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HERZ–MORREY TYPE BESOV AND TRIEBEL-LIZORKIN SPACES WITH VARIABLE EXPONENTS

BAOHUA DONG AND JINGSHI XU*

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Abstract. In the article, the boundedness of vector-valued sublinear operators in Herz–Morrey spaces with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ are obtained. Then Herz–Morrey type Besov and Triebel-Lizorkin spaces with variable exponents are introduced. Finally, we prove the equivalent quasi-norms on these spaces by Peetre's maximal operators.

1. Introduction

Recent decades, many attentions are paid to the variable exponent spaces and their applications. Let $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $p(\cdot)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable} : \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} \mathrm{d}x < \infty \text{ for some } \ \lambda > 0 \right\}.$$

It is also a Banach space when it equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0, \int_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leqslant 1 \right\}.$$

The Lebesgue and Sobolev spaces with variable exponent were studied by Kováčik and Rákosník in [20].

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* Corresponding author.
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Let $L^1_{loc}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Given a function $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy–Littlewood maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M}f(x) := \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)|\mathrm{d}y, \ \forall x \in \mathbb{R}^n,$$

where $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$. The boundedness of the Hardy–Littlewood maximal operator plays a great important role in variable exponent spaces. For example, it is well known that many results in classical harmonic analysis and function theory are also hold for the variable exponent case if the Hardy–Littlewood maximal operator is bounded in variable exponent Lebesgue space; see [4, 5, 6, 26]. In addition, many variable exponent spaces are introduced, such as: variable exponent Bessel potential spaces, Besov and Trieble–Lizorkin spaces, Hardy spaces, Herz spaces, Herz–Morrey spaces, Morrey spaces, Morrey type Besov and Trieble–Lizorkin spaces, Trieble–Lizorkin-Morrey spaces, and so on; see [1, 2, 3, 7, 10, 11, 13, 15, 16, 17, 18, 19, 27, 29, 30, 32, 33, 41, 42] and references therein. And lots of results about boundedness of sublinear operators in these spaces have been proved; see [12, 21, 22, 23, 24].

If a sublinear operator T is bounded on $L^{p(\cdot)}$ and obeys the size condition

$$|Tf(x)| \leqslant C \int_{\mathbb{R}^n} |x-y|^{-n}|f(y)|\mathrm{d}y$$

for all $f \in L^1_{loc}(\mathbb{R}^n)$ with compact support and a.e. $x \notin \operatorname{supp} f$, then T is bounded on both of the homogeneous and non-homogeneous Herz space, see the monograph [25] by S. Lu, D. Yang and G. Hu. This result is extended to the weighted vector-valued case by L. Tang and D. Yang in [35]. In [33], C. Shi and the second author introduced Herz type Besov and Triebel-Lizorkin spaces with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)B^s_\beta$ and $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)B^s_\beta$ and $\dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)F^s_\beta$ and $K^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)F^s_\beta$ and obtained their equivalent quasi-norms. For the constant exponent Herz type Besov and Triebel-Lizorkin spaces we refer the reader to [37, 38, 39, 40]. In [14], M. Izuki obtained the vector-valued boundedness for some sublinear operators satisfying the size condition on Herz–Morrey spaces with variable exponent $M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$. And in [8], the authors established the boundedness of vector-valued Hardy–Littlewood maximal operator in Herz spaces with variable exponents and characterized Herz type Besov and Triebel-Lizorkin spaces with variable exponents by Peetre's maximal operators.

Inspired by the works above, the present paper is to consider the boundedness of vector-valued Hardy–Littlewood maximal operator on Herz–Morrey type Besov and Triebel-Lizorkin spaces with variable exponents. The structure of the paper is as follows. In the rest of the section, we give some conventions. In Section 2 we shall give our main results which are the generalization of related results in [14] and [8]. In Section 3 we give proofs of our results.

During the paper, we denote by $|S|$ and χ_S the Lebesgue measure of S and the characteristics function of a measurable set $S \subset \mathbb{R}^n$ respectively. We also use the notation $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leqslant Cb$. If $a \lesssim b$ and $b \lesssim a$ we will write $a \approx b$. Finally we claim that C is always a positive constant but it may change from line to line. Other notations will be explained when we meet it.

2. Main results

In this section, firstly we shall establish the boundedness of sublinear operators in Herz–Morrey spaces with variable exponents. Before stating our result, we need to recall some definitions, notations and a lemma.

Definition 2.1. The local Lebesgue space with variable exponent is defined by

$$L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n) := \{f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n) \ \textit{for all compact subsets } K \subset \mathbb{R}^n\}.$$

We also use the following notation: $p_- := \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\}$ and $p_+ := \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\}$. $\mathcal{P}(\mathbb{R}^n)$ is the set of $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. The set $\mathcal{B}(\mathbb{R}^n)$ consists of all $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the condition that $\mathcal{M}$ is bounded on $L^{p(\cdot)}$. Moreover, we denote by $\mathcal{P}^0(\mathbb{R}^n)$ the set of measurable functions $p(\cdot)$ on $\mathbb{R}^n$ with the range in $(0, \infty]$ such that $p_- := \operatorname{ess\,inf}\{p(x) : x \in \mathbb{R}^n\} > 0$ and $p_+ := \operatorname{ess\,sup}\{p(x) : x \in \mathbb{R}^n\} \leqslant \infty$. Given $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, one can define the space $L^{p(\cdot)}(\mathbb{R}^n)$ by the case $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. For more details, it is equivalent to define the set of all functions f such that $|f|^{p_0} \in L^{\frac{p(\cdot)}{p_0}}(\mathbb{R}^n)$, where $0 < p_0 < p_-$ and $\frac{p(\cdot)}{p_0} \in \mathcal{P}(\mathbb{R}^n)$ with a quasi-norm $\|f\|_{L^{p(\cdot)}} =: \||f|^{p_0}\|^{1/p_0}_{L^{\frac{p(\cdot)}{p_0}}}$.

Definition 2.2. Let $\alpha(\cdot)$ be a real function on $\mathbb{R}^n$.

(i) $\alpha(\cdot)$ is called log-Hölder continuous on $\mathbb{R}^n$ if there exists $C > 0$ such that

$$|\alpha(x) - \alpha(y)| \leqslant \frac{C}{\log(e + 1/|x-y|)}, \ \forall x, y \in \mathbb{R}^n, |x-y| < \frac{1}{2}.$$

(ii) $\alpha(\cdot)$ is called log-Hölder continuous at the origin if there exists $C > 0$ such that

$$|\alpha(x) - \alpha(0)| \leqslant \frac{C}{\log(e + 1/|x|)}, \ \forall x \in \mathbb{R}^n.$$

(iii) $\alpha(\cdot)$ is called log-Hölder continuous at the infinity if there exist $\alpha_\infty \in \mathbb{R}$ and a constant $C > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leqslant \frac{C}{\log(e + |x|)}, \ \forall x \in \mathbb{R}^n.$$

We denote by $\mathcal{P}^{\log}_0(\mathbb{R}^n)$ and $\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ the class of all variable exponents $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ which are log-Hölder continuous at the origin and at the infinity respectively.

For giving the definition of the Herz–Morrey spaces with variable exponents, let us introduce the following notations. Let $k \in \mathbb{Z}$, $B_k := \{x \in \mathbb{R}^n : |x| \leqslant 2^k\}$, $D_k := B_k \backslash B_{k-1}$, $\chi_k := \chi_{D_k}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For any $m \in \mathbb{N}_0$, we denote $\tilde{\chi}_m := \chi_{D_m}, m \geqslant 1$ and $\tilde{\chi}_0 := \chi_{B_0}$ respectively.

Definition 2.3. Let $0 < q < \infty$, $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, $0 \leq \lambda < \infty$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$.

(i) The homogeneous Herz–Morrey space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) := \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} < \infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} := \sup_{L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} \|2^{k\alpha(\cdot)} f\chi_k\|^q_{L^{p(\cdot)}}\right)^{\frac{1}{q}}.$$

(ii) The non-homogeneous Herz–Morrey space $MK^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ with variable exponents is defined by

$$MK^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) := \{f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{MK^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} < \infty\},$$

where

$$\|f\|_{MK^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} := \sup_{L\in\mathbb{N}_0} 2^{-L\lambda}\left(\sum_{k=0}^{L} \|2^{k\alpha(\cdot)} f\chi_k\|^q_{L^{p(\cdot)}}\right)^{\frac{1}{q}}.$$

We want to state that if $\alpha(\cdot)$ is a constant, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = M\dot{K}^{\alpha,\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$ was defined in [14]. If $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(\mathbb{R}^n)$. If both $\alpha(\cdot)$ and $p(\cdot)$ are constant and $\lambda = 0$, then $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n) = \dot{K}^{\alpha}_{q,p}(\mathbb{R}^n)$ is the classical Herz space. We also want to say that there is a analog for the non-homogeneous case.

