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DYNAMICAL SYSTEMS ON ARITHMETIC FUNCTIONS DETERMINED BY PRIMES

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ABSTRACT. In this paper, we study an algebra $\mathcal{A}$ consisting of all arithmetic functions, and corresponding dynamical systems acting on $\mathcal{A}$ determined by a fixed prime p. Starting from free probabilistic models on $\mathcal{A}$ determined by p, we construct certain group dynamical systems induced by the additive group $\mathbb{R}$ of all real numbers. We investigate the basic properties and free-probabilistic data of such dynamical systems by constructing corresponding crossed product algebras.

1. INTRODUCTION

Recently, relations between operator theory and number theory have been studied (e.g., [9] through [16, 31, 20, 5, 7]). In particular, we apply *free probability* (which is one of branches of operator algebra theory, e.g., [29, 30, 32]) to modern *number theory* (e.g., [21, 22, 8, 23, 19, 6, 26, 27]).

Arithmetic functions are functions f defined from the *natural numbers* $\mathbb{N}$ into the *complex numbers* $\mathbb{C}$. In particular, they induce (classical) *Dirichlet series*,

$$L_f(s) = \sum_{k=1}^{\infty} \frac{f(k)}{k^s}, \text{ for all } s \in \mathbb{C}, \text{ for } f \in \mathcal{A}.$$

These are used in modern number theory; *combinatorial number theory*, *L-function theory*, and *analytic number theory*, etc (e.g., [21, 22, 31, 8, 23, 19, 6]). Entireness and analyticity of L-functions are interesting topics in pure analysis, too.

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Recall that if f_1, f_2 are arithmetic functions, then the *convolution* $f_1 * f_2$ is again an arithmetic function, where

$$f_1 * f_2(n) \overset{def}{=} \sum_{d|n} f_1(d)\ f_2\left(\frac{n}{d}\right),$$

for all $n \in \mathbb{N}$, where "$d \mid n$" means "d divides n," or "n is divisible by d," for $d \in \mathbb{N}$.

The collection $\mathcal{A}$ of all arithmetic functions forms an algebra, under the usual functional addition and convolution. The convolution $(*)$ on arithmetic functions provides the usual multiplication on the set of L-functions, i.e.,

$$\left(L_{f_1}(s)\right)\left(L_{f_2}(s)\right) = L_{f_1 * f_2}(s).$$

Recently, the author and Jorgensen showed in [15, 16] that all arithmetic functions are understood as Krein-space operators on a certain Krein space, for a fixed prime. Start from constructing a free probabilistic model $(\mathcal{A},\ g_p)$ as in [11, 13], we construct an indefinite pseudo-inner product $[,]$ on $\mathcal{A}$,

$$[f,\ h] = g_p\left(f * h^*\right), \text{ for all } f,\ h \in \mathcal{A}.$$

Then, by the free-distributional data obtained in [11, 13], the indefinite pseudo-inner product structure of $\mathcal{A}$ is embedded in an indefinite inner product space $\mathbb{C}^2_{A_o} = (\mathbb{C}^2,\ [,]_{A_o})$, under certain quotient relation, where

$$[(t_1,\ s_1),\ (t_2,\ s_2)]_{A_o} = \left\langle \begin{pmatrix} t_1 \\ s_1 \end{pmatrix},\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t_2 \\ s_2 \end{pmatrix} \right\rangle_2,$$

where $<,>_2$ means the (positive-definite) inner product on $\mathbb{C}^2$,

$$<(t_1,\ s_1),\ (t_2,\ s_2)>_2 = t_1\overline{t_2} + s_1\overline{s_2},$$

where $\overline{z}$ means the conjugate of z, for all $z \in \mathbb{C}$.

And this indefinite inner product space $\mathbb{C}^2_{A_o}$ is isomorphic to the Krein subspace $\mathfrak{K}_p$ of the Krein space $\mathfrak{K}^2 = \mathbb{C}^2 \oplus \mathbb{C}^2$, with its indefinite inner product $[,]_2$,

$$[(t_1,\ s_1),\ (t_2,\ s_2)]_2 = <t_1,\ t_2>_2 - <s_1,\ s_2>_2\ .$$

Thus, one can understand all arithmetic functions as Krein-space operators for fixed primes (See [16]). In [15], as an application of [16], we considered Krein-space operators induced in particular by *Dirichlet characters*.

For more about Krein spaces and Krein-space operators, we refer [20, 5, 4].

In this paper, we concentrate on a certain group action E of a *flow* $\mathbb{R}$, the additive group $(\mathbb{R}, +)$ of real numbers, acting on $\mathcal{A}$. Such an action E is introduced as a system of morphisms $\{E_z\}_{z\in\mathbb{C}}$ (over $\mathbb{C}$) in [16]. However, in [16], we did not consider detailed analytic and free-probabilistic properties of such an action. Here, we study this action and their corresponding images $\{E_z(f)\}_{f\in\mathcal{A}}$ in detail (See Section 3 below). We understand the construction of morphisms E_t as a group action E of $\mathbb{R}$, by restricting our interests to $\mathbb{R}$ from $\mathbb{C}$. i.e.,

$$t \in \mathbb{R} \longmapsto E_t : \mathcal{A} \to \mathcal{A}, \text{ for all } t \in \mathbb{R}.$$

It means that we obtain *group dynamical system* $(\mathcal{A},\ \mathbb{R},\ E)$, and hence, the corresponding *crossed product algebra* $\mathcal{A}_E = \mathcal{A} \times_E \mathbb{R}$. Representations of $\mathcal{A}_E$ will be considered.

2. Free Probability

We briefly introduce *free probability*. Free probability is a branch of *operator algebra theory*, a noncommutative probability theory on noncommutative (and hence, on commutative) algebras (e.g., pure algebraic algebras, topological algebras, topological $*$-algebras, etc).

Let $\mathfrak{A}$ be an arbitrary algebra over the complex numbers $\mathbb{C}$, and let $\psi : \mathfrak{A} \to \mathbb{C}$ be a linear functional on $\mathfrak{A}$. The pair $(\mathfrak{A}, \psi)$ is called a free probability space (*over* $\mathbb{C}$). All operators $a \in (\mathfrak{A}, \psi)$ are called *free random variables* (See [30, 32]). Note that free probability spaces are dependent upon the choice of linear functionals.

Let $a_1, \cdots, a_s$ be a free random variable in a $(\mathfrak{A}, \psi)$, for $s \in \mathbb{N}$. The *free moments of* $a_1, \cdots, a_s$ are determined by the quantities

$$\psi(a_{i_1} \cdots a_{i_n}),$$

for all $(i_1, \cdots, i_n) \in \{1, \cdots, s\}^n$, for all $n \in \mathbb{N}$.

And the *free cumulants* $k_n(a_{i_1}, \cdots, a_{i_n})$ *of* $a_1, \cdots, a_s$ is determined by the *Möbius inversion*;

$$\begin{aligned} k_n(a_{i_1}, \cdots, a_{i_n}) &= \sum_{\pi \in NC(n)} \psi_\pi(a_{i_1}, \cdots, a_{i_n}) \mu(\pi, 1_n) \\ &= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \psi_V(a_{i_1}, \cdots, a_{i_n}) \mu\left(0_{|V|},\ 1_{|V|}\right) \right), \end{aligned}$$

for all $(i_1, \cdots, i_n) \in \{1, \cdots, s\}^n$, for all $n \in \mathbb{N}$, where $\psi_\pi(\cdots)$ means the *partition-depending moments*, and $\psi_V(\cdots)$ means the *block-depending moment*; for example, if

$$\pi = \{(1,\ 5,\ 7),\ (2,\ 3,\ 4),\ (6)\} \text{ in } NC(7),$$

with three blocks (1, 5, 7), (2, 3, 4), and (6), then

$$\begin{aligned} &\psi_\pi\left(a_{i_1}^{r_1}, \cdots, a_{i_7}^{r_7}\right) \\ &= \psi_{(1,5,7)}(a_{i_1}^{r_1}, \cdots, a_{i_7}^{r_7})\ \psi_{(2,3,4)}(a_{i_1}^{r_1}, \cdots, a_{i_7}^{r_7})\ \psi_{(6)}(a_{i_1}^{r_1}, \cdots, a_{i_7}^{r_7}) \\ &= \psi(a_{i_1}^{r_1} a_{i_5}^{r_5} a_{i_7}^{r_7})\ \psi(a_{i_2}^{r_2} a_{i_3}^{r_3} a_{i_4}^{r_4})\ \psi(a_{i_6}^{r_6}). \end{aligned}$$

Here, the set $NC(n)$ denotes the *noncrossing partition set* over $\{1, \cdots, n\}$. It is a *lattice* with inclusion as $\leq$, such that

$$\theta \leq \pi \overset{def}{\Longleftrightarrow} \forall\ V \in \theta,\ \exists\ B \in \pi, \text{ s.t., } V \subseteq B,$$

where $V \in \theta$ or $B \in \pi$ means that V is a *block of* θ, respectively, B is a block of π, and $\subseteq$ means the usual set inclusion, having its minimal element $0_n = \{(1), (2), \cdots, (n)\}$, and its maximal element $1_n = \{(1, \cdots, n)\}$.

