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MORITA EQUIVALENCE OF HILBERT C^*-MODULES

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Abstract. We give a direct definition of Morita equivalence for Hilbert C^*-modules, by introducing an explicit list of axioms for an imprimitivity bimodule. We show that Hilbert C^*-modules with unit vectors, over Morita equivalent unital C^*-algebras, are Morita equivalent, and Morita equivalence is an equivalence relation in the category of left Hilbert C^*-modules with unit vectors.

1. Introduction

The notion of Morita equivalence of C^*-algebras was first introduced by Rieffel [5]. Two C^*-algebras A and B are Morita equivalent if there exist a full Hilbert left A and right B module such that the module actions commute with inner products. This module is called an A-B-imprimitivity bimodule. Morita equivalence preserves some properties of C^*-algebras but is weaker than C^*-isomorphism. Also two unital C^*-algebras are Morita equivalent if and only if they are Morita equivalent as rings [1]. Skeide introduced a notion of Morita equivalence between Hilbert C^*-modules in [7], where two Hilbert C^*-modules E and F over C^*-algebras A and B, respectively, are said to be Morita equivalent if there exist an A-B-imprimitivity bimodule M, such that $E \otimes M = F$. Two full Hilbert C^*-modules E and F are Morita equivalent in the sense of Skeide, if and only if the C^*-algebras $K_A(E)$ and $K_B(F)$ are isomorphic [6]. If two C^*-algebras A and B are Morita equivalent as Hilbert C^*-modules over themselves, they will

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be Morita equivalent as C^*-algebras, but the converse is not true. In [3], Joiţa and Moslehian introduced a notion of Morita equivalence for Hilbert C^*-modules, based on the equivalence of the corresponding C^*-algebras of compact operators, which in case of full countably generated Hilbert C^*-modules over σ-unital C^*-algebras coincides with Skeide's definition of stable Morita equivalence. In this paper, we introduce a direct and constructive notion of Morita equivalence for Hilbert C^*-modules, based on the notion of imprimitivity bimodules.

The main advantage of our definition is that we give a explicit list of axioms for the imprimitivity bimodule (like the original case of C^*-algebras). However there are certain drawbacks: we could show that our notion is an equivalence relation only for Hilbert C^*-modules with unit elements (Proposition 2.12). In this case, we show that the three notions of Morita equivalence coincide.

2. Morita Equivalence

We give an explicit list of axioms to define imprimitivity bimodules for Morita equivalent Hilbert C^*-modules (compare with [7] and [3]).

Definition 2.1. Let A and B be C^*-algebras, and E and F be left and right Hilbert C^*-modules over C^*-algebras A and B, respectively. We say that E and F are Morita equivalent, and write $E \sim_{Mor} F$, if there exist an A-B imprimitivity bimodule X, such that the following holds:

(i) X is a left E-module and a right F-module such that all the compatibility conditions hold, for example

$$(a.e).x = a.(e.x), \quad x.(f.b) = (x.f).b, \quad (a \in A, b \in B, e \in E, f \in F, x \in X).$$

(ii) There exist bilinear maps ${}_E\langle , \rangle : X \times X \to E$ and $\langle , \rangle_F : X \times X \to F$, linear with respect to the second variable and conjugate linear with respect to the first variable, such that

$${}_E\langle a.x, y\rangle = a.{}_E\langle x, y\rangle, \quad \langle x, y.b\rangle_F = \langle x, y\rangle_F.b, \quad (x, y \in X, a \in A, b \in B).$$

(iii) For each $x, y, z \in X$,

$${}_E\langle x, y\rangle.z = x.\langle y, z\rangle_F$$

(iv) For each $x \in X, e \in E$ and $f \in F$,

$$|\langle e.x, e.x\rangle_F| \le \|e\|^2 |\langle x, x\rangle_F|, \quad |{}_E\langle x.f, x.f\rangle| \le \|f\|^2 |{}_E\langle x, x\rangle|,$$

where in the left-hand sides, the absolute values are in B and A, respectively (for instance, for $a \in F$, $|a| = \langle a, a\rangle_B^{1/2} \in B$).