Lemma 2.1. (see [14, Remark 4.1]) Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$. Then there exist $0 < \delta_1, \delta_2 < 1$ depending only on $p(\cdot)$ and n such that for any ball B in $\mathbb{R}^n$ and any measurable subset $S \subset B$,

$$\frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_1} \quad \text{and} \quad \frac{\|\chi_S\|_{L^{p'(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)}} \leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_2}. \tag{2.1}$$

Now we give the boundedness of vector-valued sublinear operators in Herz–Morrey spaces with variable exponents, which generalizes the result in [14].

Theorem 2.1. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, $1 < r < \infty$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}^{\log}_0(\mathbb{R}^n) \cap \mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ with $\alpha(0), \alpha_\infty \in (-n\delta_1, n\delta_2)$, where $\delta_1, \delta_2 \in (0,1)$ are constants appearing in (2.1) and $0 \leqslant \lambda < \min\{(n\delta_1+\alpha(0))/2, (n\delta_1+\alpha_\infty)/2\}$. Suppose that T is a sublinear operator satisfying vector-valued inequality on $L^{p(\cdot)}(\mathbb{R}^n)$

$$\left\|\left(\sum_{j=1}^{\infty} |Tf_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}} \leqslant C\left\|\left(\sum_{j=1}^{\infty} |f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}} \tag{2.2}$$

for all sequences $\{f_j\}_{j=1}^\infty$ of locally integrable functions on $\mathbb{R}^n$ and size condition

$$|Tf(x)| \leqslant C\int_{\mathbb{R}^n} |x-y|^{-n}|f(y)|dy$$

for all $f \in L^1_{loc}(\mathbb{R}^n)$ and a.e. $x \notin \operatorname{supp} f$. Then we have the vector-valued inequality on $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\mathbb{R}^n)$

$$\left\| \left(\sum_{k=1}^{\infty} |Tf_k|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} \leqslant C \left\| \left(\sum_{k=1}^{\infty} |f_k|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}$$

for all sequences $\{f_j\}_{j=1}^{\infty}$ of locally integrable functions on $\mathbb{R}^n$, where C is independent of $\{f_j\}_{j=1}^{\infty}$.

Remark 2.1. Here and below, we only declare our main results in the homogeneous Herz–Morrey space with variable exponents because the proof for the non-homogeneous case can be treated by the similar way and is much more easier.

Lemma 2.2. (see [4, Corollary 2.1]) Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $1 < r < \infty$, then there exists a positive constant C such that for all sequences $\{f_j\}_{j=1}^{\infty}$ of locally integrable functions on $\mathbb{R}^n$,

$$\left\| \left(\sum_{j=1}^{\infty} |\mathcal{M}f_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}} \leqslant C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^{p(\cdot)}}.$$

From Theorem 2.1 and Lemma 2.2, we obtain the following result for the Hardy–Littlewood maximal operator.

Corollary 2.1. Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $0 < q < \infty$, $1 < r < \infty$, $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $\alpha(0), \alpha_\infty \in (-n\delta_1, n\delta_2)$, where $\delta_1, \delta_2 \in (0,1)$ are constants appearing in (2.1) and $0 \leqslant \lambda < \min\{(n\delta_1 + \alpha(0))/2, (n\delta_1 + \alpha_\infty)/2\}$, then

$$\left\| \left(\sum_{k=1}^{\infty} |\mathcal{M}f_k|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} \leqslant C \left\| \left(\sum_{k=1}^{\infty} |f_k|^r \right)^{\frac{1}{r}} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}$$

with the constant C independent of sequences $\{f_j\}_{j=1}^{\infty}$ of locally integrable functions on $\mathbb{R}^n$.

Next we will use Corollary 2.1 to prove the equivalent quasi-norms in Herz–Morrey type Besov and Triebel-Lizorkin spaces with variable exponents. In order to do it, we need some notations to give the definition of these spaces.

Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ its dual space on $\mathbb{R}^n$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\varphi}$ and $\varphi^\vee$ represent its Fourier transform and inverse Fourier transform respectively. Suppose $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(x) \geq 0$ and

$$\varphi_0(x) = \begin{cases} 1 & , |x| \leqslant 1, \\ 0 & , |x| \geq 2, \end{cases}$$

then we denote $\varphi(x) := \varphi_0(x) - \varphi_0(2x)$ and set $\varphi_j(x) := \varphi(2^{-j}x)$ for all $j \in \mathbb{N}$. Then $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity, namely $\sum_{j=0}^{\infty} \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$.

Now, we introduce the Herz–Morrey type Besov spaces and Triebel-Lizorkin spaces with variable exponents below.

Definition 2.4. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity as above, $s\in\mathbb{R}$, $0<\beta, q\le\infty$, $p(\cdot)\in\mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ with $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$.

(i) The Herz–Morrey type Besov space with variable exponents is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}B^s_\beta(\mathbb{R}^n) := \left\{ f\in\mathcal{S}'(\mathbb{R}^n):\ \|f\|_{\ell_\beta(M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)})}<\infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}B^s_\beta} := \left\| \left\{2^{sj}\varphi_j^\vee * f\right\}_{j=0}^\infty \right\|_{\ell_\beta(M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)})};$$

(ii) For $p_+<\infty$, the Herz-type Triebel-Lizorkin space with variable exponents is defined by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta(\mathbb{R}^n) := \left\{ f\in\mathcal{S}'(\mathbb{R}^n):\ \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\ell_\beta)}<\infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta} := \left\| \left\{2^{sj}\varphi_j^\vee * f\right\}_{j=0}^\infty \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\ell_\beta)}.$$

Here we denote respectively by $\ell_\beta(M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)})$ and $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\ell_\beta)$ the spaces of all sequences $\{g_j\}$ of measurable functions on $\mathbb{R}^n$ with finite quasi-norms

$$\|\{g_j\}_{j=0}^\infty\|_{\ell_\beta(M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)})} := \left(\sum_{j=0}^\infty \|g_j\|^\beta_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}\right)^{\frac{1}{\beta}},$$

and

$$\|\{g_j\}_{j=0}^\infty\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\ell_\beta)} := \left\| \left(\sum_{j=0}^\infty |g_j|^\beta\right)^{1/\beta} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}.$$

In order to make sure these spaces are well defined, we need to prove the definition of these spaces are independent of the choice related to the resolution of unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$. To achieve this, we need more notations.

Let $\Psi_0, \Psi\in\mathcal{S}(\mathbb{R}^n), \varepsilon>0$ and integer $S\geqslant -1$ satisfy

$$|\widehat{\Psi_0}(\xi)|>0 \quad \text{on} \quad \{|\xi|<2\varepsilon\}, \tag{2.3}$$

$$|\widehat{\Psi}(\xi)|>0 \quad \text{on} \quad \{\varepsilon/2<|\xi|<2\varepsilon\}, \tag{2.4}$$

and

$$D^\tau\widehat{\Psi}(0)=0 \quad \text{for all} \quad |\tau|\leqslant S, \tag{2.5}$$

where (2.3) and (2.4) are Tauberian conditions and (2.5) represents vanishing moment conditions on Ψ. If $S=-1$, we need not vanishing moment conditions.

In [28], J. Peetre introduced the classical Peetre's maximal operator

$$(\Psi_k^* f)_a(x) := \sup_{y\in\mathbb{R}^n} \frac{|\Psi_k * f(x+y)|}{(1+2^k|y|)^a}, \qquad x\in\mathbb{R}^n, k\in\mathbb{Z},$$

where the constant $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\{\Psi_k\}_{k\in\mathbb{Z}}$ is a sequence of function satisfying $\{\Psi_k\}_{k\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^n)$.

Because $\Psi_k * f(y)$ make sense pointwise, everything is well-defined. We also use dilates $\Psi_k(x) := 2^{kn}\Psi(2^k x)$ with a fixed function $\Psi \in \mathcal{S}(\mathbb{R}^n)$ and $\Psi_0(x)$ might be given by a separate function. Additionally, continuous expands are needed as well. Let $\Psi_t := t^{-n}\Psi(t^{-1}\cdot)$. We define $(\Psi_t^* f)_a(x)$ by

$$(\Psi_t^* f)_a(x) := \sup_{y\in\mathbb{R}^n} \frac{|\Psi_t * f(x+y)|}{(1+|y|/t)^a} \qquad x \in \mathbb{R}^n, t > 0.$$

Now the theorem below gives the equivalent quasi-norms of these new spaces which illustrated that they are well defined.