A *partition-dependent free moment* $\psi_\pi(a, \cdots, a)$ is given by

$$\psi_\pi(a, \cdots, a) = \prod_{V \in \pi} \psi\left(a^{|V|}\right),$$

where $|V|$ means the cardinality of V.

Also, μ is the *Möbius functional* from $NC \times NC$ into $\mathbb{C}$, where $NC = \bigcup_{n=1}^{\infty} NC(n)$. i.e., μ satisfies

$$\mu(\pi,\ \theta) = 0, \text{ for all } \pi > \theta \text{ in } NC(n),$$

and

$$\mu(0_n,\ 1_n) = (-1)^{n-1}\ c_{n-1}, \text{ and } \sum_{\pi \in NC(n)} \mu(\pi,\ 1_n) = 0,$$

for all $n \in \mathbb{N}$, where

$$c_k = \frac{1}{k+1} \begin{pmatrix} 2k \\ k \end{pmatrix} = \frac{1}{k+1} \frac{(2k)!}{k!k!}$$

means the *k-th Catalan numbers*, for all $k \in \mathbb{N}$. Notice that since each $NC(n)$ is a well-defined lattice, if $\pi < \theta$ are given in $NC(n)$, one can decide the "interval"

$$[\pi, \theta] = \{\delta \in NC(n) : \pi \leq \delta \leq \theta\},$$

and it is always lattice-isomorphic to

$$[\pi, \theta] = NC(1)^{k_1} \times NC(2)^{k_2} \times \cdots \times NC(n)^{k_n},$$

for some $k_1, \cdots, k_n \in \mathbb{N}$, where $NC(l)^{k_t}$ means "l blocks of π generates k_t blocks of θ," for $k_j \in \{0, 1, \cdots, n\}$, for all $n \in \mathbb{N}$. By the multiplicativity of μ on $NC(n)$, for all $n \in \mathbb{N}$, if an interval $[\pi, \theta]$ in $NC(n)$ satisfies the above set-product relation, then we have

$$\mu(\pi, \theta) = \prod_{j=1}^{n} \mu(0_j, 1_j)^{k_j}.$$

(For details, see [30]).

Free moments of free random variables and the free cumulants of them provide equivalent free distributional data. For example, if a free random variable a in $(\mathfrak{A}, \psi)$ is a self-adjoint operator in the von Neumann algebra $\mathfrak{A}$ in the sense that: $a^* = a$, then both free moments $\{\psi(a^n)\}_{n=1}^{\infty}$ and free cumulants $\{k_n(a, \cdots, a)\}_{n=1}^{\infty}$ give its spectral distributional data.

However, their uses are different. For instance, to study the free distribution of fixed free random variables, the computation of free moments is better; and to study the freeness of distinct free random variables in the structures, the computation and observation of free cumulants is better (See [30, 29]).

Definition 2.1. We say two subalgebras A_1 and A_2 of $\mathfrak{A}$ are *free in* $(\mathfrak{A}, \psi)$, if all "mixed" free cumulants of A_1 and A_2 vanish.. Similarly, two subsets X_1 and X_2 of $\mathfrak{A}$ are *free in* $(\mathfrak{A}, \psi)$, if two subalgebras A_1 and A_2, generated by X_1 and X_2 respectively, are free in $(\mathfrak{A}, \psi)$. Two free random variables x_1 and x_2 are *free in* $(\mathfrak{A}, \psi)$, if $\{x_1\}$ and $\{x_2\}$ are free in $(\mathfrak{A}, \psi)$.

Suppose A_1 and A_2 are free subalgebras in $(\mathfrak{A}, \psi)$. Then the subalgebra A generated both by these free subalgebras A_1 and A_2 is denoted by

$$A \overset{def}{=} A_1 *_{\mathbb{C}} A_2.$$

Assume that $\mathfrak{A}$ is generated by its family $\{A_i\}_{i \in \Lambda}$ of subalgebras, and suppose the subalgebras A_i are free from each other in $(\mathfrak{A}, \psi)$, for $i \in \Lambda$. i.e.,

$$\mathfrak{A} = \underset{i \in \Lambda}{*_{\mathbb{C}}} A_i.$$

Then, we call $\mathfrak{A}$ the *free product algebra of* $\{A_i\}_{i \in \Lambda}$.

3. Free Probabilistic Models of $\mathcal{A}$ Induced by The Primes

In this section, we introduce free probabilistic models $(\mathcal{A}, g_p)$ on the arithmetic algebra $\mathcal{A}$ determined by fixed primes p (See [11, 12, 13]). And, we put topologies on $\mathcal{A}$ determined by primes to make our dynamical systems act on $\mathcal{A}$ properly.

3.1. **Arithmetic p-Prime Probability Spaces** $(\mathcal{A},\ g_p)$**.** Let $\mathcal{A}$ be the set of all arithmetic functions, as a vector space over $\mathbb{C}$. Define the convolution $(*)$ on $\mathcal{A}$ by

$$f_1 * f_2(n) \overset{def}{=} \sum_{d|n} f_1(d)\ f_2\left(\frac{n}{d}\right), \text{ for all } n \in \mathbb{N}.$$

Then $\mathcal{A}$ becomes an algebra over $\mathbb{C}$. We call $\mathcal{A}$ the *arithmetic(-functional) algebra.*

Define a linear functional g_p on $\mathcal{A}$ by the point-evaluation at p;

(3.1.1)

$$g_p(f) \overset{def}{=} f(p), \text{ for all } f \in \mathcal{A},$$

for any fixed prime p.

Definition 3.1. Let $\mathcal{A}$ be the arithmetic algebra, and let g_p be the linear functional (3.1.1), for a prime p. Then the free probability space $(\mathcal{A},\ g_p)$ is called the arithmetic p-prime probability space.

We study primes p as linear functionals g_p on arithmetic functions, and then arithmetic functions have corresponding free-distributional data induced by primes.

Proposition 3.2. *(See [11]) Let $(\mathcal{A},\ g_p)$ be the arithmetic p-prime-probability space, for a fixed prime p. If $f,\ f_1,\ f_2$ are free random variable in $(\mathcal{A},\ g_p)$, then*

(3.1.2) $g_p(f_1 * f_2) = g_p(f_1)\ f_2(1) + f_1(1)\ g_p(f_2)$.

(3.1.3) $g_p\left(f^{(n)}\right) = nf(1)^{n-1}\ f(p)$, *for all* $n \in \mathbb{N}$,

where

$$f^{(n)} \overset{def}{=} \underbrace{f * \cdots\cdots .. * f}_{n\text{-}times},$$

for all $n \in \mathbb{N}$. □

The free moment computation (3.1.3) is obtained by (3.1.2), inductively. Also, one has that

$(3.1.2)'$

$$g_p\left(\underset{j=1}{\overset{n}{*}} f_j\right) = \sum_{j=1}^{n} f_j(p)\left(\prod_{l \neq j \in \{1,\cdots,n\}} f_l(1)\right),$$

for all $f_1, \cdots, f_n \in (\mathcal{A},\ g_p)$, for $n \in \mathbb{N}$.

From the above proposition, one can verify that free-distributional data of arithmetic functions f in $(\mathcal{A},\ g_p)$ is completely determined by quantities $f(1)$ and $f(p)$. It motivates the main result of [13].

Proposition 3.3. *(See [13]) Let $\mathcal{A}$ be the arithmetic algebra and p, an arbitrary fixed prime. Then, for a fixed p, the algebra $\mathcal{A}$ is decomposed by*

$$\mathcal{A} = \bigsqcup_{(a,b)\in\mathbb{C}\times\mathbb{C}} [a,\ b],$$

where

$$[a,\ b] \overset{def}{=} \{f \in \mathcal{A} : f(1) = a, \text{ and } f(p) = b \text{ in } \mathbb{C}\}.$$

□

We considered the following morphism Exp_t^* in [16], for "$t \in \mathbb{C}$."

