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ON THE SPECTRAL RADIUS OF HADAMARD PRODUCTS OF NONNEGATIVE MATRICES

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Abstract. We present some spectral radius inequalities for nonnegative matrices. Using the ideas of Audenaert, we then prove the inequality which may be regarded as a Cauchy–Schwarz inequality for spectral radius of nonnegative matrices

$$\rho(A\circ B)\le[\rho(A\circ A)]^{\frac{1}{2}}[\rho(B\circ B)]^{\frac{1}{2}}.$$

In addition, new proofs of some related results due to Horn and Zhang, Huang are also given. Finally, we interpolate Huang's inequality by proving

$$\begin{aligned}\rho(A_1\circ A_2\circ\cdots\circ A_k) &\le [\rho(A_1A_2\cdots A_k)]^{1-\frac{2}{k}}[\rho((A_1\circ A_1)\cdots(A_k\circ A_k))]^{\frac{1}{k}}\\ &\le \rho(A_1A_2\cdots A_k).\end{aligned}$$

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1. Introduction and preliminaries

Let M_n denote the set of complex matrices of order n. For matrices $A=(a_{ij}), B=(b_{ij})\in M_n$, we denote by $\rho(A)$ the spectral radius of A, by $A\circ B=(a_{ij}b_{ij})$ the Hadamard product of A and B. The notation $A\le B$ means that $B-A$ is entrywise nonnegative.

Zhan [10] conjectured that $\rho(A\circ B)\le\rho(AB)$ for nonnegative matrices $A,B\in M_n$. This conjecture was confirmed by Audenaert in [1] by proving

$$\rho(A\circ B)\le\rho^{\frac{1}{2}}((A\circ A)(B\circ B))\le\rho(AB). \tag{1.1}$$

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These inequalities were established via a trace description of the spectral radius. Using the fact that the Hadamard product of two matrices is a principal submatrix of the Kronecker product, Horn and Zhang proved [5] the inequalities

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}(AB \circ BA) \leq \rho(AB). \tag{1.2}$$

Huang [6] generalized the inequality $\rho(A \circ B) \leq \rho(AB)$ to an arbitrary number of nonnegative matrices:

$$\rho(A_1 \circ A_2 \circ \cdots \circ A_k) \leq \rho(A_1 A_2 \cdots A_k). \tag{1.3}$$

Recently, Peperko [8] proved the inequalities for nonnegative matrices A, B,

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}(AB \circ BA) \leq \rho^{\frac{1}{4}}(AB \circ AB)\rho^{\frac{1}{4}}(BA \circ BA) \leq \rho(AB). \tag{1.4}$$

The paper is organized as follows. Using the similar idea due to Peperko, we first give a new proof for (1.3). We then prove the inequality which may be regarded as a Cauchy–Schwarz inequality for spectral radius of nonnegative matrices

$$\rho(A \circ B) \leq [\rho(A \circ A)]^{\frac{1}{2}}[\rho(B \circ B)]^{\frac{1}{2}}. \tag{1.5}$$

In addition, new proofs of some related results due to Horn and Zhang, Huang are also given. Finally, we interpolate Huang's inequality by proving

$$\begin{aligned}\rho(A_1 \circ A_2 \circ \cdots \circ A_k) &\leq [\rho(A_1 A_2 \cdots A_k)]^{1-\frac{2}{k}}[\rho((A_1 \circ A_1)(A_2 \circ A_2)\cdots(A_k \circ A_k)]^{\frac{1}{k}} \\ &\leq \rho(A_1 A_2 \cdots A_k).\end{aligned}$$

Note that the inequality (1.1) corresponds to the case $k = 2$ of the above inequalities.

2. Main results

The two inequalities in the following lemma are classical and can be found in [4, 11].

Lemma 2.1. *Let $A, B \in M_n$ be nonnegative matrices.*
(1) If $A \leq B$, then $\rho(A) \leq \rho(B)$.
(2) $\rho(A \circ B) \leq \rho(A)\rho(B)$.

The following result appeared in [2, 7, 8]. A new and straightforward proof is presented. The similar manner can be found in [9].