Theorem 2.2. Let $\beta, q \in (0,\infty)$, $a \in \mathbb{R}$, $s < S+1$, $0 \leqslant \lambda < \min\{(n\delta_1 + \alpha(0)p_0)/2, (n\delta_1+\alpha_\infty p_0)/2\}$, $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ for $p(\cdot)/p_0 \in \mathcal{B}(\mathbb{R}^n)$ with $p_0 < \min(p_-, 1)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ with $\alpha(0)p_0, \alpha_\infty p_0 \in (-n\delta_1, n\delta_2)$, where $\delta_1, \delta_2 \in (0,1)$ are constants appearing in (2.1) for $p(\cdot)/p_0$. Suppose that Φ_0, Φ belong to $\mathcal{S}(\mathbb{R}^n)$ given by (2.3), (2.4) and (2.5). Then

(i) For $a > n/p_0$, then the space $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} B_\beta^s(\mathbb{R}^n)$ can be characterized by

$$M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} B_\beta^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \ \|f\|^{(i)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} B_\beta^s} < \infty \right\}, \quad i = 1, \cdots, 4,$$

where

$$\begin{aligned}
\|f\|^{(1)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} B_\beta^s} &:= \|\Phi_0 * f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} + \left(\int_0^1 t^{-s\beta} \|\Phi_t * f\|^\beta_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} \frac{\mathrm{d}t}{t} \right)^{1/\beta}, \\
\|f\|^{(2)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} B_\beta^s} &:= \|(\Phi_0^* f)_a\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} + \left(\int_0^1 t^{-s\beta} \|(\Phi_t^* f)_a\|^\beta_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} \frac{\mathrm{d}t}{t} \right)^{1/\beta}, \\
\|f\|^{(3)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} B_\beta^s} &:= \left(\sum_{k=0}^\infty 2^{sk\beta} \|(\Phi_k^* f)_a\|^\beta_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} \right)^{1/\beta}, \\
\|f\|^{(4)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} B_\beta^s} &:= \left(\sum_{k=0}^\infty 2^{sk\beta} \|\Phi_k * f\|^\beta_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} \right)^{1/\beta}.
\end{aligned}$$

Then, $\{\|\cdot\|^{(i)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} B_\beta^s}\}_{i=1}^4$ are equivalent.

(ii) If $p_0 < \beta$, then for $a > n/p_0$ the space $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} F_\beta^s(\mathbb{R}^n)$ can be characterized by

$$M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} F_\beta^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \ \|f\|^{(i)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda} F_\beta^s} < \infty \right\}, \ i = 1, \ldots, 5,$$

where

$$\|f\|^{(1)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta} := \|\Phi_0 * f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} + \left\| \left(\int_0^1 t^{-s\beta} \left|\Phi_t * f\right|^\beta \frac{\mathrm{d}t}{t} \right)^{1/\beta} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}, \tag{2.6}$$

$$\|f\|^{(2)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta} := \|(\Phi_0^* f)_a\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} + \left\| \left(\int_0^1 [t^{-s}(\Phi_t^* f)_a]^\beta \frac{\mathrm{d}t}{t} \right)^{1/\beta} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}, \tag{2.7}$$

$$\|f\|^{(3)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta} := \|\Phi_0 * f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} \left\| \left(\int_0^1 t^{-s\beta} \times \int_{|z|<t} |(\Phi_t * f)(\cdot + z)|^\beta \mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}} \right)^{1/\beta} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}, \tag{2.8}$$

$$\|f\|^{(4)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta} := \left\| \left(\sum_{k=0}^{\infty} [2^{ks\beta}(\Phi_k^* f)_a]^\beta \right)^{1/\beta} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}, \tag{2.9}$$

$$\|f\|^{(5)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta} := \left\| \left(\sum_{k=0}^{\infty} 2^{ks\beta} |\Phi_k * f|^\beta \right)^{1/\beta} \right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}. \tag{2.10}$$

Then, $\{\|\cdot\|^{(i)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta}\}_{i=1}^5$ are equivalent.

It is easy to see that Definition 2.4 is a special case of (2.10). Therefore, under the conditions of Theorem 2.2, those spaces in Definition 2.4 are independent of the choice of the resolution of unity $\{\varphi_j\}_{j\in\mathbb{N}_0}$.

3. Proofs of the main results

In this section, we will prove Theorems 2.1 and 2.2. Firstly, let us begin with Theorem 2.1. To do so, we need the following lemmas.

Lemma 3.1. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q \in (0,\infty)$, $0 \le \lambda < \infty$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, then

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}} \approx \max\left\{ \sup_{L\le 0, L\in\mathbb{Z}} 2^{-L\lambda} \left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}}, \sup_{L>0, L\in\mathbb{Z}} \left[2^{-L\lambda} \left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q} \|f\chi_k\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}} + 2^{-L\lambda} \left(\sum_{k=0}^{L} 2^{k\alpha_\infty q} \|f\chi_k\|^q_{L^{p(\cdot)}} \right)^{\frac{1}{q}} \right] \right\}.$$

Lemma 3.1 is similar to Proposition 3.8 in [1]. Indeed, when $\alpha(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, there exist positive constants C_1, C_2 such that if $k \leq 0$ and $x \in D_k$ then $C_1 2^{k\alpha(0)} \leq 2^{k\alpha(x)} \leq C_2 2^{k\alpha(0)}$; if $k > 1$ and $x \in D_k$ then $C_1 2^{k\alpha_\infty} \leq 2^{k\alpha(x)} \leq C_2 2^{k\alpha_\infty}$. Thus, we obtain Lemma 3.1.

Lemma 3.2. (see [14, Lemma 3.4]) Let $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that for all balls B in $\mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}}\|\chi_B\|_{L^{p'(\cdot)}} \leqslant C.$$

Proof of Theorem 2.1. We only consider sequences $\{f_j\}_{j=1}^\infty$ of locally measurable functions on $\mathbb{R}^n$ such that $(\sum_{k=1}^\infty |f_k|^r)^{\frac{1}{r}} \in M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$. Firstly we use Lemma 3.1 to obtain

$$\begin{aligned}
&\left\|\left(\sum_{j=1}^\infty |Tf_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} \\
&\approx \max\left\{\sup_{L\leq 0, L\in\mathbb{Z}} 2^{-L\lambda}\left(\sum_{k=-\infty}^{L} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^\infty |Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{\frac{1}{q}},\right. \\
&\quad \sup_{L>0, L\in\mathbb{Z}}\left[2^{-L\lambda}\left(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^\infty |Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{\frac{1}{q}}\right. \\
&\quad \left.\left. +2^{-L\lambda}\left(\sum_{k=0}^{L} 2^{k\alpha_\infty q}\left\|\left(\sum_{j=1}^\infty |Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}}^q\right)^{\frac{1}{q}}\right]\right\} \\
&=: \max\{E_T, F_T\}.
\end{aligned}$$

We also denote F_T by $F_T := \sup_{L>0, L\in\mathbb{Z}}[G_T + H_T]$ with

$$G_T := 2^{-L\lambda}\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^\infty |Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}},$$

$$H_T := 2^{-L\lambda}\left\{\sum_{k=0}^{L} 2^{k\alpha_\infty q}\left\|\left(\sum_{j=1}^\infty |Tf_j|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}.$$

Next we need to show estimates $E_T \lesssim E_f$, $G_T \lesssim G_f$ and $H_T \lesssim H_f$ respectively with

$$
\begin{aligned}
E_f &:= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_k\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}},\\
G_f &:= 2^{-L\lambda}\left\{\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}},\\
H_f &:= 2^{-L\lambda}\left\{\sum_{k=0}^{L} 2^{k\alpha_\infty q}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_k\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Then we obtain $E_T \lesssim E_f$ and $F_T \lesssim F_f$ which F_f represents $\sup_{L>0,L\in\mathbb{Z}}[G_f + H_f]$.

From above all and using Lemma 3.1 again, we have

$$
\begin{aligned}
\left\|\left(\sum_{j=1}^{\infty}|Tf_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} &\approx \max\{E_T, F_T\}\\
&\lesssim \max\{E_f, F_f\}\\
&\approx \left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}}.
\end{aligned}
$$

Now let us start to show estimates $E_T \lesssim E_f$, $G_T \lesssim G_f$ and $H_T \lesssim H_f$ respectively. We declare that the proof of $G_T \lesssim G_f$ is similar to $E_{\mathcal{M}} \lesssim E_f$. So we only prove $E_{\mathcal{M}} \lesssim E_f$ and $H_{\mathcal{M}} \lesssim H_f$ respectively. To continue, we let $f_j^i = \chi_i f_j$.