Corollary 3.4. *Let $t \in \mathbb{C}$. Define a morphism Exp_t^* on $\mathcal{A}$ by*

$$Exp_t^*(f) \overset{def}{=} \sum_{n=1}^{\infty} \frac{t^n}{n!} f^{(n)}, \text{ for all } f \in \mathcal{A}.$$

Then
(3.1.4)

$$g_p\left(Exp_t^*(f)\right) = \left(te^{tf(1)}\right) f(p).$$

Proof. Observe that:

$$g_p\left(Exp_x^*(f)\right) = g_p\left(\sum_{n=1}^{\infty} \frac{t^n}{n!} f^{(n)}\right)$$

$$= \sum_{n=1}^{\infty} \frac{t^n}{n!} \left(f^{(n)}(p)\right) = \sum_{n=1}^{\infty} \frac{t^n}{n!} \left(nf(1)^{n-1}f(p)\right)$$

by (3.1.3)

$$= \sum_{n=1}^{\infty} \frac{t^n f(1)^{n-1}}{(n-1)!} f(p) = \sum_{n=1}^{\infty} \frac{t\left(t^{n-1}f(1)\right)^{n-1}}{(n-1)!} f(p)$$

$$= f(p)\left(t\sum_{k=0}^{\infty} \frac{(tf(1))^k}{k!}\right)$$

$$= \left(te^{tf(1)}\right) f(p) = \left(te^{tf(1)}\right) g_p(f),$$

for all $f \in \mathcal{A}$. □

Also, the above morphism $Exp_t^*(\bullet)$ on $\mathcal{A}$ satisfies a certain co-cycle property for g_p.

Corollary 3.5. *Let $Exp_t^*(\bullet)$ be as above in (3.1.4). Then*
(3.1.5)

$$g_p\left(Exp_t^*(f_1 + f_2)\right) = g_p\left((Exp_t^*(f_1)) * (Exp_t^*(f_2))\right)$$

$$+ g_p\left(Exp_t^*(f_1)\right) + g_p\left(Exp_t^*(f_2)\right),$$

for all $f_1,\ f_2 \in \mathcal{A}$, for all primes p.

Proof. Let f_j be arithmetic functions in the arithmetic p-prime probability space $(\mathcal{A},\ g_p)$, and let $Exp_t^*(f_j)$ be the corresponding elements of $(\mathcal{A},\ g_p)$, for $j = 1,\ 2$, where $Exp_t^*(\bullet)$ is a morphism introduced as above, for all $t \in \mathbb{C}$. Observe that:

$$g_p\left((Exp_t^*(f_1)) * (Exp_t^*(f_2))\right)$$

$$= g_p\left(\left(\sum_{n=1}^{\infty} \frac{t^n}{n!} f_1^{(n)}\right)\left(\sum_{k=1}^{\infty} \frac{t^k}{k!} f_2^{(k)}\right)\right)$$

$$= g_p\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{t^{n+k}}{n!k!} f_1^{(n)} * f_2^{(k)}\right)$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{t^{n+k}}{n!k!} g_p\left(f_1^{(n)} * f_2^{(k)}\right)$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{t^{n+k}}{n!k!} \left(f_1^{(n)}(1)f_2^{(k)}(p) + f_1^{(n)}(p)f_2^{(k)}(1)\right)$$

by (3.1.2)

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{t^{n+k}}{n!k!} \left((f_1(1))^n f_2^{(k)}(p) + f_1^{(n)}(p)\,(f_2(1))^k\right)$$

since $h^{(n)}(1) = (h(1))^n$, for all $h \in \mathcal{A}$, and $n \in \mathbb{N}$

$$= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{t^{n+k}}{n!k!} \left((f_1(1))^n \left(k f_2(1)^{k-1} f_2(p) \right) \right.$$
$$\left. (f_2(1))^k \left(n f_1(1)^{n-1} f_1(p) \right) \right)$$

by (3.1.3)

$$= t \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{t^{n+k-1}}{(n-1)!k!} (f_1(1))^{n-1} (f_2(1))^k f_1(p)$$

$$+ t \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{t^{n+k-1}}{n!(k-1)!} (f_1(1))^n (f_2(1))^{k-1} f_2(p)$$

$$= \left(te^{t(f_1(1)+f_2(1))} f_1(p) - te^{tf_1(1)} f_1(p) \right)$$

$$+ \left(te^{t(f_1(1)+f_2(1))} f_2(p) - te^{tf_2(1)} f_2(p) \right)$$

$$= te^{t(f_1(1)+f_2(1))} (f_1(p) + f_2(p))$$

$$- te^{tf_1(1)} f_1(p) - te^{tf_2(1)} f_2(p)$$

$$= g_p\left(Exp_t^*(f_1 + f_2)\right) - g_p\left(Exp_t^*(f_1)\right) - g_p\left(Exp_t^*(f_2)\right),$$

by (3.1.4). □

Let $1_{\mathcal{A}}$ be the identity element of the arithmetic algebra $\mathcal{A}$, i.e.,

$$1_{\mathcal{A}}(n) \overset{def}{=} \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$. Motivated by the morphism $Exp_t^*(\bullet)$ on $\mathcal{A}$, define a morphism

$$E_t : \mathcal{A} \to \mathcal{A}$$

for $t \in \mathbb{C}$, by

(3.1.6)

$$E_t(f) \overset{def}{=} 1_{\mathcal{A}} + \sum_{n=1}^{\infty} \frac{t^n}{n!} f^{(n)}, \text{ for all } f \in \mathcal{A},$$

i.e.,

$$E_t(f) = 1_{\mathcal{A}} + Exp_t^*(f) \text{ in } \mathcal{A}, \text{ for all } f \in \mathcal{A},$$

for $t \in \mathbb{C}$. Also, by identifying $f^{(0)}$ with $1_{\mathcal{A}}$, one has

(3.1.6)′

$$E_t(f) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}, \text{ for } f \in \mathcal{A}.$$

Then, by the above corollary, one obtains that:

Corollary 3.6. *Let $E_t : \mathcal{A} \to \mathcal{A}$ be the morphism as above, for $t \in \mathbb{C}$.*

(3.1.7) $E_1(f_1) * E_1(f_2) = E_1(f_1 + f_2)$ in $\mathcal{A}$, for all $f_1, f_2 \in \mathcal{A}$.

(3.1.8) For all $f \in \mathcal{A}$, the $\mathbb{C}$-valued function $t \mapsto g_p(E_t(f))$ is entire on $\mathbb{C}$, for all primes p.

(3.1.9) For all $f \in \mathcal{A}$, the corresponding arithmetic function $E_t(f)$ is the unique solution to the differential equation;

(i) $E_t(f) \in \mathcal{A}$, *for all* $t \in \mathbb{C}$,
(ii) $\frac{d}{dt}E_t(f) = f * E_t(f) = E_t(f) * f$,
(iii) $E_0(f) = 1_{\mathcal{A}}$,

Proof. Observe that
$E_1(f_1) * E_1(f_1)$

$$= \left(1_{\mathcal{A}} + \sum_{n=1}^{\infty} \frac{1}{n!}\ f_1^{(n)}\right) * \left(1_{\mathcal{A}} + \sum_{k=1}^{\infty} \frac{1}{k!}\ f_2^{(k)}\right)$$

$$= 1_{\mathcal{A}} + \sum_{k=1}^{\infty} \frac{1}{k!}\ f_2^{(k)} + \sum_{n=1}^{\infty} \frac{1}{n!}\ f_1^{(n)}$$

$$+ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n!k!} \left(f_1^{(n)} * f_2^{(k)}\right)$$

since $1_{\mathcal{A}}$ is the identity element of $\mathcal{A}$ (under convolution)

$$= 1_{\mathcal{A}} + Exp_1^*(f_2) + Exp_1^*(f_1) + Exp_1^*(f_1) * Exp_1^*(f_2)$$
$$= E_1(f_1 + f_2),$$

by (3.1.5). Thus, the statement (3.1.7) holds true.

Now, consider the function

$$t \in \mathbb{C} \longmapsto g_p\left(E_t(f)\right) \in \mathcal{A},$$

for an arbitrary fixed arithmetic function $f \in \mathcal{A}$. Notice that

$$g_p\left(E_t(f)\right) = g_p\left(1_{\mathcal{A}} + Exp_t^*(f)\right) = g_p\left(Exp_t^*(f)\right)$$
$$= te^{tf(1)}g_p(f) = \left(tf(p)\right)e^{tf(1)},$$

by (3.1.4). Since $f(p)$ and $f(1)$ are constants in $\mathbb{C}$, the maps

$$t \mapsto tf(p) \text{ and } t \mapsto e^{tf(1)}$$

are entire on $\mathbb{C}$, and hence,

$$t \longmapsto tf(p)e^{tf(1)}$$

is entire on $\mathbb{C}$. Equivalently, the statement (3.1.8) holds.