Lemma 2.2. *Let $\{A_{1m}\}_{m=1}^t, \{A_{2m}\}_{m=1}^t, \ldots, \{A_{sm}\}_{m=1}^t$ be nonnegative matrices of order n. Then*

$$\begin{aligned}&(A_{11} \circ A_{21} \circ \cdots \circ A_{s1})(A_{12} \circ A_{22} \circ \cdots \circ A_{s2}) \cdots (A_{1t} \circ A_{2t} \circ \cdots \circ A_{st}) \\ &\leq (A_{11}A_{12} \cdots A_{1t}) \circ (A_{21}A_{22} \cdots A_{2t}) \circ \cdots \circ (A_{s1}A_{s2} \cdots A_{st}).\end{aligned}$$

Proof. For $p = 1, \ldots, s,\ q = 1, \ldots, t$, we denote by $a_{ij}^{(pq)}$ the entry of the matrix A_{pq} in the position (i, j). The (i, j)-th entry of

$$(A_{11} \circ A_{21} \circ \cdots \circ A_{s1})(A_{12} \circ A_{22} \circ \cdots \circ A_{s2}) \cdots (A_{1t} \circ A_{2t} \circ \cdots \circ A_{st})$$

equals

$$\sum_{j_1,j_2,\ldots,j_{t-1}} (a^{(11)}_{ij_1}a^{(21)}_{ij_1}\cdots a^{(s1)}_{ij_1})(a^{(12)}_{j_1j_2}a^{(22)}_{j_1j_2}\cdots a^{(s2)}_{j_1j_2})\cdots(a^{(1t)}_{j_{t-1}j}a^{(2t)}_{j_{t-1}j}\cdots a^{(st)}_{j_1j}).$$

Denote

$$L=\sum_{j_1,j_2,\ldots,j_{t-1}} (a^{(11)}_{ij_1}a^{(21)}_{ij_1}\cdots a^{(s1)}_{ij_1})(a^{(12)}_{j_1j_2}a^{(22)}_{j_1j_2}\cdots a^{(s2)}_{j_1j_2})\cdots(a^{(1t)}_{j_{t-1}j}a^{(2t)}_{j_{t-1}j}\cdots a^{(st)}_{j_1j}).$$

Then

$$\begin{aligned} L &= \sum_{j_1,j_2,\ldots,j_{t-1}} (a^{(11)}_{ij_1}a^{(12)}_{j_1j_2}\cdots a^{(1t)}_{j_{t-1}j})(a^{(21)}_{ij_1}a^{(22)}_{j_1j_2}\cdots a^{(2t)}_{j_{t-1}j})\cdots(a^{(s1)}_{ij_1}a^{(s2)}_{j_1j_2}\cdots a^{(st)}_{j_{t-1}j}) \\ &\le \sum_{j_1,j_2,\ldots,j_{t-1}} (a^{(11)}_{ij_1}a^{(12)}_{j_1j_2}\cdots a^{(1t)}_{j_{t-1}j})\cdots \sum_{j_1,j_2,\ldots,j_{t-1}} (a^{(s1)}_{ij_1}a^{(s2)}_{j_1j_2}\cdots a^{(st)}_{j_{t-1}j}).\end{aligned}$$

Note that for $p=1,\ldots,s$, each term

$$\sum_{j_1,j_2,\ldots,j_{t-1}} (a^{(p1)}_{ij_1}a^{(p2)}_{j_1j_2}\cdots a^{(pt)}_{j_{t-1}j})$$

is the entry in the (i,j)-th position of $A_{p1}A_{p2}\cdots A_{pt}$. This completes the proof. □

It should be noted that its proof of (1.3) is included for the sake of completeness, since some of the proofs below can be proved by using similar ideas. The same proofs were given in [8, Th.3.16] in greater generality for sets of matrices. In the present paper only the special case for usual spectral radius is presented with the same proof.