Firstly, we estimate E_T by sublinear of T and Minkowski's inequality.

$$\begin{aligned}
E_T &= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_k\left(\sum_{j=1}^{\infty}|Tf_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_k\left(\sum_{j=1}^{\infty}|T\sum_{i=-\infty}^{\infty} f_j^i|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&\leqslant \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_k\sum_{i=-\infty}^{\infty}\left(\sum_{j=1}^{\infty}|Tf_j^i|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_k\sum_{i=-\infty}^{k-2}\left(\sum_{j=1}^{\infty}|Tf_j^i|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&\quad+\sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_k\sum_{i=k-1}^{k+1}\left(\sum_{j=1}^{\infty}|Tf_j^i|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&\quad+\sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_k\sum_{i=k+2}^{\infty}\left(\sum_{j=1}^{\infty}|Tf_j^i|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&=: E_T^1+E_T^2+E_T^3.
\end{aligned}$$

By the same way we consider H_T.

$$\begin{aligned}
H_T &\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L} 2^{\alpha_\infty qk}\left\|\chi_k\sum_{i=-\infty}^{k-2}\left(\sum_{j=1}^{\infty}|Tf_j^i|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&\quad+2^{-L\lambda}\left\{\sum_{k=0}^{L} 2^{\alpha_\infty qk}\left\|\chi_k\sum_{i=k-1}^{k+1}\left(\sum_{j=1}^{\infty}|Tf_j^i|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&\quad+2^{-L\lambda}\left\{\sum_{k=0}^{L} 2^{\alpha_\infty qk}\left\|\chi_k\sum_{i=k+2}^{\infty}\left(\sum_{j=1}^{\infty}|Tf_j^i|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&=: H_T^1+H_T^2+H_T^3.
\end{aligned}$$

Secondly, we will prove E_T^i and H_T^i, $i=1,2,3$ and divide the proof into three steps.

Step 1. First of all, we consider E_T^2. Since T satisfies (2.2), then

$$
\begin{aligned}
E_T^2 &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\sum_{i=k-1}^{k+1}\left\|\left(\sum_{j=1}^{\infty}|f_j^i|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
&= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\sum_{i=k-1}^{k+1}\left\|\chi_i\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_{k-1}\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
&\quad+ \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_{k}\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
&\quad+ \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_{k+1}\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_{k}\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
&=: E_f.
\end{aligned}
$$

Then turn to H_T^2, similarly, we have

$$
\begin{aligned}
H_T^2 &\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L} 2^{\alpha_\infty qk}\left\|\chi_{k-1}\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
&\quad+ 2^{-L\lambda}\left\{\sum_{k=0}^{L} 2^{\alpha_\infty qk}\left\|\chi_{k}\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
&\quad+ 2^{-L\lambda}\left\{\sum_{k=0}^{L} 2^{\alpha_\infty qk}\left\|\chi_{k+1}\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
&\lesssim H_f.
\end{aligned}
$$

Step 2. We consider E_T^1. Since the sublinear operator T satisfies the size condition

$$
Tf(x) \leqslant C\int_{\mathbb{R}^n}|x-y|^{-n}|f(y)|\mathrm{d}y
$$

for all $f\in L^1_{loc}(\mathbb{R}^n)$ and a.e $x\notin \operatorname{supp} f$.

Then, for $\forall i \leqslant k-2, x \in R_k, 1 < r < \infty$, by the size condition and the generalized Minkowski inequality, we obtain

$$\left(\sum_{j=1}^{\infty} T^r(f_j^i)(x)\right)^{\frac{1}{r}} \lesssim \left[\sum_{j=1}^{\infty}\left(2^{-kn}\int_{\mathbb{R}^n}|f_j^i(y)|\mathrm{d}y\right)^r\right]^{\frac{1}{r}}$$
$$\lesssim 2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^i|^r\right)^{\frac{1}{r}}\mathrm{d}y.$$

By Hölder's inequality we get

$$\begin{aligned} E_T^1 &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\left\|\chi_k\sum_{i=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^i|^r\right)^{\frac{1}{r}}\mathrm{d}y\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}} \\ &= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\|\chi_k\|_{L^{p(\cdot)}}^q\left(\sum_{i=-\infty}^{k-2}2^{-kn}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^i|^r\right)^{\frac{1}{r}}\mathrm{d}y\right)^q\right\}^{\frac{1}{q}} \\ &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)qk}\|\chi_k\|_{L^{p(\cdot)}}^q\right. \\ &\quad\left.\times\left(\sum_{i=-\infty}^{k-2}2^{-kn}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}\|\chi_i\|_{L^{p'(\cdot)}}\right)^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{3.1}$$

On the other hand, by using Lemmas 2.1 and 3.2, we have

$$2^{-kn}\|\chi_k\|_{L^{p(\cdot)}}\|\chi_i\|_{L^{p'(\cdot)}} \lesssim 2^{-kn}\|\chi_{B_i}\|_{L^{p'(\cdot)}}|B_k|\|\chi_{B_k}\|_{L^{p'(\cdot)}}^{-1} \lesssim 2^{n\delta_2(i-k)}. \tag{3.2}$$

We put (3.2) into (3.1) and get

$$\begin{aligned} E_T^1 &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)kq}\left(\sum_{i=-\infty}^{k-2}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}2^{n\delta_2(i-k)}\right)^q\right\}^{\frac{1}{q}} \\ &= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{i=-\infty}^{k-2}2^{\alpha(0)k}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}2^{n\delta_2(i-k)}\right)^q\right\}^{\frac{1}{q}} \\ &= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{i=-\infty}^{k-2}2^{\alpha(0)i}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}2^{b(i-k)}\right)^q\right\}^{\frac{1}{q}}, \end{aligned} \tag{3.3}$$

here $b := n\delta_2 - \alpha(0) > 0$.

For $1<q<\infty$, we can make (3.3) further calculation by Hölder's inequality.

$$\begin{aligned}
E_T^1 \lesssim & \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{i=-\infty}^{k-2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^{q} 2^{\frac{bq(i-k)}{2}}\right)\right.\\
&\left.\times\left(\sum_{i=-\infty}^{k-2} 2^{\frac{bq'(i-k)}{2}}\right)^{\frac{q}{q'}}\right\}^{\frac{1}{q}}\\
\lesssim & \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=-\infty}^{k-2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^{q} 2^{\frac{bq(i-k)}{2}}\right\}^{\frac{1}{q}}\\
= & \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{i=-\infty}^{L-2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^{q}\sum_{k=i+2}^{L} 2^{\frac{bq(i-k)}{2}}\right\}^{\frac{1}{q}}\\
\lesssim & \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{i=-\infty}^{L-2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
\lesssim & E_f.
\end{aligned}$$

For $0<q\leqslant 1$, we use the following inequality

$$\left(\sum_{i=1}^{\infty} a_i\right)^q \leqslant \sum_{i=1}^{\infty} a_i^q, \quad a_i\geqslant 0, i\in\mathbb{N} \tag{3.4}$$

and obtain

$$\begin{aligned}
E_T^1 \lesssim & \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=-\infty}^{k-2} 2^{\alpha(0)qi}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^{q} 2^{bq(i-k)}\right\}^{\frac{1}{q}}\\
= & \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{i=-\infty}^{L-2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^{q}\sum_{k=i+2}^{L} 2^{bq(i-k)}\right\}^{\frac{1}{q}}\\
\lesssim & \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{i=-\infty}^{L-2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^{q}\right\}^{\frac{1}{q}}\\
\lesssim & E_f.
\end{aligned}$$

Similarly, we have

$$H_T^1 \lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{\infty}\left(\sum_{i=-\infty}^{k-2} 2^{\alpha_\infty i}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}} 2^{b_1(i-k)}\right)^q\right\}^{\frac{1}{q}},$$

here $b_1 = n\delta_2 - \alpha_\infty > 0$.