By (3.1.8) and (3.1.7), the statement (3.1.9) holds true. In particular, one can get that:

$$\frac{t}{dt}\left(E_t(f)\right) = \frac{t}{dt}\left(1_{\mathcal{A}} + \sum_{n=1}^{\infty} \frac{t^n}{n!} f^{(n)}\right)$$

$$= \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} f^{(n)} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!}\left(f^{(n-1)} * f\right)$$

$$= f * \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} f^{(k)}\right)$$

by identifying $h^{(0)} = 1_{\mathcal{A}}$

$$= f * \left(1_{\mathcal{A}} + Exp_t^*(f)\right)$$

$$= f * E_t(f) = E_t(f) * f.$$

□

By (3.1.7), one can obtain that:
(3.1.7)′

$$g_p\left(E_1(f_1) * E_1(f_2)\right) = g_p\left(E_1(f_1 + f_2)\right),$$

for all $f_1,\ f_2 \in \mathcal{A}$, for all primes p.

The above special case (3.1.7) will be extended to our future works below. Also, motivated by (3.1.7) and (3.1.7)′, we obtain the following theorem, too.

Theorem 3.7. *Let $E_t : \mathcal{A} \to \mathcal{A}$ be in the sense of (3.1.6). Define a subset (3.1.10)*

$$\Gamma \overset{def}{=} \{E_1(f) \in \mathcal{A} : f \in \mathcal{A}\}$$

of $\mathcal{A}$. Then the subset Γ of (3.1.10) is an infinite abelian group under convolution. i.e.,

Proof. Define a subset Γ of $\mathcal{A}$ as above. Then, under convolution, it satisfies that

$$E_1(f_1) * E_1(f_2) = E_1(f_1 + f_2),$$

in Γ, by (3.1.7), and hence, the operation $(*)$ is closed in Γ.

$$\begin{aligned}(E_1(f_1) * E_1(f_2)) * E_1(f_3) \\ &= E_1\left(f_1 + f_2 + f_3\right) \\ &= E_1(f_1) * \left(E_1(f_2) * E_1(f_3)\right),\end{aligned}$$

by (3.1.7), for all $f_1,\ f_2,\ f_3 \in \mathcal{A}$. Thus, Γ is associative. i.e., it is a semigroup under $(*)$.

Moreover, there exists an arithmetic function $0_{\mathcal{A}}$ in $\mathcal{A}$,

$$0_{\mathcal{A}}(n) \overset{def}{=} 0, \text{ for all } n \in \mathbb{N},$$

such that

$$E_t\left(0_{\mathcal{A}}\right) = 1_{\mathcal{A}} + Exp_t^*\left(0_{\mathcal{A}}\right) = 1_{\mathcal{A}}, \text{ for all } t \in \mathbb{C}.$$

So, one has $E_1\left(0_{\mathcal{A}}\right) = 1_{\mathcal{A}}$ in Γ, and hence,

$$E_1\left(0_{\mathcal{A}}\right) * E_1(f) = 1_{\mathcal{A}} * E_1(f) = E_1\left(0_{\mathcal{A}} + f\right) = E_1(f),$$

for all $f \in \mathcal{A}$. Therefore, there exists the $(*)$-identity $1_{\mathcal{A}} = E_1\left(0_{\mathcal{A}}\right)$ in Γ. Thus, Γ is a monoid.

For all $f \in \mathcal{A}$, there exists $-f \in \mathcal{A}$. Again, by (3.1.7), we have

$$E_1(f) * E_1(-f) = E_1\left(f + (-f)\right) = E_1\left(0_{\mathcal{A}}\right) = 1_{\mathcal{A}},$$

in Γ. It shows that, for any $E_1(f) \in \Gamma$, there exists a unique inverse $E_1(-f)$ in Γ. Therefore, the subset Γ forms a group under $(*)$ in $\mathcal{A}$.

Furthermore, since the convolution $(*)$ is commutative in $\mathcal{A}$, it is commutative in Γ, too. Therefore, the group Γ is an abelian group in $\mathcal{A}$. □

The above theorem (3.1.10) shows that the group Γ is a Lie group in a Lie algebra $\mathcal{A}$.

And, by (3.1.2) and (3.1.3), we obtain the following joint free moment computation (3.1.6).

Proposition 3.8. *(See [11, 13]) Let $f_1, \cdots, f_s$ be free random variables of the arithmetic p-prime-probability space $(\mathcal{A},\ g_p)$, for $s \in \mathbb{N}$. Then*

(3.1.11)

$$g_p\left(\underset{j=1}{\overset{n}{*}} f_{i_j}\right) = \sum_{j=1}^{n} \left(f_{i_j}(p) \left(\prod_{k \in \{1, \cdots, n\},\, k \neq j} f_{i_k}(1) \right) \right),$$

for all $(i_1, \cdots, i_n) \in \{1, \cdots, s\}^n$, for all $n \in \mathbb{N}$, where the Π on the right-hand side of (3.1.4) means the usual multiplication of $\mathbb{C}$. $\square$

Now, let $f_1, \cdots, f_s$ be free random variables in the arithmetic p-prime-probability space $(\mathcal{A},\ g_p)$, for a prime p, for $s \in \mathbb{N}$. Observe that

$$\begin{aligned} & k_n\left(f_{i_1}, \cdots, f_{i_n}\right) \\ &= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} (g_p)_V \left(f_{i_1}, \cdots, f_{i_n}\right) \mu\left(0_{|V|},\ 1_{|V|}\right) \right) \\ &= \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1, \cdots, j_k) \in \pi} g_p \left(\underset{l=1}{\overset{k}{*}} f_{i_{j_l}} \right) \mu(0_k,\ 1_k) \right) \\ &= \sum_{\pi \in NC(n)} \left(\prod_{V=(j_1, \cdots, j_k) \in \pi} \left(\sum_{t=1}^{k} f_{i_{j_t}}(p) \left(\prod_{u \in \{1, \cdots, k\}, u \neq t} f_{i_{j_u}}(1) \right) \right) \mu(0_k,\ 1_k) \right), \end{aligned}$$

by (3.1.4). So, we obtain the following free-cumulant computation as equivalent free-distributional data of (3.1.11).

Proposition 3.9. *Let $f_1, \cdots, f_s$ be free random variables in the arithmetic p-prime-probability space $(\mathcal{A},\ g_p)$. Then*

(3.1.12)

$$\begin{aligned} & k_n\left(f_{i_1}, \cdots, f_{i_n}\right) \\ &\quad = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \left(\sum_{t \in V} f_{i_{j_t}}(p) \left(\underset{u \in V \setminus \{t\}}{*} f_{i_{j_u}}(1) \right) \right) \mu\left(0_{|V|},\ 1_{|V|}\right) \right), \end{aligned}$$

for all $(i_1, \cdots, i_n) \in \{1, \cdots, s\}^n$, for all $n \in \mathbb{N}$. $\square$

Also, by (3.1.5) and (3.1.12), one obtains the following necessary freeness conditions on $(\mathcal{A},\ g_p)$, for all primes p.

Theorem 3.10. *(See [11]) Let $f_1,\ f_2 \in (\mathcal{A},\ g_p)$. Then f_1 and f_2 are free in $(\mathcal{A},\ g_p)$, if and only if either (3.1.13) or (3.1.14) holds, where*

(3.1.13) $f_1(p) = 0 = f_2(p)$,

(3.1.14) $f_i(1) = 0 = f_j(p)$, where $i \neq j \in \{1,\ 2\}$. $\square$

3.2. **Norm Topologies on $\mathcal{A}$.** Let $(\mathcal{A},\ g_p)$ be the arithmetic p-prime probability space. For a fixed prime p and its corresponding linear functional g_p, define a norm $\|.\|_p$ on $\mathcal{A}$ by

(3.2.1)

$$\|f\|_p \overset{def}{=} \sqrt{|f(1)|^2 + |f(p)|^2},$$

for all $f \in (\mathcal{A}, g_p)$. The definition of morphism $\|.\|_p$ of (3.2.1) is motivated by the structures of [11, 13, 15, 16], where $|.|$ on the right-hand side of (3.2.1) means the *modulus on* $\mathbb{C}$. As we have seen in Proposition 3.2 (and [13]), whenever a prime p is fixed, then free random variables f of the arithmetic p-prime probability space $(\mathcal{A}, g_p)$ are classified by $(f(1), f(p)) \in \mathbb{C}^2$.