(v) For $F_0 = \langle X, X\rangle_F$ and $E_0 =_E \langle X, X\rangle$, $\langle F_0, F_0\rangle_B$ is dense in B and $\langle E_0, E_0\rangle_A$ is dense in A.

(vi) For $x, y \in X$,

$$|{}_E\langle x, y\rangle|^2 \le \|{}_E\langle x, x\rangle\| |{}_E\langle y, y\rangle|, \quad |\langle x, y\rangle_F|^2 \le \|\langle x, x\rangle_F\| |\langle y, y\rangle_F|.$$

(vii) For $x, y, z, w \in X$,

$$\langle {}_E\langle x, y\rangle.z, w\rangle_F = \langle z, {}_E\langle y, x\rangle.w\rangle_F, \quad {}_E\langle x.\langle y, z\rangle_F, w\rangle =_E \langle x, w.\langle z, y\rangle_F\rangle.$$

In this case, X is called an E-F-imprimitivity bimodule.

From condition (v), it follows that Morita equivalent Hilbert C^*-modules are full. Moreover, if E and F are Morita equivalent, then the C^*-algebras A and B are Morita equivalent.

Now for $x \in X$, let

$$\|x\|_E := \|_E\langle x,x\rangle\|_E^{1/2}, \quad \|x\|_F := \|\langle x,x\rangle_F\|_F^{1/2}.$$

We claim that these define seminorms on X. We should only check the triangle inequality.

Lemma 2.2. *For $e \in E$, we have $\|e\|_E = \||e|\|_A$.*

Proof. For each $e \in E$, $\|e\|_E^2 = \|_A\langle e,e\rangle\| = \||e|^2\|_A = \||e|\|_A^2$. □

Proposition 2.3. *For $x, y \in X$, $\|x+y\|_E \le \|x\|_E + \|y\|_E$.*

Proof. Given $x, y \in X$, we have

$$\begin{aligned}\|x+y\|_E^2 &= \|_E\langle x+y, x+y\rangle\|_E \\ &\le \|_E\langle x,x\rangle\|_E + \|_E\langle y,y\rangle\|_E + \||_E\langle x,y\rangle|^2\|_A^{1/2} + \||_E\langle y,x\rangle|^2\|_A^{1/2}.\end{aligned}$$

By (vi), $|_E\langle x,y\rangle|^2 \le \|_E\langle x,x\rangle\||_E\langle y,y\rangle|$. Hence

$$\|_E\langle x,y\rangle\|_E^2 = \||_E\langle x,y\rangle|^2\|_A \le \|_E\langle x,x\rangle\|_E.\||_E\langle y,y\rangle|\|_A = \|x\|_E\|y\|_E.$$

Therefore, $\|x+y\|_E^2 \le \|x\|_E^2 + \|y\|_E^2 + 2\|x\|_E\|y\|_E = (\|x\|_E + \|y\|_E)^2$. □

We have defined equivalence of a left and a right Hilbert C^*-module. If we define the equivalence between two left (or two right) Hilbert C^*-modules, this yields an equivalence relation. If E is a right Hilbert A-module, then E is a left Hilbert A-module with the same inner product and the left module action $a.e = e.a^*$ and $\lambda.e = e.\overline{\lambda}$, $(\lambda \in \mathbb{C}, e \in E, a \in A)$. We denote this module by $\widetilde{E}$.

Definition 2.4. Two left Hilbert C^*-modules E and F over C^*-algebras A and B are Morita equivalent if $E \sim_{Mor} \widetilde{F}$. In this case, we write $E \approx_{Mor} F$.

For C^*-algebras A and B, it is clear that A and B are Morita equivalent as C^*-algebras if and only if A and B are Morita equivalent as Hilbert C^*-modules over themselves, in the sense of Definition 2.1.

As discussed earlier, Morita equivalent Hilbert C^*-modules in the sense of Definition 2.4 are automatically full. On the other hand, in the category of full Hilbert C^*-modules, two Hilbert C^*-modules E and F for C^*-algebras A and B are Morita equivalent in the sense of [3] if and only if the C^*-algebras A and B are Morita equivalent [3, Proposition 2.8]. Therefore, the notion of Morita equivalence in the sense Definition 2.4 is stronger than the notion of Morita equivalence introduced in [3]. In terms of imprimitivity bimodules, for equivalence of E and F, the authors in [3] require the existence of an A-B imprimitivity bimodule X, but we also require X to have compatible module structures over E and F and to be an E-F imprimitivity bimodule.