For $1 < q < \infty$, using Hölder's inequality, we have

$$\begin{aligned}
H_T^1 &\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=-\infty}^{k-2} 2^{\alpha_\infty iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{\frac{bq(i-k)}{2}}\left(\sum_{i=-\infty}^{k-2}2^{\frac{b_1q'(i-k)}{2}}\right)^{\frac{q}{q'}}\right\}^{\frac{1}{q}}\\
&\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=-\infty}^{k-2} 2^{\alpha_\infty iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{\frac{b_1q(i-k)}{2}}\right\}^{\frac{1}{q}}\\
&\leqslant 2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=-2}^{k-2} 2^{\alpha_\infty iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{\frac{b_1q(i-k)}{2}}\right\}^{\frac{1}{q}}\\
&\quad + 2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=-\infty}^{-3} 2^{\alpha_\infty iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{\frac{b_1q(i-k)}{2}}\right\}^{\frac{1}{q}}\\
&=: I_1 + I_2.
\end{aligned}$$

Now we consider I_1 and I_2 respectively. Due to $b_1 > 0$, we have

$$\begin{aligned}
I_1 &= 2^{-L\lambda}\left\{\sum_{i=-2}^{L-2} 2^{\alpha_\infty iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q \sum_{k=i+2}^{L} 2^{\frac{b_1q(i-k)}{2}}\right\}^{\frac{1}{q}}\\
&\lesssim 2^{-L\lambda}\left\{\sum_{i=-2}^{L-2} 2^{\alpha_\infty iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&\lesssim H_f.
\end{aligned}$$

Because $L > 0$, $b_1 > 0$ and $\lambda \geqslant 0$, we obtain

$$\begin{aligned}
I_2 &\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=-\infty}^{-3} 2^{\frac{b_1q(i-k)}{2}}\left[\sum_{m=\infty}^{i} 2^{\alpha_\infty mq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_m\right\|_{L^{p(\cdot)}}^q\right]\right\}^{\frac{1}{q}}\\
&= 2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=-\infty}^{-3} 2^{\frac{b_1q(i-k)}{2}}\cdot 2^{iq\lambda}H_f^q\right\}^{\frac{1}{q}}\\
&= 2^{-L\lambda}\left\{\left(\sum_{k=0}^{L}2^{-kb_1q/2}\right)\left(\sum_{i=-\infty}^{-3}2^{(b_1/2+\lambda)qi}\right)H_f^q\right\}^{\frac{1}{q}}\\
&\lesssim 2^{-L\lambda}\left\{\left(\sum_{k=0}^{L}2^{-kb_1q/2}\right)\left(\sum_{i=-\infty}^{L}2^{(b_1/2+\lambda)qi}\right)H_f^q\right\}^{\frac{1}{q}}\\
&\lesssim 2^{-L\lambda}2^{-Lb_1/2}2^{(b_1/2+\lambda)L}H_f\\
&= H_f.
\end{aligned}$$

For $0<q\leqslant 1$, using (3.4) again, we have

$$\begin{aligned}
H_T^1 &\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{\infty}\sum_{i=-\infty}^{k-2}2^{\alpha_\infty iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{b_1(i-k)}\right\}^{\frac{1}{q}}\\
&\lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=-2}^{k-2}2^{\alpha_\infty iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{b_1(i-k)}\right\}^{\frac{1}{q}}\\
&\quad+2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=-\infty}^{-3}2^{\alpha_\infty iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{b_1(i-k)}\right\}^{\frac{1}{q}}\\
&=: J_1+J_2.
\end{aligned}$$

Because it is similar to the process of case $1<q<\infty$ in Step 2 above, we can conclude that $H_T^2\lesssim H_f$ is also true for $0<q\leqslant 1$.

Step 3. First of all, we estimate E_T^3. For $\forall i\geq k+2, x\in R_k$, by the size condition and generalized Minkowski's inequality again, we have

$$\begin{aligned}
\left(\sum_{j=1}^{\infty}T^r(f_j^i)(x)\right)^{\frac{1}{r}} &\lesssim \left[\sum_{j=1}^{\infty}\left(2^{-in}\int_{\mathbb{R}^n}|f_j^i(y)|\mathrm{d}y\right)^r\right]^{\frac{1}{r}}\\
&= 2^{-in}\left(\sum_{j=1}^{\infty}\left(\int_{\mathbb{R}^n}|f_j^i(y)|\mathrm{d}y\right)^r\right)^{\frac{1}{r}}\\
&\lesssim 2^{-in}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^i|^r\right)^{\frac{1}{r}}\mathrm{d}y.
\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}
E_T^3 &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}}2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}2^{\alpha(0)qk}\left\|\chi_k\sum_{i=k+2}^{\infty}2^{-in}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^i|^r\right)^{\frac{1}{r}}\mathrm{d}y\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&= \sup_{L\leqslant 0, L\in\mathbb{Z}}2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}2^{\alpha(0)qk}\,\|\chi_k\|_{L^{p(\cdot)}}^q\left(\sum_{i=k+2}^{\infty}2^{-in}\int_{\mathbb{R}^n}\left(\sum_{j=1}^{\infty}|f_j^i|^r\right)^{\frac{1}{r}}\mathrm{d}y\right)^q\right\}^{\frac{1}{q}}\\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}}2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}2^{\alpha(0)qk}\|\chi_k\|_{L^{p(\cdot)}}^q\right.\\
&\qquad\times\left.\left(\sum_{i=k+2}^{\infty}2^{-in}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}\|\chi_i\|_{L^{p'(\cdot)}}\right)^q\right\}^{\frac{1}{q}}.
\end{aligned}\tag{3.5}$$

Using Lemmas 2.1 and 3.2 again, we obtain

$$
\begin{aligned}
2^{-in}\|\chi_k\|_{L^{p(\cdot)}}\|\chi_i\|_{L^{p'(\cdot)}} &\leqslant 2^{-in}\|\chi_{B_k}\|_{L^{p(\cdot)}}\|\chi_{B_i}\|_{L^{p'(\cdot)}} \\
&\lesssim 2^{-in}\|\chi_{B_k}\|_{L^{p(\cdot)}}|B_i|\|\chi_{B_i}\|_{L^{p(\cdot)}}^{-1} \\
&\lesssim 2^{n\delta_1(k-i)}. \qquad (3.6)
\end{aligned}
$$

We put (3.6) into (3.5) and get

$$
\begin{aligned}
E_T^3 &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L} 2^{\alpha(0)kq}\left(\sum_{i=k+2}^{\infty}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}} 2^{n\delta_1(k-i)}\right)^q\right\}^{\frac{1}{q}} \\
&= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{i=k+2}^{\infty} 2^{\alpha(0)k}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}} 2^{n\delta_1(k-i)}\right)^q\right\}^{\frac{1}{q}} \\
&= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{i=k+2}^{\infty} 2^{\alpha(0)i}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}} 2^{d(k-i)}\right)^q\right\}^{\frac{1}{q}}, \qquad (3.7)
\end{aligned}
$$

where $d := n\delta_1 + \alpha(0) > 0$.

To make (3.7) further calculation, we divide it into two case below.

If $1 < q < \infty$, by using Hölder's inequality we have

$$
\begin{aligned}
E_T^3 &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\left(\sum_{i=k+2}^{\infty} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{\frac{dq(k-i)}{2}}\right)\right. \\
&\qquad \left.\times\left(\sum_{i=k+2}^{\infty} 2^{\frac{dq'(k-i)}{2}}\right)^{\frac{q}{q'}}\right\}^{\frac{1}{q}} \\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=k+2}^{\infty} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{\frac{dq(k-i)}{2}}\right\}^{\frac{1}{q}} \\
&= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=k+2}^{L+2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{\frac{dq(k-i)}{2}}\right\}^{\frac{1}{q}} \\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=L+3}^{\infty} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{\frac{dq(k-i)}{2}}\right\}^{\frac{1}{q}} \\
&=: I_3 + I_4.
\end{aligned}
$$

Now we consider I_3 and I_4 respectively. Since $d > 0$, we have

$$
\begin{aligned}
I_3 &= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{i=-\infty}^{L+2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q \sum_{k=-\infty}^{i-2} 2^{\frac{dq(k-i)}{2}}\right\}^{\frac{1}{q}}\\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{i=-\infty}^{L+2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q\right\}^{\frac{1}{q}}\\
&\lesssim E_f.
\end{aligned}
$$

Because $d>0$ and $\lambda-d/2<0$, we obtain

$$
\begin{aligned}
I_4 &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=L+3}^{\infty} 2^{\frac{dq(k-i)}{2}}\left[\sum_{m=-\infty}^{i} 2^{\alpha(0)mq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_m\right\|_{L^{p(\cdot)}}^q\right]\right\}^{\frac{1}{q}}\\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=L+3}^{\infty} 2^{\frac{dq(k-i)}{2}}\cdot 2^{iq\lambda}E_f^q\right\}^{\frac{1}{q}}\\
&= \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\left(\sum_{k=-\infty}^{L} 2^{dqk/2}\right)\left(\sum_{i=L+3}^{\infty} 2^{(\lambda-d/2)qi}\right)E_f^q\right\}^{\frac{1}{q}}\\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} \left\{2^{-Lq\lambda}2^{dqL/2}2^{(\lambda-d/2)qL}E_f^q\right\}^{\frac{1}{q}}\\
&= E_f.
\end{aligned}
$$

Thus, we have $E_T^3 \lesssim E_f$.