One may understand (3.2.1) as a process;

$$f \in \mathcal{A} \longmapsto (f(1), f(p)) \in \mathbb{C}^2 \longmapsto \|(f(1), \ f(p))\|_2 \in \mathbb{R}_0^+,$$

where $\|.\|_2$ means the *usual Euclidean norm on* $\mathbb{C}^2$, where $\mathbb{R}_0^+$ is the subset of $\mathbb{R}$, consisting of all positive real numbers or 0, i.e.,

(3.2.1)′

$$\|f\|_p = \|(f(1), \ f(p))\|_2 \text{, for all } f \in (\mathcal{A}, g_p).$$

Proposition 3.11. *The morphism* $\|.\|_p : \mathcal{A} \to \mathbb{R}_0^+$ *of (3.2.1) is a well-defined pseudo-norm on* $\mathcal{A}$ *with respect to a fixed prime* p.

Proof. By (3.2.1)′, indeed, $\|.\|_p$ is a pseudo-norm on $\mathcal{A}$, since the Euclidean norm $\|.\|_2$ is a well-defined norm on $\mathbb{C}^2$.

Assume now that $f_1 \neq f_2$ in $\mathcal{A} - \{0_{\mathcal{A}}\}$, and assume further that

$$f_j(1) = 0 = f_j(p), \text{ for } j = 1, 2.$$

Then

$$\|f_1\|_p = 0 = \|f_2\|_p \, .$$

Therefore, the morphism $\|.\|_p$ of (3.2.1) is a pseudo-norm, which is not a norm on $\mathcal{A}$. □

Now, define a subset $\mathcal{N}_p$ of $\mathcal{A}$ by

(3.2.2)

$$\mathcal{N}_p \overset{def}{=} \{f \in \mathcal{A} : \|f\|_p = 0\},$$

equivalently,

$$\mathcal{N}_p = \{f \in \mathcal{A} : f(1) = 0 = f(p)\}.$$

Proposition 3.12. *The subset* $\mathcal{N}_p$ *of* $\mathcal{A}$ *is an (two-sided) ideal of* $\mathcal{A}$.

Proof. Let $f_1, f_2 \in \mathcal{N}_p$, and $t_1, t_2 \in \mathbb{C}$. Then

$$(t_1 f_1 + t_2 f_2)(1) = 0,$$

and

$$(t_1 f_1 + t_2 f_2)(p) = 0,$$

so, $t_1 f_1 + t_2 f_2 \in \mathcal{N}_p$, too. Thus, the subset $\mathcal{N}_p$ is a (pure-algebraic) subspace of $\mathcal{A}$.

Now, let $f \in \mathcal{N}_p$, and $h \in \mathcal{A}$. Then

$$(f * h)(1) = f(1)h(1) = 0,$$

and

$$(f * h)(p) = f(1)h(p) + f(p)h(1) = 0.$$

Therefore, $f * h \in \mathcal{N}_p$, too. So, the subspace $\mathcal{N}_p$ is a left ideal of $\mathcal{A}$.

By the commutativity of the convolution $(*)$ on $\mathcal{A}$, the subset $\mathcal{N}_p$ of $\mathcal{A}$ is a (two-sided) ideal. □

Construct now a quotient space $\mathcal{A}_p$ of $\mathcal{A}$ quotient by $\mathcal{N}_p$ as

(3.2.3)

$$\mathcal{A}_p = \mathcal{A} \,/\, \mathcal{N}_p.$$

Then the *normed space* $(\mathcal{A}_p,\ \|.\|_p)$ is well-defined. i.e., the inherited pseudo-norm $\|.\|_p$ of (3.2.1) on $\mathcal{A}$ becomes a well-defined norm on $\mathcal{A}_p$. All elements of $\mathcal{A}_p$ are of the forms

$$[f]_{\mathcal{N}_p} = \{h \in \mathcal{A} : \|h - f\|_p = 0\},$$

as equivalence classes, determined by the quotienting $\mathcal{N}_p$. But, for convenience, we will denote $[f]_{\mathcal{N}_p}$ simply by f, if there is no confusion.

We denote this normed space $(\mathcal{A}_p,\ \|.\|_p)$ simply by $\mathcal{A}_p$. Also, construct the norm-completion $\mathfrak{A}_p$ of $\mathcal{A}_p$,

(3.2.4)

$$\mathfrak{A}_p \overset{def}{=} \overline{\mathcal{A}_p}^{\|.\|_p} \text{ in } \mathcal{A}.$$

where $\overline{X}^{\|.\|_p}$ means the $\|.\|_p$-norm-closure of X in $\mathcal{A}$. i.e., we constructed the corresponding Banach space $\mathfrak{A}_p$ from the normed space $\mathcal{A}_p$ of (3.2.3).

Definition 3.13. The Banach space $\mathfrak{A}_p$ of (3.2.4) induced by the arithmetic p-prime probability space $(\mathcal{A},\ g_p)$ is called the p(-prime)-Banach space of $\mathcal{A}$.

By definition, if f is a "nonzero" element of $\mathfrak{A}_p$, then neither $f(1) = 0$, nor $f(p) = 0$, equivalently,

(3.2.5)

$$\text{either } f(1) \neq 0 \text{ or } f(p) \neq 0.$$

So, without loss of generality, if we mention "$f \in \mathfrak{A}_p$," then one can understand f as an (certain limit of) arithmetic function(s) of $\mathcal{A}$, satisfying (3.2.5).

Hence, the linear functional g_p acts well on $\mathfrak{A}_p$ (under quotient). i.e., we have a Banach probability space $(\mathfrak{A}_p,\ g_p)$.

Definition 3.14. The Banach probability space $(\mathfrak{A}_p,\ g_p)$ is said to be the (arithmetic) p-(prime-)Banach probability space of $\mathcal{A}$.

Let $f \in (\mathcal{A},\ g_p)$, and let $E_t(f) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}$ be in the sense of (3.1.6) and $(3.1.6)'$, for $t \in \mathbb{C}$, with identity: $f^{(0)} = 1_{\mathcal{A}}$. Then

$$\left(E_t(f)\right)(1) = e^{tf(1)},$$

and

$$\left(E_t(f)\right)(p) = te^{tf(1)} f(p).$$

So, if $f \in (\mathcal{A},\ g_p)$, for any arbitrary fixed $t \in \mathbb{C}$,

$$0 \leq \|E_t(f)\|_p < \infty \text{ in } \mathbb{R}_0^+.$$

Thus, $E_t(f) \in \mathfrak{A}_p$, whenever $f \in \mathfrak{A}_p$.

Proposition 3.15. *For any arbitrary fixed $t \in \mathbb{C}$, if $f \in \mathfrak{A}_p$, then $E_t(f) \in \mathfrak{A}_p$, too. Thus, $E_t(f)$ is a free random variable in the p-Banach probability space $(\mathfrak{A}_p, g_p)$.* □

Later, in this paper, we restrict our interests to the case where $t \in \mathbb{R}$.

4. Krein-Space Representations of $(\mathcal{A}, g_p)$

In this section, we briefly introduce a Krein-space representation of $\mathcal{A}$, determined by a fixed prime p, and the corresponding arithmetic p-prime probability space $(\mathcal{A}, g_p)$. For more details, see [15, 16].

In [15], we showed that $\mathbb{C}^2_{A_o} = (\mathbb{C}^2, [,]_{2:A_o})$ is an "indefinite" inner product space, where

$$[(t_1,\ t_2),\ (s_1,\ s_2)]_{2:A_o} = \left\langle \begin{pmatrix} t_1 \\ t_2 \end{pmatrix},\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right\rangle_2,$$

where $<,>_2$ is the inner product on $\mathbb{C}^2$;

$$< (t_1,\, t_2),\, (s_1,\, s_2) >_2 = t_1\overline{s_1} + t_2\overline{s_2},$$

and where

$$A_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Also, there exists a vector-space epimorphism $\pi_p : \mathcal{A} \to \mathbb{C}^2_{A_o}$, such that

$$\pi_p(h) = (h(1),\, h(p)), \text{ for all } f \in \mathcal{A}.$$

Then we have

$$[\pi_p(f),\ \pi_p(h)]_{2:A_o} = g_p\left(f * h^*\right),$$

where

$$h^*(n) \overset{def}{=} \overline{h(n)} \text{ in } \mathbb{C}, \text{ for all } n \in \mathbb{N}.$$

By [7], this indefinite inner product $\mathbb{C}^2_{A_o}$ is isomorphic to the Krein-subspace $\mathfrak{K}_p = \Delta_2 \oplus \Delta_2^-$ of the Krein space $\mathfrak{K}^2 = \mathbb{C}^2 \oplus \mathbb{C}^2$, where

$$\Delta_2 = \{(t,\, t) : t \in \mathbb{C}\}$$

and

$$\Delta_2^- = \{(t,\, -\overline{t}) : t \in \mathbb{C}\}.$$

i.e., $\mathbb{C}^2_{A_o}$ is a Krein space, too under $[,]_{2:A_o}$ (Also, see [16]).

