pi &= \|ji_Z^{-1}T^{**}\Phi^* i_X jS\|_\pi \\ &\le \lambda \|ji_Z^{-1}T^{**}\Phi^* i_X\|\|jS\|_{\mathcal{I}(Z,X)} \\ &\le \lambda\|T\|\|S\|_{\mathcal{I}(Z,Z)}.\end{aligned}$$

(b)⇒(c) Let Y be a separable closed subspace of X. Then by [11, Theorem] there exists a separable ideal Z in X such that $Y \subset Z$. Hence by (b) Z has the λ-BWAP.

(c)⇒(a) This proof is due to the one of [4, Lemma 3(b)]. Let $T \in \mathcal{K}(X)$. We use Lemma 3.2 to prove that $T \in \overline{\{S \in \mathcal{F}(X) : \|S\| \le \lambda\|T\|\}}^{\tau_c}$. Let F be a finite-dimensional subspace of X and let $\varepsilon > 0$ be given. Then by [9, Lemma 1] we see that there exists a separable subspace Y of X such that for every finite-dimensional subspace E of X with $F \subset E$ there exists an operator $T_E : E \to Y$ satisfying that $\|T_E\| \le 1 + 1/\dim E$ and the restriction $T_E|_F$ is the identity map. Consider the separable subspace $\overline{\text{span}}(T(X) \cup Y)$ of X. Then by (c) there exists a separable closed subspace Z of X with $\overline{\text{span}}(T(X) \cup Y) \subset Z$ such that Z has the λ-BWAP. Since the restriction $T|_Z \in \mathcal{K}(Z)$, there exists an $S \in \mathcal{F}(Z)$ with $\|S\| \le \lambda\|T\|$ such that

$$\|Sf - Tf\| \le \varepsilon\|f\|$$

for every $f \in F$. We define the map $S_E : X \to X$ by

$$S_E x = ST_E x \text{ if } x \in E,\ S_E = 0 \text{ otherwise},$$

for every finite-dimensional subspace E of X with $F \subset E$. By compactness, there is a subnet which converges pointwise to a finite rank linear operator $\widetilde{S}$ on X with $\|\widetilde{S}\| \le \lambda\|T\|$ and for every $f \in F$, we have

$$\|\widetilde{S}f - Tf\| = \lim_G \|S_G f - Tf\| = \lim_G \|ST_G f - Tf\| = \|Sf - Tf\| \le \varepsilon\|f\|.$$

□

Remark 3.3. In view of the proof of Theorem 1.2(a)⇒(b), we see that for every $T \in \mathcal{W}(X)$, the space of all weakly compact operators on X,

$$T \in \overline{\{S \in \mathcal{F}(X) : \|S\| \le \lambda\|T\|\}}^{\tau_c}$$

if and only if for every ideal Z in X, for every $T \in \mathcal{W}(Z)$,

$$T \in \overline{\{S \in \mathcal{F}(Z) : \|S\| \le \lambda\|T\|\}}^{\tau_c}.$$

Proof of Theorem 1.3. (a)⇒(b) follows from the proof of Theorem 1.1. (b)⇒(c) is trivial. The proof of (c)⇒(d) is similar to (b)⇒(c) in Theorem 1.2.

(d)⇒(a) This result was inspired from [4, Lemma 3(a)]. Let $T \in \mathcal{K}(X)$. Let K be a compact subset of X and let $\varepsilon > 0$ be given. Consider the separable subspace $\overline{\text{span}}(K \cup T(X))$ of X. Then by (c) there exists a separable closed subspace Z

of X with $\overline{\mathrm{span}}(K \cup T(X)) \subset Z$ such that Z has the WAP. Since the restriction $T|_Z \in \mathcal{K}(Z)$, there exists an $S \in \mathcal{F}(Z)$ such that

$$\sup_{x \in K} \|Sx - Tx\| \leq \varepsilon.$$

Then by an application of the Hahn–Banach theorem there exists an extension $\widehat{S} : X \to Z$ of S. Then $j\widehat{S} \in \mathcal{F}(X)$, where $j : Z \to X$ is the inclusion map, and

$$\sup_{x \in K} \|j\widehat{S}x - Tx\| = \sup_{x \in K} \|Sx - Tx\| \leq \varepsilon.$$

Hence X has the WAP. □

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