Proposition 2.5. *Let E_1 and E_2 be two left Hilbert C^*-modules over C^*-algebras A_1 and A_2, and let F_1 and F_2 be right Hilbert C^*-modules over C^*-algebras B_1 and B_2. If $E_1 \sim_{Mor} F_1$ and $E_2 \sim_{Mor} F_2$ then $E_1 \otimes E_2 \sim_{Mor} F_1 \otimes F_2$.*

Proof. Let X_1 and X_2 be E_1-F_1 and E_2-F_2 imprimitivity bimodules, respectively. Then $X_1 \otimes X_2$ is an $A_1 \otimes_{\min} A_2$-$B_1 \otimes_{\min} B_2$ imprimitivity bimodule. Moreover, $X_1 \otimes X_2$ is a left Banach $E_1 \otimes E_2$-module and right Banach $F_1 \otimes F_2$-module with $(e_1 \otimes e_2)(x_1 \otimes x_2) = e_1x_1 \otimes e_2x_2$, and $(x_1 \otimes x_2)(f_1 \otimes f_2) = x_1f_1 \otimes x_2f_2$, respectively. The compatibility conditions are easily verified.

Define ${}_{E_1\otimes E_2}\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = {}_{E_1}\langle x_1, y_1\rangle \otimes_{E_2}\langle x_2, y_2\rangle$ and $\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle_{F_1\otimes F_2}$ $= \langle x_1, y_1\rangle_{F_1} \otimes \langle x_2, y_2\rangle_{F_2}$. Then all the axioms are satisfied. □

If F_1 and F_2 are right Hilbert C^*-modules over C^*-algebras B_1 and B_2, then the left Hilbert C^*-modules $\widetilde{F_1 \otimes F_2}$ and $\widetilde{F_1} \otimes \widetilde{F_2}$ can be identified. Using this fact we have the following corollary.

Corollary 2.6. *Let E_1, E_2 F_1 and F_2 be left Hilbert C^*-modules over C^*-algebras A_1, A_2, B_1 and B_2. If $E_1 \approx_{Mor} F_1$ and $E_2 \approx_{Mor} F_2$ then $E_1 \otimes E_2 \approx_{Mor} F_1 \otimes F_2$.*

Definition 2.7. Let E be a left Hilbert C^*-module over a unital C^*-algebra A. An element $e \in E$ is called a unit vector if ${}_A\langle e, e\rangle = 1_A$, and similarly for right Hilbert modules.

Theorem 2.8. *If unital C^*-algebras A and B are Morita equivalent, and E and F are left and right Hilbert C^*-modules on A and B, with unit vectors e_0 and f_0, respectively, then $E \sim_{Mor} F$.*

Proof. Suppose that X is an A-B-imprimitivity bimodule. We claim that X is an E-F-imprimitivity bimodule. For this, first, we show that X is a left E-module and a right F-module.

For $e \in E$ and $x \in X$, let $e.x =_A \langle e, e_0\rangle.x$. This module action has compatibility conditions, for example

$$(a.e).x =_A \langle a.e, e_0\rangle.x = a_A\langle e, e_0\rangle.x = a.(e.x).$$

Similarly, we can define module actions for F using f_0. We define an E-valued inner product for X by

$${}_E\langle x, y\rangle =_A \langle x, y\rangle.e_0, \quad (x, y \in X, e \in E).$$