Define now an algebra-action Θ of $\mathcal{A}$ acting on $\mathbb{C}^2_{A_o}$ by

(4.1)

$$f \in \mathcal{A} \longmapsto \Theta_f : \mathbb{C}^2_{A_o} \to \mathbb{C}^2_{A_o}$$

by

$$\Theta_f = \begin{pmatrix} f(1) & 0 \\ f(p) & f(1) \end{pmatrix}, \text{ for all } f \in \mathcal{A}.$$

Then Θ is indeed a well-defined algebra-action of $\mathcal{A}$ acting on $\mathbb{C}^2_{A_o}$. Thus, we can act Θ for $\mathfrak{A}_p$ (under topology).

Moreover, it satisfies that:

(4.2)

$$\Theta_f^* = \Theta_{f^*} = \begin{pmatrix} f^*(1) & 0 \\ f^*(p) & f^*(1) \end{pmatrix},$$

for all $f \in \mathcal{A}$. Remark that, we are using the inner product $[,]_{2:A_o}$ on $\mathbb{C}^2$, not the usual ones.

Indeed, one can check that:

$$[\Theta_f(\xi),\ \eta]_{2:A_o} = [\xi,\ \Theta_{f^*}(\eta)]_{2:A_o},$$

for all ξ, $\eta \in \mathbb{C}^2$.

Also, we have the following multiplication rule;

(4.3)

$$\Theta_{f_1}\Theta_{f_2} = \Theta_{f_1 * f_2}, \text{ for all } f_1,\ f_2 \in \mathcal{A}.$$

The fundamental properties of Θ_f are considered in [15]. The equivalent operators θ_f acting on the isomorphic Krein space $\mathfrak{K}_p$ of $\mathbb{C}^2_{A_o}$ are studied in detail, in [16].

If we take a vector $(1,\ 0)$ in $\mathbb{C}^2_{A_o}$, then it is identified as $\pi_p(h)$, for some $h \in \mathcal{A}$, such that $h(1) = 1$, and $h(p) = 0$. So, one can understand the vector $(1,\ 0)$ of $\mathbb{C}^2_{A_o}$ as the image $\pi_p(1_{\mathcal{A}})$ (e.g., [13]). Denote $(1,\ 0)$ by Ω_p. i.e.,

$$\Omega_p = (1,\ 0) \in \mathbb{C}^2_{A_o}.$$

Then one can define a linear functional φ_p on the operator algebra $B(\mathbb{C}^2_{A_o})$ by

(4.4)

$$\varphi_p(T) \overset{def}{=} [T\Omega_p,\ \Omega_p]_{2:A_o}, \text{ for all } T \in B(\mathbb{C}^2_{A_o}).$$

Then one has

(4.5)

$$\varphi_p(\Theta_f^n) = g_p\left(f^{(n)}\right), \text{ for all } n \in \mathbb{N},$$

for all $f \in \mathcal{A}$, by [15, 16].

So, the free probabilistic model $(\mathcal{A},\ g_p)$ corresponds a free probabilistic model $(B(\mathbb{C}^2_{A_o}),\ \varphi_p)$ (under quotient). By Section 3.2, we can conclude that the p-Banach probability space $(\mathfrak{A}_p,\ g_p)$ induced by $(\mathcal{A},\ g_p)$ corresponds $(B(\mathbb{C}^2_{A_o}),\ \varphi_p)$. i.e., there exists well-defined Krein-space representations

$$f \in (\mathcal{A},\ g_p) \longmapsto \Theta_f \in \left(B(\mathbb{C}^2_{A_o}),\ \varphi_p\right),$$

and

$$f \in (\mathfrak{A}_p,\ g_p) \longmapsto \Theta_f \in \left(B(\mathbb{C}^2_{A_o}),\ \varphi_p\right),$$

under free-probabilistic equivalence (in the sense of Voiculescu, e.g., [30, 32]).

If one constructs a subalgebra $\mathbb{A}_p$, generated by $\{\Theta_f\}_{f \in \mathcal{A}}$, in $B(\mathbb{C}^2_{A_o})$, then $(\mathcal{A},\ g_p)$ is equivalent to $(\mathbb{A}_p,\ \varphi_p)$ "up to quotient," "under a topology of Section 3.2), equivalently, we can get that:

Theorem 4.1. *(See [16]) Free probability spaces $(\mathfrak{A}_p,\ g_p)$ and $(\mathbb{A}_p,\ \varphi_p)$ are equivalent.* □

5. Embedding E of $\mathbb{R}$ on $\mathfrak{A}_p$

Let $E_t : \mathfrak{A}_p \to \mathfrak{A}_p$ be a morphism in the sense of (3.1.6) and (3.1.6)′, for all "$t \in \mathbb{R}$." As we discussed and assumed in Section 3.2, we understand $E_t(f)$ as elements of the p-Banach probability space $(\mathfrak{A}_p,\, g_p)$ of Section 4. Note here that we are restricting our interests to the cases where t are in $\mathbb{R}$ (not in $\mathbb{C}$).

As in $(3.1.6)'$, let

$$E_t\,(f) = \sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}, \text{ for } t \in \mathbb{R},$$

with identity:

$$f^{(0)} = 1_{\mathcal{A}}, \text{ for all } f \in \mathfrak{A}_p.$$

Theorem 5.1. *For any $t,\, s \in \mathbb{R}$, we have*

(5.1)

$$E_{t+s}(f) = E_t(f) * E_s(f) \textit{ in } \mathfrak{A}_p \textit{ for all } f \in \mathfrak{A}_p.$$

Proof. Observe that:

$$\begin{aligned} E_{t+s}(f) &= \sum_{n=0}^{\infty} \frac{(t+s)^n}{n!}\; f^{(n)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} t^k s^{n-k} \right) f^{(n)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k s^{n-k}\; f^{(n)} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{t^k s^{n-k}}{k!(n-k)!} \right) f^{(n)} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k,l \in \mathbb{N}\, \cup\, \{0\},\; n=k+l} \frac{t^k}{k!} \frac{s^l}{l!} \right) f^{(n)}, \end{aligned} \tag{5.2}$$

for all $f \in \mathcal{A}$, for $t,\, s \in \mathbb{R}$. Also, observe that,

$$\begin{aligned} E_t(f) * E_s(f) &= \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} f^{(k)} \right) \left(\sum_{l=0}^{\infty} \frac{s^l}{l!}\; f^{(l)} \right) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^k}{k!} \frac{s^l}{l!}\; f^{(k+l)} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k,l \in \mathbb{N}\, \cup\, \{0\},\; n=k+l} \frac{t^k}{k!} \frac{s^l}{l!} \right) f^{(n)}, \end{aligned} \tag{5.3}$$

for all $f \in \mathfrak{A}_p$, for $t,\, s \in \mathbb{R}$. Therefore, by (5.2) and (5.3), one can conclude that

$$E_{t+s}(f) = E_t(f) * E_s(f),$$

for all $f \in \mathfrak{A}_p$, for all $t,\, s \in \mathbb{R}$. □

The system $\{E_t\}_{t \in \mathbb{R}}$ of morphisms E_t's satisfies

$$E_{t+s}(\bullet) = E_t(\bullet) * E_s(\bullet) \text{ on } \mathfrak{A}_p,$$

by (5.1). Let's understand $\mathbb{R}$ as its maximal additive subgroup $(\mathbb{R}, +)$, which is identical to $\mathbb{R}$, set-theoretically. In *dynamical system*, sometimes, this group is said to be the "*flow*" (up to group-isomorphisms).

Motivated by (5.1), define a group-action E of the flow $\mathbb{R} = (\mathbb{R}, +)$ on the p-Banach algebra $\mathfrak{A}_p$ by

(5.4)
$$E : t \in \mathbb{R} \longmapsto E_t \text{ on } \mathfrak{A}_p.$$

Then E is indeed a well-defined group-action of $\mathbb{R}$ on $\mathfrak{A}_p$, because (i) each E_t is a well-defined function on $\mathfrak{A}_p$, sending an element f of $\mathfrak{A}_p$ to an element $E_t(f)$ of $\mathfrak{A}_p$, and (ii) E satisfies the relation (5.1). i.e., one can get that:

Corollary 5.2. *The morphism E of (5.4) is a group-action of the flow $\mathbb{R}$ acting on $\mathfrak{A}_p$.* □

The above group-action E of the flow $\mathbb{R}$ on $\mathfrak{A}_p$ satisfies the following property.