Proof of Huang's Inequality (1.3). Since the Hadamard product is commutative, we have

$$\begin{aligned}(A_1\circ A_2\circ\cdots\circ A_k)^k &= (A_1\circ A_2\circ\cdots\circ A_k)(A_2\circ\cdots A_k\circ A_1)\cdots(A_k\circ A_1\circ\cdots\circ A_{k-1}) \\ &\le (A_1A_2\cdots A_k)\circ(A_2\cdots A_kA_1)\circ\cdots\circ(A_kA_1\cdots A_{k-1}),\end{aligned}$$

where the inequality follows from Lemma 2.2. By Lemma 2.1, we have

$$\begin{aligned}\rho^k(A_1\circ A_2\circ\cdots\circ A_k) &= \rho((A_1\circ A_2\circ\cdots\circ A_k)^k) \\ &\le \rho((A_1A_2\cdots A_k)\circ(A_2\cdots A_kA_1)\circ\cdots\circ(A_kA_1\cdots A_{k-1})) \\ &\le \rho(A_1A_2\cdots A_k)\rho(A_2\cdots A_kA_1)\cdots\rho(A_kA_1\cdots A_{k-1}) \\ &= \rho^k(A_1A_2\cdots A_k).\end{aligned}$$

In the last equality above we use the fact that

$$\rho(A_2\cdots A_kA_1)=\rho(A_3A_4\cdots A_2)=\cdots=\rho(A_kA_1\cdots A_{k-1})=\rho(A_1A_2\cdots A_k).$$

This completes the proof. □

Denote by $\|A\|$ the spectral norm of $A\in M_n$, which equals to the largest singular value. The following interesting inequality is also due to Huang [6, Corollary 6]. We give a new proof.

Corollary 2.3. [6] *Let $A,B\in M_n$ be nonnegative. Then*

$$\|A\circ B\|\le\rho(A^TB). \tag{2.1}$$

Proof. Note that $\rho(C^TC) = \|C\|^2$. Then

$$\|A \circ B\| = \rho^{\frac{1}{2}}((A^T \circ B^T)(A \circ B)) = \rho^{\frac{1}{2}}((A^T \circ B^T)(B \circ A)).$$

By Lemma 2.1, we have

$$\rho((A^T \circ B^T)(B \circ A)) \leq \rho((A^TB) \circ (B^TA)) \leq \rho(A^TB)\rho(B^TA) = \rho^2(A^TB).$$

This completes the proof. □

The next inequality refines the inequality due to Huang [6]:

$$\|A_1 \circ A_2 \circ \cdots \circ A_k\| \leq \rho^{\frac{1}{2}}(A_1A_1^TA_2A_2^T \cdots A_kA_k^T).$$

Proposition 2.4. *Let $A_1, A_2, \ldots, A_k \in M_n$ be nonnegative matrices. Then*

$$\|A_1 \circ A_2 \circ \cdots \circ A_k\| \leq \rho^{\frac{1}{2}}((A_1A_1^T) \circ (A_2A_2^T) \circ \cdots \circ (A_kA_k^T)).$$

Proof. Note that for any nonnegative square matrix C, $\rho(C^TC) = \|C\|^2$. Then

$$\|A_1 \circ A_2 \circ \cdots \circ A_k\| = \rho^{\frac{1}{2}}((A_1 \circ A_2 \circ \cdots \circ A_k)(A_1 \circ A_2 \circ \cdots \circ A_k)^T).$$

Since

$$(A_1 \circ A_2 \circ \cdots \circ A_k)(A_1 \circ A_2 \circ \cdots \circ A_k)^T = (A_1 \circ A_2 \circ \cdots \circ A_k)(A_1^T \circ A_2^T \circ \cdots \circ A_k^T),$$

by Lemma 2.2, we have

$$(A_1 \circ A_2 \circ \cdots \circ A_k)(A_1^T \circ A_2^T \circ \cdots \circ A_k^T) \leq (A_1A_1^T) \circ (A_2A_2^T) \circ \cdots \circ (A_kA_k^T).$$

Therefore

$$\begin{aligned} \|A_1 \circ A_2 \circ \cdots \circ A_k\| &= \rho^{\frac{1}{2}}((A_1 \circ A_2 \circ \cdots \circ A_k)(A_1^T \circ A_2^T \circ \cdots \circ A_k^T)) \\ &\leq \rho^{\frac{1}{2}}((A_1A_1^T) \circ (A_2A_2^T) \circ \cdots \circ (A_kA_k^T)). \end{aligned}$$

This completes the proof. □

We need the following lemma whose proof can be found in [1, Lemma 1].