We have

$${}_E\langle x, y\rangle.z = ({}_A\langle x, y\rangle.e_0).z =_A \langle x, y\rangle.(e_0.z) = ({}_A\langle x, y\rangle_A\langle e_0, e_0\rangle).z =_A \langle x, y\rangle.z,$$

and

$$x.\langle y, z\rangle_F = x.(f_0.\langle y, z\rangle_B) = (x.f_0).\langle y, z\rangle_B = (x.\langle f_0, f_0\rangle_B).\langle y, z\rangle_B = x.\langle y, z\rangle_B,$$

which gives (iii). For (iv),

$$\langle e.x, e.x\rangle_F = f_0.\langle e.x, e.x\rangle_B = f_0.\langle {}_A\langle e, e_0\rangle.x, {}_A\langle e, e_0\rangle.x\rangle_B.$$

Let $b = \langle {}_A\langle e, e_0\rangle.x, {}_A\langle e, e_0\rangle.x\rangle_B$ and $a = {}_A\langle e, e_0\rangle$. Then

$$\begin{aligned}|\langle e.x, e.x\rangle_F|_B^2 &= \langle f_0.b, f_0.b\rangle_B = b^*\langle f_0, f_0\rangle_B b = b^*b = |b|^2\\ &= |\langle {}_A\langle e, e_0\rangle.x, {}_A\langle e, e_0\rangle.x\rangle_B|^2 = |\langle a.x, a.x\rangle_B|^2\\ &\leq \|a\|^4|\langle x, x\rangle_B|^2 = \|{}_A\langle e, e_0\rangle\|^4|\langle x, x\rangle_B|^2\\ &= \|{}_A\langle e, e_0\rangle^*_A\langle e, e_0\rangle\|^2|\langle x, x\rangle_B|^2\\ &\leq \|{}_A\langle e, e\rangle\|^2.\|{}_A\langle e_0, e_0\rangle\|^2.|\langle x, x\rangle_B|^2 = \|e\|^4|\langle x, x\rangle_B|^2.\end{aligned}$$

On the other hand,

$$|\langle x, x\rangle_F|^2 = \langle f_0.\langle x, x\rangle_B, f_0.\langle x, x\rangle_B\rangle_B = \langle x, x\rangle_B^*.1_B.\langle x, x\rangle_B = |\langle x, x\rangle_B|^2.$$

For (vi),

$$\begin{aligned}|{}_E\langle x, y\rangle|^2 &= {}_A\langle {}_E\langle x, y\rangle, {}_E\langle x, y\rangle\rangle = {}_A\langle e_0.{}_A\langle x, y\rangle, e_0.{}_A\langle x, y\rangle\rangle\\ &= {}_A\langle x, y\rangle^*_A\langle e_0, e_0\rangle_A\langle x, y\rangle \leq \|{}_A\langle x, x\rangle\|_A\langle y, y\rangle.\end{aligned}$$

On the other hand, $\|{}_E\langle x, x\rangle\| = \||{}_E\langle x, x\rangle|\|_A$, and

$$|{}_E\langle x, x\rangle|_A^2 = {}_A\langle {}_A\langle x, x\rangle.e_0, {}_A\langle x, x\rangle.e_0\rangle = {}_A\langle x, x\rangle_A\langle e_0, e_0\rangle_A\langle x, x\rangle^* = |{}_A\langle x, x\rangle|^2,$$

and (vi) follows.

For (vii), we have

$$\begin{aligned}\langle {}_E\langle x, y\rangle.z, w\rangle_F &= \langle {}_A\langle x, y\rangle.e_0.z, w\rangle_F = \langle {}_A\langle x, y\rangle_A\langle e_0, e_0\rangle.z, w\rangle_F\\ &= \langle {}_A\langle x, y\rangle.z, w\rangle_F = \langle z, {}_A\langle y, x\rangle.w\rangle_F.\end{aligned}$$

This completes the proof. □

Clearly, if f_0 is a unit vector for the right Hilbert C^*-module F over a unital C^*-algebra B, then f_0 is a unit vector for the left Hilbert C^*-module $\widetilde{F}$ over B.