Proposition 5.3. *Let E be the group action (5.4) of the flow $\mathbb{R}$ acting on $\mathfrak{A}_p$. Then*

(5.5)
$$g_p\left(E_t(f) * E_s(f)\right) = (t+s)e^{(t+s)f(1)}f(p),$$

for all primes p, for all $f \in \mathfrak{A}_p$, and $t,\ s \in \mathbb{R}$.

Proof. By (5.1), one has that

$$E_t(f) * E_s(f) = E_{t+s}(f), \text{ for all } f \in \mathfrak{A}_p,$$

for $t,\ s \in \mathbb{R}$. Thus,

$$\begin{aligned}(E_t(f) * E_s(f))\,(p) &= g_p\left(E_t(f) * E_s(f)\right) \\ &= g_p\left(E_{t+s}(f)\right) = (t+s)e^{(t+s)f(1)}f(p),\end{aligned}$$

by (3.1.4), because $g_p\left(E_t(f)\right) = g_p\left(Exp_t^*(f)\right)$, for all primes p, for all $f \in \mathcal{A}$. □

The above relation (5.5) (with the general formula (3.11)) guarantees that:

$$\begin{aligned} & g_p\left(E_t(f) * E_s(f)\right) \\ &= E_t(f)(1)\ E_s(f)(p) + E_t(f)(p)\ E_s(f)(1) \end{aligned}$$

by (3.1.1)

$$\begin{aligned} &= \left(\sum_{n=0}^{\infty} \tfrac{t^n}{n!} f^{(n)}(1)\right)\left(se^{sf(1)}f(p)\right) \\ &\qquad + \left(te^{tf(1)}f(p)\right)\left(\sum_{k=0}^{\infty} \tfrac{s^k}{k!} f^{(k)}(1)\right) \end{aligned}$$

by (3.1.4)

$$\begin{aligned} &= \left(\sum_{n=0}^{\infty} \tfrac{(tf(1))^n}{n!}\right)\left(se^{sf(1)}f(p)\right) \\ &\qquad + \left(te^{tf(1)}f(p)\right)\left(\sum_{k=0}^{\infty} \tfrac{(sf(1))^k}{k!}\right) \end{aligned}$$

since $h^{(m)}(1) = (h(1))^m$, for all $h \in \mathfrak{A}_p$, for all $m \in \mathbb{N}$

$$\begin{aligned} &= \left(e^{tf(1)}\right)\left(se^{sf(1)}f(p)\right) + \left(te^{tf(1)}f(p)\right)\left(e^{sf(1)}\right) \\ &= se^{(t+s)f(1)}f(p) + te^{(t+s)f(1)}f(p) \\ &= \left(se^{(t+s)f(1)} + te^{(t+s)f(1)}\right)(f(p)) \\ &= (t+s)\,e^{(t+s)f(1)}f(p) = g_p\left(E_{t+s}(f)\right). \end{aligned}$$

Recall that, for any arithmetic function $f \in \mathfrak{A}_p$, one can get f^* in $\mathfrak{A}_p$, such that

$$f^*(n) = \overline{f(n)} \text{ in } \mathbb{C}, \text{ for all } n \in \mathbb{N}.$$

The group-action E also satisfies that:

Proposition 5.4. *Let $f \in \mathfrak{A}_p$, and let $E_t(f)$ be the corresponding element in $\mathfrak{A}_p$, for $t \in \mathbb{R}$. Then $(E_t(f))^* = E_t(f^*)$.*

Proof. Observe that:

$$(E_t(f))^*(k) = \overline{(E_t(f))(k)}$$

$$= \overline{\sum_{n=0}^{\infty} \frac{t^n}{n!} f^{(n)}(k)} = \sum_{n=0}^{\infty} \overline{\left(\frac{t^n}{n!}\right)} \overline{f^{(n)}(k)}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left((f^*)^{(n)}(k) \right)$$

since $t \in \mathbb{R}$, and since $\left(f^{(n)}\right)^* = (f^*)^{(n)}$, for all $k \in \mathbb{N}$

$$= E_t(f^*),$$

for all $t \in \mathbb{R}$, for all $f \in \mathfrak{A}_p$, and $k \in \mathbb{N}$. And hence, one can obtain that

$$(E_t(f))^* = E_t(f^*).$$

for all $f \in \mathfrak{A}_p$, and $t \in \mathbb{R}$. ☐

Consider now that, for f, $h \in \mathfrak{A}_p$, and for $t \in \mathbb{R}$,

$$E_t(f+h) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (f+h)^{(n)}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \left(f^{(k)} * h^{(n-k)}\right) \right)$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left(f^{(k)} * h^{(n-k)}\right) \right)$$

$$= \sum_{n=0}^{\infty} t^n \left(\sum_{k,l \in \mathbb{N}\cup\{0\},\, n=k+l} \frac{1}{k!l!} \left(\frac{f^{(k)}}{k!} * \frac{h^{(l)}}{l!} \right) \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k,l \in \mathbb{N}\cup\{0\},\, n=k+l} \frac{t^n}{k!l!} \left(\frac{f^{(k)}}{k!} * \frac{h^{(l)}}{l!} \right) \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k,l \in \mathbb{N}\cup\{0\},\, n=k+l} \frac{t^k t^l}{k!l!} \left(f^{(k)} * h^{(l)} \right) \right)$$

(5.6)

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^k t^l}{k!l!} \left(f^{(k)} * h^{(l)} \right).$$

Therefore, one can obtain the following theorem, generalizing (3.1.7).

Theorem 5.5. *Let f, $h \in \mathfrak{A}_p$, and let E be in the sense of (5.4). Then*

(5.7)

$$E_t(f) * E_t(h) = E_t(f+h), \text{ for all } t \in \mathbb{R}.$$

Proof. The proof of (5.7) is done by the above computation (5.6). By (5.6), we have

$$E_t(f+h) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^k t^l}{k!l!} \left(f^{(k)} * h^{(l)} \right).$$

By definition, one can get that:

$$E_t(f) * E_t(h) = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} f^{(k)}\right) * \left(\sum_{l=0}^{\infty} \frac{t^l}{l!} h^{(l)}\right)$$
$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^k t^l}{k!l!} \left(f^{(k)} * h^{(l)}\right).$$

Therefore,

$$E_t(f+h) = E_t(f) * E_t(h),$$

for all $t \in \mathbb{R}$, for $f \in \mathfrak{A}_p$. □

Definition 5.6. We call the images E_t of the group-action E of the flow $\mathbb{R}$ acting on $\mathfrak{A}_p$, the t-th exponential on $\mathfrak{A}_p$. Also, we call the group-action E, the flowed exponential on $\mathfrak{A}_p$.

The following theorem generalizes (5.1) and (5.7) together.

Theorem 5.7. *Let f, $h \in \mathfrak{A}_p$, and t, $s \in \mathbb{R}$. Then*
(5.8)

$$E_t(f) * E_s(h) = E_1(tf + sh) \text{ in } \mathfrak{A}_p.$$

Proof. Observe that:

$$E_t(f) * E_s(h) = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} f^{(n)}\right) * \left(\sum_{l=0}^{\infty} \frac{s^l}{l!} h^{(l)}\right)$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^k s^l}{k!l!} \left(f^{(k)} * h^{(l)}\right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k,l \in \mathbb{N} \cup \{0\},\ n=k+l} \frac{t^k s^l}{k!l!} \left(f^{(k)} * h^{(l)}\right)\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k,l \in \mathbb{N} \cup \{0\},\ n=k+l} n!\ \frac{t^k s^l}{k!l!} \left(f^{(k)} * h^{(l)}\right)\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{j=0}^{\infty} \frac{n!}{j!(n-j)!} t^j s^{n-j} \left(f^{(j)} * h^{(n-j)}\right)\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{j=0}^{\infty} \begin{pmatrix} n \\ j \end{pmatrix} \left((tf)^{(j)} * (sh)^{(n-j)}\right)\right)$$

because $(ra)^{(n)} = r^n\, a^{(n)}$, for all $a \in \mathcal{A}$, $r \in \mathbb{R}$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (tf + sh)^{(n)}$$

$$= E_1(tf + sh).$$

□

By (5.8), one can verify that

$$E_t(f) * 0_{\mathcal{A}} = E_t(f) * E_s(0_{\mathcal{A}}) = E_1(tf + s \cdot 0_{\mathcal{A}}) = E_1(tf),$$

i.e.,

$$E_t(f) = E_1(tf),$$

for all $f \in \mathfrak{A}_p$, $t \in \mathbb{R}$.