Lemma 2.5. [1] *Let $A \in M_n$ be a positive matrix. Then $\rho(A) = \lim_{m\to\infty} (\mathrm{Tr}A^m)^{\frac{1}{m}}$.*

Next, using Lemma 2.5 we prove the inequality which may be regarded as a Cauchy–Schwarz inequality for spectral radius of nonnegative matrices. This inequality is a consequence of a well known result for nonnegative matrices due to Elsner, Johnson and Silva [3], i.e.,

$$\rho(A^{(\frac{1}{2})} \circ B^{(\frac{1}{2})}) \leq \rho(A)^{\frac{1}{2}}\rho(B)^{\frac{1}{2}},$$

where $A^{(t)}$ denotes the Hadamard–Schur power. It should be noted that a new proof is included for the sake of completeness.

Proposition 2.6. *Let $A, B \in M_n$ be nonnegative matrices. Then*

$$\rho(A \circ B) \leq [\rho(A \circ A)]^{\frac{1}{2}}[\rho(B \circ B)]^{\frac{1}{2}}. \tag{2.2}$$

Proof. Without loss of generality we may assume that $A, B \in M_n$ are positive, since the spectral radius is continuous in matrices. Let $A = (a_{ij}), B = (b_{ij}) \in M_n$. For any positive integer k, we have

$$\begin{aligned}
\operatorname{Tr}((A \circ B)^{2k}) &= \sum_{i_1, i_2, \ldots, i_{2k}} (a_{i_1 i_2} b_{i_1 i_2})(a_{i_2 i_3} b_{i_2 i_3}) \cdots (a_{i_{2k} i_1} b_{i_{2k} i_1}) \\
&= \sum_{i_1, i_2, \ldots, i_{2k}} (a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{2k} i_1})(b_{i_1 i_2} b_{i_{2k} i_1} b_{i_2 i_3} \cdots b_{i_{2k} i_1}) \\
&\le \left(\sum_{i_1, i_2, \ldots, i_{2k}} (a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{2k} i_1})^2 \right)^{\frac{1}{2}} \left(\sum_{i_1, i_2, \ldots, i_{2k}} (b_{i_1 i_2} b_{i_2 i_3} \cdots b_{i_{2k} i_1})^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{i_1, i_2, \ldots, i_{2k}} a^2_{i_1 i_2} a^2_{i_2 i_3} \cdots a^2_{i_{2k} i_1} \right)^{\frac{1}{2}} \left(\sum_{i_1, i_2, \ldots, i_{2k}} b^2_{i_1 i_2} b^2_{i_2 i_3} \cdots b^2_{i_{2k} i_1} \right)^{\frac{1}{2}} \\
&= [\operatorname{Tr}(A \circ A)^{2k}]^{\frac{1}{2}} [\operatorname{Tr}(B \circ B)^{2k}]^{\frac{1}{2}}.
\end{aligned}$$

The above inequality follows from the Cauchy–Schwarz inequality. Taking the $2k$-th root, we have

$$[\operatorname{Tr}((A \circ B)^{2k})]^{\frac{1}{2k}} \le [\operatorname{Tr}(A \circ A)^{2k}]^{\frac{1}{4k}} [\operatorname{Tr}(B \circ B)^{2k}]^{\frac{1}{4k}}.$$

Taking the limit $k \to \infty$, and invoking Lemma 2.5 we obtain the inequality (2.2). This completes the proof. □

Remark 2.7. Recall that Audenaert [1] proved

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}((A \circ A)(B \circ B)).$$

We remark that the above inequality and the inequality (2.2) are not comparable in general. Here are two examples. Consider

$$A = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\rho^{\frac{1}{2}}((A \circ A)(B \circ B)) = \sqrt{2} > 0 = [\rho(A \circ A)]^{\frac{1}{2}} [\rho(B \circ B)]^{\frac{1}{2}}.$$

But for

$$A = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$\rho^{\frac{1}{2}}((A \circ A)(B \circ B)) = 0 < \sqrt{2} = [\rho(A \circ A)]^{\frac{1}{2}} [\rho(B \circ B)]^{\frac{1}{2}}.$$

Combining Lemma 2.2 and Proposition 2.6, we obtain the inequality due to Peperko [8].