Corollary 2.9. *Let E and F be left Hilbert C^*-modules over the unital C^*-algebras A and B, with unit vectors. Then A and B are Morita equivalent, as C^*-algebras if and only if E and F are Morita equivalent, as Hilbert C^*-modules.*

Corollary 2.10. *Any two left Hilbert C^*-modules E and F over a unital C^*-algebra A, with unit vectors, are Morita equivalent. In particular, $E \approx_{Mor} A$, where A is considered as a left Hilbert A-module.*

Corollary 2.11. *Every two Hilbert spaces are Morita equivalent as Hilbert $\mathbb{C}$-modules.*

It follows from Theorem 2.8 that in the category of full Hilbert C^*-modules with unit vectors over unital C^*-algebras, the notion of Morita equivalence in the sense of Definition 2.4 coincides with Morita equivalence in the sense of [3]. Also, in the category of full countably generated Hilbert C^*-modules with unit vectors over unital C^*-algebras, these two notions coincide with the Skeide's notion of stable Morita equivalence [6].

Proposition 2.12. *In the category of full left Hilbert C^*-modules with unit vectors over unital C^*-algebras, Morita equivalence is an equivalence relation.*

Proof. Let E and F and G be full left Hilbert C^* -modules having unit vectors over the unital C^*-algebras A and B and C, respectively. By Corollary 2.9, $E \approx_{Mor} E$. If $E \approx_{Mor} F$, then A is Morita equivalent to B, and since the Morita equivalence of C^*-algebras is an equivalence relation, B is Morita equivalent to A and by Corollary 2.9, $F \approx_{Mor} E$. Transitivity follows similarly. □

Example 2.13. (i) For unital C^*-algebras A, B, and Hilbert spaces H, K, let $E = A \otimes H$ and $F = K \otimes B$, then E and F are left and right Hilbert C^*-modules on A and B, respectively, with module actions

$$a.(a' \otimes h) = aa' \otimes h, \quad (k \otimes b').b = k \otimes b'b, \quad (a, a' \in A, b, b' \in B, h \in H, k \in K).$$

and inner products

$${}_E\langle a \otimes h, a' \otimes h' \rangle = \langle h, h' \rangle aa'^*, \quad \langle k \otimes b, k' \otimes b' \rangle_F = \langle k, k'^* b'.$$

Choose a vector $h \in H$ of norm one, and let $e_0 = 1 \otimes h$. Then

$${}_E\langle e_0, e_0 \rangle =_E \langle 1 \otimes h, 1 \otimes h \rangle = \|h\|^2 1 = 1.$$

Similarly, we can find $f_0 \in F$ with $\langle f_0, f_0 \rangle_F = 1$. Hence if A and B are Morita equivalent, then $A \otimes H \sim_{Mor} K \otimes B$.

(ii) For a compact topological space X and a C^*-algebra A, $E = C(X, A)$ is a left Hilbert $C(X)$-module with module actions

$$(f.g)(x) = f(x)g(x) \quad (f \in C(X), g \in E)$$

and inner product

$${}_{C(X)}\langle g, g' \rangle(x) = \varphi(g(x)g'^*) \quad (g, g' \in E, x \in X),$$

where φ is a fixed bounded positive linear functional on A. Choose $a \in A$ with $\varphi(aa^*) = 1$. Let $g_0 \in E$ be the constant function with value a. Then for each $x \in X$, $\varphi(g_0(x)g_0(x)^*) = 1$. Hence ${}_{C(X)}\langle g_0, g_0 \rangle = 1$. Therefore, if X and Y are homeomorphic, then for any C^*-algebras A and B, $C(X, A) \sim_{Mor} C(Y, B)$, as left and right Hilbert C^*-modules over $C(X)$ and $C(Y)$, respectively.

Proposition 2.14. *If A is a C^*-algebra, and E, F are left and right Hilbert A-modules such that there exist $e_0 \in E, f_0 \in F$, with $0 \neq_A \langle e_0, e_0 \rangle = \langle f_0, f_0 \rangle_A = t_0 \in Z(A)$, and $\overline{At_0} = A$, then $E \sim_{Mor} F$.*

Proof. Let ${}_A\langle e_0, e_0 \rangle = t_0$. Without loss of generality, we may assume that $\|t_0\| = 1$. We know that A is an A-A-imprimitivity bimodule, with inner products ${}_A\langle a, b \rangle = ab^*$ and $\langle a, b \rangle_A = a^*b$. We claim that A is also an E-F-imprimitivity bimodule. Define the module actions by

$$e.x =_A \langle e, e_0 \rangle x, \quad x.f = x\langle f, f_0 \rangle_A, \quad (x \in A, e \in E, f \in F).$$

and the E-valued and F-valued inner products by

$${}_E\langle a, b \rangle =_A \langle a, b \rangle.e_0, \quad \langle a, b \rangle_F = f_0.\langle a, b \rangle_A \quad (a, b \in A).$$