If $0<q\leqslant 1$, then we use (3.4) in (3.7) and get

$$
\begin{aligned}
E_T^3 &\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=k+2}^{\infty} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{dq(k-i)}\right\}^{\frac{1}{q}}\\
&\lesssim \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=k+2}^{L+2} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{dq(k-i)}\right\}^{\frac{1}{q}}\\
&\quad + \sup_{L\leqslant 0, L\in\mathbb{Z}} 2^{-L\lambda}\left\{\sum_{k=-\infty}^{L}\sum_{i=L+3}^{\infty} 2^{\alpha(0)iq}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{dq(k-i)}\right\}^{\frac{1}{q}}\\
&=: J_3+J_4.
\end{aligned}
$$

Because it is similar to the process of case $1<q<\infty$ above, we can conclude that $E_T^3 \lesssim E_f$ is also true for $0<q\leqslant 1$.

Then we consider H_T^3. Similarly, we have

$$H_T^3 \lesssim 2^{-L\lambda} \left\{ \sum_{k=0}^{L} \left(\sum_{i=k+2}^{\infty} 2^{\alpha_\infty i} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_i \right\|_{L^{p(\cdot)}} 2^{d_1(k-i)} \right)^q \right\}^{\frac{1}{q}}, \tag{3.8}$$

here $d_1 = n\delta_1 + \alpha_\infty > 0$.

For $1 < q < \infty$, we can make (3.8) further calculation by Hölder's inequality.

$$\begin{aligned} H_T^3 &\lesssim 2^{-L\lambda} \left\{ \sum_{k=0}^{L} \left(\sum_{i=k+2}^{\infty} 2^{\alpha_\infty q i} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_i \right\|_{L^{p(\cdot)}}^q 2^{\frac{d_1 q(k-i)}{2}} \right) \right. \\ &\qquad \left. \times \left(\sum_{i=k+2}^{\infty} 2^{\frac{d_1 q'(k-i)}{2}} \right)^{\frac{q}{q'}} \right\}^{\frac{1}{q}} \\ &\lesssim 2^{-L\lambda} \left\{ \sum_{k=0}^{L} \sum_{i=k+2}^{\infty} 2^{\alpha_\infty q i} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_i \right\|_{L^{p(\cdot)}}^q 2^{\frac{d_1 q(k-i)}{2}} \right\}^{\frac{1}{q}} \\ &\leqslant 2^{-L\lambda} \left\{ \sum_{k=0}^{L} \sum_{i=k+2}^{L+2} 2^{\alpha_\infty q i} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_i \right\|_{L^{p(\cdot)}}^q 2^{\frac{d_1 q(k-i)}{2}} \right\}^{\frac{1}{q}} \\ &\quad + 2^{-L\lambda} \left\{ \sum_{k=0}^{L} \sum_{i=L+3}^{\infty} 2^{\alpha_\infty q i} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_i \right\|_{L^{p(\cdot)}}^q 2^{\frac{d_1 q(k-i)}{2}} \right\}^{\frac{1}{q}} \\ &=: I_5 + I_6. \end{aligned}$$

Because $d_1 > 0$, we have

$$\begin{aligned} I_5 &= 2^{-L\lambda} \left\{ \sum_{i=2}^{L+2} 2^{\alpha_\infty q i} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_i \right\|_{L^{p(\cdot)}}^q \sum_{k=0}^{i-2} 2^{\frac{d_1 q(k-i)}{2}} \right\}^{\frac{1}{q}} \\ &\lesssim 2^{-L\lambda} \left\{ \sum_{i=2}^{L+2} 2^{\alpha_\infty q i} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_i \right\|_{L^{p(\cdot)}}^q \right\}^{\frac{1}{q}} \\ &\lesssim H_f. \end{aligned}$$

Since $d_1 > 0$ and $\lambda - d_1/2 < 0$, we obtain

$$I_6 = 2^{-L\lambda} \left\{ \sum_{k=0}^{L} \sum_{i=L+3}^{\infty} 2^{\alpha_\infty q i} \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{\frac{1}{r}} \chi_i \right\|_{L^{p(\cdot)}}^q 2^{\frac{d_1 q(k-i)}{2}} \right\}^{\frac{1}{q}}$$

$$\leqslant 2^{-L\lambda}\left\{\left(\sum_{k=0}^{L}2^{\frac{d_1qk}{2}}\right)\left(\sum_{i=L+3}^{\infty}2^{(\lambda-d_1/2)qi}\right)\right.$$
$$\left.\times\left[\sum_{m=0}^{i}2^{\alpha_\infty qm}\left\|2^{-i\lambda}\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_m\right\|_{L^{p(\cdot)}}^q\right]\right\}^{\frac{1}{q}}$$
$$\lesssim 2^{-L\lambda}2^{d_1/2L}2^{(\lambda-d_1/2)L}H_f$$
$$= H_f.$$

For $0<q\leqslant 1$, we use (3.4) in (3.8) and have

$$H_T^3 \lesssim 2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=k+2}^{\infty}2^{\alpha_\infty qi}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{d_1q(k-i)}\right\}^{\frac{1}{q}}$$
$$\leqslant 2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=k+2}^{L+2}2^{\alpha_\infty qi}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{d_1q(k-i)}\right\}^{\frac{1}{q}}$$
$$+\,2^{-L\lambda}\left\{\sum_{k=0}^{L}\sum_{i=L+3}^{\infty}2^{\alpha_\infty qi}\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{\frac{1}{r}}\chi_i\right\|_{L^{p(\cdot)}}^q 2^{d_1q(k-i)}\right\}^{\frac{1}{q}}$$
$$=: I_J + J_6.$$

Because it is similar to the process of case $1<q<\infty$ above, we can state that $H_T^3 \lesssim H_f$ is also true for $0<q\leqslant 1$.

This we finish the proof of Theorem 2.1. □

Now we turn to prove Theorem 2.2. Because the proofs of B-parts and F-parts are similar, we only prove F-parts below. Our proof will use the idea that comes from [36]. To continue, we recall some lemmas.

Lemma 3.3. (see [31, Lemma 1]) Let $\mu,\nu\in\mathcal{S}(\mathbb{R}^n)$, $-1\leqslant M\in\mathbb{Z}$,

$$D^\tau\widehat{\mu}(0)=0 \quad \text{for all} \quad |\tau|\leqslant M.$$

Then for any $N>0$ there is a constant C_N such that

$$\sup_{z\in\mathbb{R}^n}|\mu_t*\nu(z)|(1+|z|)^N\leqslant C_N t^{M+1},$$

where $\mu_t(x)=t^{-n}\mu(\frac{x}{t})$ for all $0<t\leqslant 2$.

Lemma 3.4. (see [31, Lemma 2]) Let $0<q\leqslant\infty$, $\delta>0$. For any sequence $\{g_j\}_0^\infty$ of nonnegative numbers denote

$$G_j=\sum_{k=0}^{\infty}2^{-|k-j|\delta}g_k.$$

Then there is a constant C depending only on q and δ such that

$$\|\{G_j\}_0^\infty\|_{\ell_q}\leqslant C\|\{g_j\}_0^\infty\|_{\ell_q}. \tag{3.9}$$

Lemma 3.5. Let $0 < \beta \leqslant \infty$, $\delta > 0$, $0 < q \leqslant \infty$, $0 \leqslant \lambda < \infty$, $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. For any sequence $\{g_j\}_0^\infty$ of nonnegative measurable functions on $\mathbb{R}^n$ denote

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|\delta} g_k(x), \qquad x \in \mathbb{R}^n.$$

Then there are some constants $C_1 = C_1(q, \delta)$ and $C_2 = C_2(p(\cdot), q, \delta)$ such that

$$\|\{G_j\}_0^\infty\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\ell_\beta)} \leqslant C_1 \|\{g_j\}_0^\infty\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\ell_\beta)} \tag{3.10}$$

and

$$\|\{G_j\}_0^\infty\|_{\ell_\beta(M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda})} \leqslant C_2 \|\{g_j\}_0^\infty\|_{\ell_\beta(M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda})}. \tag{3.11}$$

Proof. Firstly, (3.10) follows immediately from (3.9) by Lemma 3.4. Next we prove (3.11) for $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ and we separate it into two cases.