Corollary 2.8. [8] *Let $A, B \in M_n$ be nonnegative matrices. Then*

$$\rho(A \circ B) \le \rho^{\frac{1}{2}}(AB \circ BA) \le \rho^{\frac{1}{4}}(AB \circ AB) \rho^{\frac{1}{4}}(BA \circ BA) \le \rho(AB).$$

Finally, we interpolate Huang's inequality (1.3) by proving

Theorem 2.9. *Let $A_1, A_2, \ldots, A_k \in M_n$ be nonnegative matrices. Then*

$$\begin{aligned}\rho(A_1 \circ A_2 \circ \cdots \circ A_k) &\le [\rho(A_1A_2\cdots A_k)]^{1-\frac{2}{k}}[\rho((A_1\circ A_1)(A_2\circ A_2)\cdots(A_k\circ A_k)]^{\frac{1}{k}} \\ &\le \rho(A_1A_2\cdots A_k).\end{aligned}$$

Proof. It suffices to prove this theorem for the case of positive matrices. Because of the continuity of the spectral radius, the theorem follows for the case of non-negative matrices as well. Next, we assume that all the matrices $A_1, \ldots, A_k$ are positive.

Denote by $a_{ij}^{(t)}$ the entry of the matrix A_t in the position (i, j), for $t = 1, \ldots, k$. Then

$$\begin{aligned}&\mathrm{Tr}((A_1\circ A_2\circ\cdots\circ A_k)^{mk}) = \mathrm{Tr}[((A_1\circ\cdots A_k)(A_2\circ\cdots\circ A_1)\cdots(A_k\circ\cdots\circ A_{k-1}))^m] \\ &\le \sum_{i_1,i_2,\ldots,i_{mk}} (a^{(1)}_{i_1i_2}a^{(2)}_{i_1i_2}\cdots a^{(k)}_{i_1i_2})(a^{(2)}_{i_2i_3}\cdots a^{(k)}_{i_2i_3}a^{(1)}_{i_2i_3})\cdots(a^{(k)}_{i_{mk}i_1}a^{(1)}_{i_{mk}i_1}\cdots a^{(k-1)}_{i_{mk}i_1}) \\ &= \sum_{i_1,i_2,\ldots,i_{mk}} [(a^{(1)}_{i_1i_2}a^{(2)}_{i_2i_3}\cdots a^{(k)}_{i_{mk}i_1})(a^{(2)}_{i_1i_2}a^{(3)}_{i_2i_3}\cdots a^{(1)}_{i_{mk}i_1}) \\ &\qquad (a^{(3)}_{i_1i_2}a^{(4)}_{i_2i_3}\cdots a^{(2)}_{i_{mk}i_1})\cdots(a^{(k)}_{i_1i_2}a^{(1)}_{i_2i_3}\cdots a^{(k-1)}_{i_{mk}i_1})] \\ &\le \sum_{i_1,i_2,\ldots,i_{mk}} (a^{(1)}_{i_1i_2}a^{(2)}_{i_2i_3}\cdots a^{(k)}_{i_{mk}i_1})(a^{(2)}_{i_1i_2}a^{(3)}_{i_2i_3}\cdots a^{(1)}_{i_{mk}i_1}) \\ &\qquad \sum_{i_1,i_2,\ldots,i_{mk}} (a^{(3)}_{i_1i_2}a^{(4)}_{i_2i_3}\cdots a^{(2)}_{i_{mk}i_1})\cdots \sum_{i_1,i_2,\ldots,i_{mk}} (a^{(k)}_{i_1i_2}a^{(1)}_{i_2i_3}\cdots a^{(k-1)}_{i_{mk}i_1}).\end{aligned}$$

Note that

$$\begin{aligned}&\sum_{i_1,i_2,\ldots,i_{mk}} (a^{(3)}_{i_1i_2}a^{(4)}_{i_2i_3}\cdots a^{(2)}_{i_{mk}i_1})\cdots \sum_{i_1,i_2,\ldots,i_{mk}} (a^{(k)}_{i_1i_2}a^{(1)}_{i_2i_3}\cdots a^{(k-1)}_{i_{mk}i_1}) \\ &= \mathrm{Tr}((A_3A_4\cdots A_2)^m)\mathrm{Tr}((A_4A_5\cdots A_3)^m)\cdots\mathrm{Tr}((A_kA_1\cdots A_{k-1})^m).\end{aligned}$$