For each $x, y, z \in A$, we have

$${}_E\langle x, y \rangle.z =_A \langle x, y \rangle.e_0.z = xy^*.e_0.z = xy^*_A\langle e_0, e_0 \rangle z = xy^* t_0 z,$$

$$x.\langle y, z \rangle_F = x.f_0.\langle y, z \rangle_A = x.f_0.y^*z = xt_0y^*z = xy^*t_0z.$$

For (iv), let $e \in E, x \in A$ and let ${}_A\langle e, e_0\rangle = t$. Then

$$\begin{aligned}\langle e.x, e.x\rangle_F &= f_0.\langle e.x, e.x\rangle_A = f_0.(e.x)^*(e.x)\\ &= f_0.({}_A\langle e, e_0\rangle x))^*({}_A\langle e, e_0\rangle x) = f_0.(x^*t^*tx).\end{aligned}$$

In particular, for $a = x^*t^*tx$,

$$|\langle e.x, e.x\rangle_F|^2 = \langle f_0.a, f_0.a\rangle_A = a^*\langle f_0, f_0\rangle_A a = a^*t_0a = x^*t^*txt_0x^*t^*tx.$$

By Cauchy-Schwartz inequality,

$$t^*t =_A \langle e, e_0\rangle_A^* \langle e, e_0\rangle \leq \|{}_A\langle e, e\rangle\|_A\langle e_0, e_0\rangle = \|e\|^2t_0.$$

Hence

$$|\langle e.x, e.x\rangle_F|^2 \leq x^*\|e\|^2t_0xt_0x^*\|e\|^2t_0xt_0 \leq \|e\|^4\|t_0\|^2x^*xt_0x^*x.$$

On the other hand, $\langle x, x\rangle_F = f_0.\langle x, x\rangle_A = f_0.x^*x$. Therefore,

$$|\langle x, x\rangle_F|^2 = \langle f_0.x^*x, f_0.x^*x\rangle_A = x^*xt_0x^*x$$

and the result follows.

For (vi), we have

$$\begin{aligned}|{}_E\langle x, y\rangle|^2 &= |{}_A\langle x, y\rangle.e_0|^2 = |xy^*.e_0|^2 =_A \langle xy^*.e_0, xy^*.e_0\rangle = xy^*t_0yx^*\\ &= (xt_0^{1/4}y^*)(yt_0^{1/4}x^*)t_0^{1/2} = (ay^*)(ya^*)t_0^{1/2} \quad (a := xt_0^{1/4})\\ &=_A \langle a, y\rangle_A\langle a, y\rangle^* t_0^{1/2} \leq \|{}_A\langle a, a\rangle\|_A\langle y, y\rangle t_0^{1/2}\\ &= \|xt_0^{1/4}\|^2yy^*t_0^{1/2} = \|xt_0^{1/2}x^*\|yy^*t_0^{1/2}.\end{aligned}$$

On the other hand, ${}_E\langle x, x\rangle =_A \langle x, x\rangle.e_0$, thus

$$\begin{aligned}\|{}_E\langle x, x\rangle\| &= \|{}_A\langle {}_A\langle x, x\rangle.e_0, {}_A\langle x, x\rangle.e_0\rangle\|^{1/2}\\ &= \|{}_A\langle xx^*.e_0, xx^*.e_0\rangle\|^{1/2}\\ &= \|xx^*t_0xx^*\|^{1/2} = \|xt_0^{1/2}x^*\|,\end{aligned}$$

and

$$|{}_E\langle y, y\rangle| = |{}_A\langle yy^*.e_0, yy^*.e_0\rangle|^{1/2} = (yy^*t_0yy^*)^{1/2} = yy^*t_0^{1/2},$$

from which (vi) follows. For (vii),

$$\langle {}_E\langle x, y\rangle.z, w\rangle_F = f_0.\langle {}_A\langle x, y\rangle.z, w\rangle_A = f_0.\langle xy^*t_0z, w\rangle_A = f_0.z^*t_0^*yx^*w = f_0.z^*yx^*t_0w$$

and

$$\langle z, {}_E\langle y, x\rangle.w\rangle_F = f_0.\langle z, {}_A\langle y, x\rangle.e_0.w\rangle_A = f_0.\langle z, yx^*t_0w\rangle_A = f_0.z^*yx^*t_0w.$$