Case 1. $p_- \geqslant 1, q \geqslant 1$. Because $\|\cdot\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}}$ is a norm, we have

$$\|G_j\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} \leqslant \sum_{k=0}^{\infty} 2^{-|k-j|\delta} \|g_k\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}}.$$

By Lemma 3.4, we obtain (3.11).

Case 2. For $q < 1$, we choose a constant $p_0 < \min(p_-, q)$ and obtain

$$\begin{aligned}
\|G_j\|^{p_0}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}} &= \||G_j|^{p_0}\|_{M\dot{K}_{q/p_0,p(\cdot)/p_0}^{p_0\alpha(\cdot),p_0\lambda}} \\
&\leqslant \|\sum_{k=0}^{\infty} 2^{-|k-j|p_0\delta} |g_k|^{p_0}\|_{M\dot{K}_{q/p_0,p(\cdot)/p_0}^{p_0\alpha(\cdot),p_0\lambda}} \\
&\leqslant \sum_{k=0}^{\infty} 2^{-|k-j|p_0\delta} \||g_k|^{p_0}\|_{M\dot{K}_{q/p_0,p(\cdot)/p_0}^{p_0\alpha(\cdot),p_0\lambda}}.
\end{aligned}$$

Due to (3.4) and (3.11) proved for the space $M\dot{K}_{q/p_0,p(\cdot)/p_0}^{p_0\alpha(\cdot),p_0\lambda}(\mathbb{R}^n)$,

$$\begin{aligned}
\|\{G_j\}\|^{p_0}_{\ell_\beta(M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda})} &= \|\{|G_j|^{p_0}\}\|_{\ell_{\beta/p_0}\left(M\dot{K}_{q/p_0,p(\cdot)/p_0}^{p_0\alpha(\cdot),p_0\lambda}\right)} \\
&\lesssim \|\{|g_k|^{p_0}\}\|_{\ell_{\beta/p_0}\left(M\dot{K}_{q/p_0,p(\cdot)/p_0}^{p_0\alpha(\cdot),p_0\lambda}\right)} \\
&= \|\{g_k\}\|^{p_0}_{\ell_\beta(M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda})}.
\end{aligned}$$

Raising to power $1/p_0$ on both sides of the inequality, we have (3.11). □

Lemma 3.6. (see [9, Theorem 6]) Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity and let $R \in \mathbb{N}$. Then there exist functions $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$supp\ \theta_0,\ supp\ \theta \subseteq \{x \in \mathbb{R}^n : |x| \leqslant 1\},$$

$$|\widehat{\theta_0}(\xi)| > 0 \quad \text{on} \quad \{|\xi| < 2\epsilon\},$$

$$|\widehat{\theta}(\xi)| > 0 \quad \text{on} \quad \{\epsilon/2 < |\xi| < 2\epsilon\},$$

$$\int_{\mathbb{R}^n} x^\gamma \theta(x) dx = 0 \quad \text{for} \quad 0 < |\gamma| \leqslant R$$

such that

$$\widehat{\theta_0}(\xi)\widehat{\psi_0}(\xi)+\sum_{j=1}^{\infty}\widehat{\theta}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi)=1,\quad \text{for all}\quad \xi\in\mathbb{R}^n,$$

where the functions $\psi_0,\psi\in\mathcal{S}(\mathbb{R}^n)$ are defined via $\widehat{\psi_0}(\xi)=\frac{\varphi_0(\xi)}{\widehat{\theta}_0(\xi)}$ and $\widehat{\psi}(\xi)=\frac{\varphi_1(2\xi)}{\widehat{\theta}(\xi)}$.

Proof of Theorem 2.2. We separate the whole proof by four steps. The first step is to prove the equivalence of (2.6) and (2.7). The second step is to build the bridge from (2.7) to (2.9) and use the system (Ψ_0,Ψ) to replace the system (Φ_0,Φ). Next we will state that Definition 2.4 can be seen as a special case of (2.10) and the equivalence of (2.9) and (2.10) goes parallel to (2.6) and (2.7). Finally, we will illustrate that (2.10) is equivalent to the rest.

Step 1. For every $f\in\mathcal{S}'(\mathbb{R}^n)$, we will prove the following inequalities

$$\|f\|^{(2)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta}\lesssim\|f\|^{(1)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta}\lesssim\|f\|^{(2)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta}.$$

Since the inequality $\|f\|^{(1)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta}\lesssim\|f\|^{(2)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta}$ is easy to get by the definition of Peetre's maximal operator, we only prove the left inequality.

For any given $r<\min\{p_-,\beta\}$ and $N\in\mathbb{N}$, using Lemmas 3.3 and 3.6, there exists a constant $C>0$ such that for any $f\in\mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned}\left(\int_1^2|2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^\beta\frac{\mathrm{d}t}{t}\right)^{r/\beta}&\lesssim\sum\nolimits_{k\in l+\mathbb{N}_0}2^{(l-k)(Nr-n+rs)}2^{krs}\\&\quad\times\mathcal{M}\left[\left(\int_1^2|((\Phi_k)_t*f)(\cdot)|^\beta\frac{\mathrm{d}t}{t}\right)^{r/\beta}\right](x).\end{aligned}$$

For the proof of the inequality above, see [36].

Next, let $p_0=r\in(n/a,<min\{p_-,\beta\})$, $N>max\{0,-s\}+a$ and $\delta:=N+s-d/r>0$, then we obtain for any $l\in\mathbb{N}$

$$\begin{aligned}&\left(\int_1^2|2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^\beta\frac{\mathrm{d}t}{t}\right)^{r/\beta}\\&\quad\lesssim\sum_{k\in l+\mathbb{N}_0}2^{-\delta r|l-k|}2^{krs}\mathcal{M}\left[\left(\int_1^2|((\Phi_k)_t*f)(\cdot)|^\beta\frac{\mathrm{d}t}{t}\right)^{r/\beta}\right](x).\end{aligned}$$

Applying Lemma 3.5 in the space $M\dot{K}^{r\alpha(\cdot),r\lambda}_{q/r,p(\cdot)/r}(\ell_{\beta/r})$, we have

$$\begin{aligned}&\left\|\left\{\left(\int_1^2|2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^\beta\frac{\mathrm{d}t}{t}\right)^{r/\beta}\right\}_{l\in\mathbb{N}}\right\|_{K^{r\alpha(\cdot),q/r}_{p(\cdot)/r}(\ell_{\beta/r})}\\&\quad\lesssim\left\|\left\{\mathcal{M}\left[\left(\int_1^2|2^{ks}((\Phi_l)_t*f)(\cdot)|^\beta\frac{\mathrm{d}t}{t}\right)^{r/\beta}\right]\right\}_{l\in\mathbb{N}}\right\|_{M\dot{K}^{r\alpha(\cdot),r\lambda}_{q/r,p(\cdot)/r}(\ell_{\beta/r})}.\end{aligned}$$

Then, we use Theorem 2.1 and obtain

$$\begin{aligned}
&\left\|\left\{\left(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^\beta \frac{\mathrm{d}t}{t}\right)^{r/\beta}\right\}_{l\in\mathbb{N}}\right\|_{M\dot{K}^{r\alpha(\cdot),r\lambda}_{q/r,p(\cdot)/r}(\ell_{\beta/r})}\\
&\quad\lesssim \left\|\left\{\mathcal{M}\left[\left(\int_1^2 |2^{ks}((\Phi_l)_t * f)(\cdot)|^\beta \frac{\mathrm{d}t}{t}\right)^{r/\beta}\right]\right\}_{l\in\mathbb{N}}\right\|_{M\dot{K}^{r\alpha(\cdot),r\lambda}_{q/r,p(\cdot)/r}(\ell_{\beta/r})}\\
&\quad\lesssim \left\|\left\{\left(\int_1^2 |2^{ks}((\Phi_l)_t * f)(\cdot)|^\beta \frac{\mathrm{d}t}{t}\right)^{r/\beta}\right\}_{l\in\mathbb{N}}\right\|_{M\dot{K}^{r\alpha(\cdot),r\lambda}_{q/r,p(\cdot)/r}(\ell_{\beta/r})}\\
&\quad= \left\|\left\{\left(\int_1^2 |2^{ks}((\Phi_l)_t * f)(\cdot)|^\beta \frac{\mathrm{d}t}{t}\right)^{1/\beta}\right\}_{l\in\mathbb{N}}\right\|^r_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\ell_\beta)}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\left\|\left(\int_0^1 |\lambda^{-s}(\Phi^*_\lambda f)_a(\cdot)|^\beta \frac{d\lambda}{\lambda}\right)^{1/\beta}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}\\
&\quad\approx \left\|\left(\sum_{l=1}^\infty \int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(\cdot)|^\beta \frac{\mathrm{d}t}{t}\right)^{1/\beta}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}\\
&\quad\lesssim \left\|\left\{\left(\int_1^2 |2^{ls}\Phi_{2^{-l}t} * f(\cdot)|^\beta \frac{\mathrm{d}t}{t}\right)^{1/\beta}\right\}_{l\in\mathbb{N}}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(\ell_\beta)}\\
&\quad\approx \left\|\left(\int_0^1 |\lambda^{-s}\Phi_\lambda * f(\cdot)|^\beta \frac{d\lambda}{\lambda}\right)^{1/\beta}\right\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}}.
\end{aligned}$$

This proves $\|f\|^{(2)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta} \lesssim \|f\|^{(1)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta}$.