The expression $\sum_{i_1,i_2,\ldots,i_{mk}} (a^{(1)}_{i_1i_2}a^{(2)}_{i_2i_3}\cdots a^{(k)}_{i_{mk}i_1})(a^{(2)}_{i_1i_2}a^{(3)}_{i_2i_3}\cdots a^{(1)}_{i_{mk}i_1})$ is the standard Euclidean inner product of two nonnegative vectors in $\mathrm{R}^{n^{mk}}$, one with components $a^{(1)}_{i_1i_2}a^{(2)}_{i_2i_3}\cdots a^{(k)}_{i_{mk}i_1}$ and the other with components $a^{(2)}_{i_1i_2}a^{(3)}_{i_2i_3}\cdots a^{(1)}_{i_{mk}i_1}$. One sees that these two vectors have the same sets of components (which can be seen by performing a cyclic permutation on the indices $i_1, i_2, \ldots, i_{mk}$). Thus both vectors have the same Euclidean norm. Applying the Cauchy–Schwarz inequality then gives $\sum_{i_1,i_2,\ldots,i_{mk}} (a^{(1)}_{i_1i_2}a^{(2)}_{i_2i_3}\cdots a^{(k)}_{i_{mk}i_1})(a^{(2)}_{i_1i_2}a^{(3)}_{i_2i_3}\cdots a^{(1)}_{i_{mk}i_1})$

$$\le \sum_{i_1,i_2,\ldots,i_{mk}} ((a^{(1)}_{i_1i_2})^2(a^{(2)}_{i_2i_3})^2\cdots(a^{(k)}_{i_{mk}i_1})^2) = \mathrm{Tr}[((A_1\circ A_1)(A_2\circ A_2)\cdots(A_k\circ A_k))^m].$$

This proves that the following inequality holds for any positive integer m:

$$\begin{aligned}&\mathrm{Tr}((A_1\circ A_2\circ\cdots\circ A_k)^{mk}) \\ &\qquad \le \mathrm{Tr}((A_3A_4\cdots A_2)^m)\mathrm{Tr}((A_4A_5\cdots A_3)^m)\cdots\mathrm{Tr}((A_kA_1\cdots A_{k-1})^m) \\ &\qquad\quad \times\mathrm{Tr}[((A_1\circ A_1)(A_2\circ A_2)\cdots(A_k\circ A_k))^m].\end{aligned}$$

Taking the mk-th root on the above inequality, taking the limit $m \to \infty$, we have

$$\begin{aligned}\rho(A_1\circ A_2\circ\cdots\circ A_k) &\le \rho^{\frac{1}{k}}(A_3A_4\cdots A_2)\rho^{\frac{1}{k}}(A_4A_5\cdots A_3)\cdots\rho^{\frac{1}{k}}(A_kA_1\cdots A_{k-1}) \\ &\quad \times[\rho((A_1\circ A_1)(A_2\circ A_2)\cdots(A_k\circ A_k)]^{\frac{1}{k}}.\end{aligned}$$

Using the fact

$$\rho(A_3A_4\cdots A_2) = \rho(A_4A_5\cdots A_3) = \cdots = \rho(A_kA_1\cdots A_{k-1}) = \rho(A_1A_2\cdots A_k),$$

we obtain the first inequality.

On the other hand, by Lemma 2.2,

$$(A_1 \circ A_1)(A_2 \circ A_2)\cdots(A_k \circ A_k) \leq (A_1A_2\cdots A_k) \circ (A_1A_2\cdots A_k).$$

Applying Lemma 2.1 we have

$$[\rho\left((A_1 \circ A_1)(A_2 \circ A_2)\cdots(A_k \circ A_k)\right)]^{\frac{1}{k}} \leq \rho^{\frac{2}{k}}\left((A_1A_2\cdots A_k)\right).$$

This completes the proof. □

Remark 2.10. The inequality (1.1) corresponds to the case $k = 2$ of Theorem 2.9.

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