This completes the proof. □

The condition $\overline{At_0} = A$, in the above proposition, is necessary in order to get the condition (v) of the Definition 2.1. When this is not satisfied, one may define

$$E_0 = \overline{\{t_0.e : e \in E\}}, \quad F_0 = \overline{\{t_0.f : f \in F\}},$$

and observe that E_0 and F_0 are Hilbert $\overline{At_0}$-modules, and similar to the above argument, show that At_0 is a E_0-F_0-imprimitivity bimodule, therefore $E_0 \sim_{Mor} F_0$.

Definition 2.15. [2] Let A and B be C^*-algebras and E be a left Hilbert A-module and F be a right Hilbert B-module. We say that E and F are isomorphic if there is a bijective map $\Phi : E \to F$ and a C^*-isomorphism $\varphi : A \to B$ such that $\langle \Phi(e_2), \Phi(e_1)\rangle_B = \varphi\left({}_A\langle e_1, e_2\rangle\right)$ for all $e_1, e_2 \in E$.

Proposition 2.16. *Let A and B be σ-unital C^*-algebras, E a left A-Hilbert C^*-module and F a right B-Hilbert C^*-module. If E and F are isomorphic and there is a strictly positive element $t_0 =_A \langle e_0, e_0\rangle \in Z(A)$, then $E \sim_{Mor} F$.*

Proof. If $t_0 =_A \langle e_0, e_0\rangle \in Z(A)$ is a strictly positive element in A then

$$\varphi(t_0) = \langle \Phi(e_0), \Phi(e_0)\rangle_B \in Z(B)$$

is a strictly positive element in B.

Since A and B are isomorphic, A and B are Morita equivalent and A is an A-B-imprimitivity bimodule, with the bimodule structure $a.x = ax$ and $x.b = x\varphi^{-1}(b)$ and inner products ${}_A\langle x_1, x_2\rangle = x_1 x_2^*$ and $\langle x_1, x_2\rangle_B = \varphi(x_1^* x_2)$. Similar to the proof of Proposition 2.14, one can show that A is an E-F imprimitivity bimodule, hence $E \sim_{Mor} F$. □

The assumption on the existence of the strictly positive element t_0 in the above proposition can not be dropped, even if A and B are commutative and unital. For example, let $X = Y \cup Z$ be a compact space, where Y and Z are disjoint, non-empty, open subsets of X, which are homeomorphic. Then $C(X)$ is a left and a right $C(X)$-module with inner products

$${}_{C(X)}\langle f, g\rangle = f\bar{g}\chi_Y, \quad \langle f, g\rangle_{C(X)} = \bar{f}g\chi_Z, \quad (f, g \in C(X)).$$

In this case, ${}_{C(X)}\langle f, f\rangle$ is supported in Y, for any $f \in C(X)$, hence it could not be a strictly positive element of $C(X)$. Take an isomorphism $\psi : C(Y) \to C(Z)$ and identify $C(X)$ with $C(Y) \oplus C(Z)$. Let $\sigma : C(Y) \oplus C(Z) \to C(Z) \oplus C(Y)$ be the flip isomorphism. In Definition 2.15, put $\Phi := \sigma^{-1} \circ (\psi \oplus \psi^{-1}) : C(X) \to C(X)$ and $\varphi = id$, then $C(X)$, as a left $C(X)$-module, is isomorphic to $C(X)$, as a right $C(X)$-module, but $C(X) \nsim_{Mor} C(X)$, as no imprimitivity bimodule could satisfy condition (v) of Definition 2.4.

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