Step 2. Suppose that $\Psi_0, \Psi \in \mathcal{S}'(\mathbb{R}^n)$ are functions satisfying (3.10). First of all, we will prove for any $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|f\|^{(4)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta(\mathbb{R}^n,\Psi)} \lesssim \|f\|^{(2)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta(\mathbb{R}^n,\Phi)} \tag{3.12}$$

Let $\delta = \min\{1, S+1-s\}$, using Lemmas 3.6 and 3.3 again, there exists a constant $C > 0$ such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$2^{ls}(\Psi^*_l f)_a(x) \leqslant C \sum_{k\in\mathbb{N}_0} 2^{-|k-l|\delta} 2^{ks}(\Phi^*_{2^{-k}t}f)_a(x),\ x\in\mathbb{R}^n \text{ and } t\in[1,2]. \tag{3.13}$$

For details about (3.13), see [36].

Let $\beta \geqslant 1$. We take $(\int_1^2 |\cdot|^\beta \mathrm{d}t/t)^{1/\beta}$ on both sides of (3.13) and have

$$2^{ls}(\Psi^*_l f)_a(x) \lesssim \sum_{k\in\mathbb{N}_0} 2^{-|k-l|\delta} 2^{ks}\left(\int_1^2 |(\Phi^*_{2^{-k}t}f)_a(x)|^\beta \frac{\mathrm{d}t}{t}\right)^{1/\beta}.$$

Applying Lemma 3.5, we obtain

$$\|\{2^{ls}(\Psi_l^*f)_a\}_{l\in\mathbb{N}}\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\ell_\beta)} \lesssim \left\| \left(\sum_{k=1}^{\infty} 2^{ks\beta} \int_1^2 |(\Phi_{2^{-k}t}^*f)_a(x)|^\beta \frac{\mathrm{d}t}{t} \right)^{1/\beta} \right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}}$$

which gives the desired result.

In the case $\beta < 1$, we prove as below. Because the quantity $(\int_1^2 |\cdot|^\beta \mathrm{d}t/t)^{1/\beta}$ is not a norm, we only have

$$(2^{ls}(\Psi_l^*f)_a(x))^\beta \lesssim \sum_{k\in\mathbb{N}_0} 2^{-\beta|k-l|\delta} 2^{ks\beta} \int_1^2 |(\Phi_{2^{-k}t}^*f)_a(x)|^\beta \frac{\mathrm{d}t}{t}.$$

We know that the right-hand side of the above inequality is less than a convolution $(\gamma * \tau)_\ell$ with

$$\gamma_k = 2^{-|k|\delta\beta} \quad \textit{and} \quad \tau_k = 2^{ks\beta} \int_1^2 |(\Phi_{2^{-k}t}^*f)_a(x)|^\beta \frac{\mathrm{d}t}{t}.$$

Now applying the ℓ_1-norm to both sides, then for any $x \in \mathbb{R}^n$

$$\begin{aligned} \|2^{ls}(\Psi_l^*f)_a(x)\|_{\ell_\beta}^\beta &\leqslant \|\gamma\|_{\ell_1} \cdot \|\tau\|_{\ell_1} \\ &\lesssim \sum_{k=1}^{\infty} 2^{ks\beta} \int_1^2 |(\Phi_{2^{-k}t}^*f)_a(x)|^\beta \frac{\mathrm{d}t}{t}. \end{aligned}$$

Taking the power $(\cdots)^{1/\beta}$ to both sides of the last inequality and applying the $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}$-norm, then we have (3.12).

Similarly, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, we obtain

$$\|f\|^{(2)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}F_\beta^s(\mathbb{R}^n,\Phi)} \lesssim \|f\|^{(4)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}F_\beta^s(\mathbb{R}^n,\Psi)}.$$

Step 3. Taking $t = 1$ in Step 1 and omitting the integration over t, we see immediately

$$\|f\|^{(5)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}F_\beta^s} \lesssim \|f\|^{(4)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}F_\beta^s} \lesssim \|f\|^{(5)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}F_\beta^s}.$$

Step 4. We show (2.10) is equivalent to the rest.

First, we will show that for any $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|f\|^{(2)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}F_\beta^s(\mathbb{R}^n)} \lesssim \|f\|^{(3)}_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}F_\beta^s}. \tag{3.14}$$

For $0 < r < \min\{p_-, \beta\}$, see [36], there exists a positive constant C such that for any $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned} &\left(\int_1^2 |(\Psi_{2^{-l}t}^*f)_a(x)|^\beta \frac{\mathrm{d}t}{t} \right)^{r/\beta} \\ &\leqslant C \sum_{k\in\mathbb{N}_0} 2^{-kNs} 2^{(k+l)n} \int_{\mathbb{R}^n} \frac{\left(\int_1^2 \int_{|z|<2^{-(k+l)}t} |((\Phi_{k+l})_t * f)(z+y)|^\beta \mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}} \right)^{r/\beta}}{(1+2^l|x-y|)^{ar}} \mathrm{d}y. \end{aligned}$$

Suppose that $ar > n$ and denote

$$g_l(y) := \frac{2^{nl}}{(1+2^l|y|)^{ar}}, \ \forall y \in \mathbb{R}^n.$$

Then $g_l(y) \in L_1(\mathbb{R}^n)$ and we have

$$\left(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^\beta \frac{\mathrm{d}t}{t}\right)^{r/\beta}$$
$$\lesssim \sum_{k\in\mathbb{N}_0} 2^{-kNr}2^{kn}2^{lsr}\left[g_l * \left(\int_1^2 \int_{|z|<2^{-(k+l)}t} |((\Phi_{k+l})_t * f)(z+\cdot)|^\beta \mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/\beta}\right](x).$$

By using the majorant property of the Hardy–Littlewood maximal operator in [34], we have

$$\left(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^\beta \frac{\mathrm{d}t}{t}\right)^{r/\beta}$$
$$\lesssim \sum_{k\in\mathbb{N}_0} 2^{lsr}2^{k(-Nr+n)}\mathcal{M}\left[\left(\int_1^2 \int_{|z|<2^{-(k+l)}t} |((\Phi_{k+l})_t * f)(z+\cdot)|^\beta \mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/\beta}\right](x).$$

An index shift on the right-hand side gives

$$\left(\int_1^2 |2^{ls}(\Phi^*_{2^{-l}t}f)_a(x)|^\beta \frac{\mathrm{d}t}{t}\right)^{r/\beta}$$
$$\lesssim \sum_{k\in l+\mathbb{N}_0} 2^{lsr}2^{(k-l)(-Nr+n)}\mathcal{M}\left[\left(\int_1^2 \int_{|z|<2^{-k}t} |((\Phi_{k})_t * f)(z+\cdot)|^\beta \mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/\beta}\right](x)$$
$$= \sum_{k\in l+\mathbb{N}_0} 2^{(l-k)(Nr-n+rs)}2^{krs}\mathcal{M}\left[\left(\int_1^2 \int_{|z|<2^{-k}t} |((\Phi_{k})_t * f)(z+\cdot)|^\beta \mathrm{d}z \frac{\mathrm{d}t}{t^{n+1}}\right)^{r/\beta}\right](x).$$

By the similar way as after (3.13), we have (3.14).

Second, it is easy to see that $\|f\|^{(3)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta} \lesssim \|f\|^{(2)}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}F^s_\beta}$, since for any $t > 0$

$$\frac{1}{t^n}\int_{|z|<t} |(\Phi_t * f)(x+z)|\mathrm{d}z \lesssim \sup_{|z|<t} \frac{|(\Phi_t * f)(x+z)|}{(1+1/t|z|)^a} \lesssim (\Phi^*_t f)_a(x).$$

This we finish the proof of Theorem 2.2. □

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Department of Mathematics, Hainan Normal University, Haikou 571158, China.
E-mail address: baohua_dong@126.com; jingshixu@126.com