Corollary 5.8. *Let $f \in \mathfrak{A}_p$, and $t \in \mathbb{R}$. Then*
(5.9)

$$E_t(f) = E_1(tf).$$

□

Indeed, from (5.8) and (5.9), one can re-obtain (5.1) and (5.7) as follows:

$$E_t(f) * E_s(f) = E_1(tf + sf) = E_1((t+s)f) = E_{t+s}(f),$$

and

$$E_t(f) * E_t(h) = E_1(tf + th) = E_1(t(f+h)) = E_t(f+h),$$

for all f, $h \in \mathfrak{A}_p$, and t, $s \in \mathbb{R}$.

Remark 5.9. In fact, the relations (5.8) and (5.9) hold if t is taken in $\mathbb{C}$.

Also, by (5.9) and by the above remark, one can obtain the following corollary.

Corollary 5.10. *Let Γ be a group in the sense of (3.1.10), and let*

$$\Gamma' \overset{def}{=} \{E_t(f) : \forall f \in \mathfrak{A}_p, \forall t \in \mathbb{C}\}$$

be a subset of $\mathfrak{A}_p$. Then Γ' is a group-isomorphic to Γ as groups.

Proof. The proof is done by (5.9). i.e.,

$$\begin{aligned} \Gamma' &= \{E_t(f) = E_1(tf) : \forall f \in \mathfrak{A}_p, \forall t \in \mathbb{C}\} \\ &\subseteq \{E_1(f) : \forall f \in \mathfrak{A}_p\} = \Gamma. \end{aligned}$$

Thus, Γ' is a subset of Γ, set-theoretically. Moreover, under convolution, Γ' is homomorphic to Γ, by (5.8). i.e.,

$$E_{t_1}(f_1) * E_{t_2}(f_2) = E_1(t_1 f_1 + t_2 f_2)$$

$$\longmapsto E_1(t_1 f_1) * E_1(t_2 f_2),$$

for all f_1, $f_2 \in \mathfrak{A}_p$, and t_1, $t_2 \in \mathbb{R}$. So, Γ' is a subgroup of Γ.

Observe that Γ is a subset of Γ'. Indeed, if $h \in \Gamma$, then $h = E_1(f)$, for some $f \in \mathfrak{A}_p$. Moreover, if $f = tf_1$ in $\mathfrak{A}_p$, for $t \in \mathbb{C}$, and $f_1 \in \mathfrak{A}_p$, then it is identical to $E_t(f_1)$ in Γ. i.e., a group Γ is a subset of Γ' (which is homomorphic to Γ).

Therefore, Γ is group-isomorphic to Γ'. □

So, we can get a subgroup Γ_+ of Γ, defined by

$$\Gamma_+ = \{E_t(f) : f \in \mathfrak{A}_p, t \in \mathbb{R}\}.$$

Then it is a (classical) Lie group.

Let $f_0 \in \mathfrak{A}_p$ be a fixed nonzero arithmetic function, i.e., $f_0 \neq 0_{\mathfrak{A}_p}$. For this fixed $f_0 \in \mathfrak{A}_p$, define a subset Γ_{f_0} of Γ_+ by
(5.9)

$$\Gamma_{f_0} \overset{def}{=} \{E_t(f_0) : t \in \mathbb{R}\}.$$

Clearly, Γ_{f_0} is a subset of the group Γ_+. Moreover, it satisfies that:
(5.10)

$$E_t(f_0) * E_s(f_0) = E_{t+s}(f_0),$$

for all t, $s \in \mathbb{R}$, and $E_0(f_0)$ acts as the $(*)$-identity on Γ_{f_0}, i.e.,
(5.11)

$$\begin{aligned} E_t(f_0) * E_0(f_0) &= E_t(f_0) * 1_{\mathcal{A}} = E_t(f_0) \\ &= 1_{\mathcal{A}} * E_t(f_0) = E_0(f_0) * E_t(f_0), \end{aligned}$$

for all $t \in \mathbb{R}$. Indeed,

$$E_0(f_0) = E_1(0 \cdot f_0) = E_1(0_{\mathcal{A}}) = \sum_{n=0}^{\infty} \frac{1}{n!} 0_{\mathcal{A}}^{(n)} = 1_{\mathcal{A}}.$$

Also, each element $E_t(f_0)$ has its $(*)$-inverse $E_{-t}(f_0)$, such that:
(5.12)

$$E_t(f_0) * E_{-t}(f_0) = E_0(f_0) = E_{-t}(f_0) * E_t(f_0).$$

Proposition 5.11. *Let Γ_{f_0} be a subset of the group Γ_+, in the sense of (5.9), for nonzero $f_0 \in \mathfrak{A}_p$. Then it is a subgroup of Γ under convolution $(*)$. Moreover, it is group-isomorphic to the flow $\mathbb{R}$. i.e.,*
(5.13)

$$\Gamma_{f_0} = (\Gamma_{f_0}, *) \overset{\text{Group}}{=} (\mathbb{R}, +) = \mathbb{R}.$$

Proof. By (5.10), the convolution $(*)$ is closed in Γ_{f_0}. Also, the operation is associative;

$$\begin{aligned} &(E_{t_1}(f_0) * E_{t_2}(f_0)) * E_{t_3}(f_0) \\ &\quad = E_{t_1+t_2+t_3}(f_0) \\ &\quad = E_{t_1}(f_0) * (E_{t_2}(f_0) * E_{t_3}(f_0)), \end{aligned}$$

by (5.1) and (5.8).

Also, the $(*)$-identity $1_{\mathcal{A}} = E_0(f_0)$ is contained in Γ_{f_0}, by (5.11). Finally, every element $E_t(f_0)$ is $(*)$-invertible with its $(*)$-inverse $E_{-t}(f_0)$, for all $t \in \mathbb{R}$. Therefore, the subset Γ_{f_0}, for a fixed $f_0 \in \mathcal{A}$, of Γ is a subgroup.

This subgroup Γ_{f_0} is group-isomorphic to the flow $\mathbb{R}$. Indeed, one can define a group-isomorphism,

$$\varphi : E_t(f_0) \in \Gamma_{f_0} \longmapsto t \in \mathbb{R}.$$

□

By the above proposition, one can realize that the Lie group Γ_+ is generated (or sectionized) by the system $\{\Gamma_f\}_{f \in \mathcal{A}}$ of subgroups Γ_f in the sense of (5.9). i.e., Γ_+ is filtered by $\mathcal{A}$.

6. Flowed Exponential E on $\mathcal{A}$ as Krein-Space Operators

As we have seen in Section 4, each arithmetic function f, as a free random variable of the p-Banach probability space $(\mathfrak{A}_p, g_p)$ (under quotient), is understood as a Krein-space operator Θ_f acting on the Krein-space $\mathbb{C}^2_{A_o} \overset{\text{Krein}}{=} \mathfrak{K}_p$, satisfying that:

$$\Theta_f = \begin{pmatrix} f(1) & 0 \\ f(p) & f(1) \end{pmatrix},$$

with

$$\Theta_f^* = \Theta_{f^*} \text{ and } \Theta_f \Theta_h = \Theta_{f*h}, \text{ on } \mathbb{C}^2_{A_o},$$

for all $f,\, h \in \mathcal{A}$,

So, if f is a free random variable of the p-Banach probability space $(\mathfrak{A}_p,\, g_p)$, then the corresponding Krein-space operator Θ_f is well-defined on $\mathbb{C}^2_{A_o}$.

Now, let $f \in \mathfrak{A}_p$, and $t \in \mathbb{R}$, and suppose $E_t(f)$ is the t-th exponential of f in $(\mathfrak{A}_p,\, g_p)$. Then

(6.1)

$$\Theta_{E_t(f)} = \begin{pmatrix} (E_t(f))\,(1) & 0 \\ (E_t(f))\,(p) & (E_t(f))\,(1) \end{pmatrix} = \begin{pmatrix} e^{tf(1)} & 0 \\ te^{tf(1)}f(p) & e^{tf(1)} \end{pmatrix}$$

on $\mathbb{C}^2_{A_o}$.

Proposition 6.1. *Let $E_t(f) \in \Gamma_+$, for $f \in \mathfrak{A}_p$, and $t \in \mathbb{R}$. Then*

(6.2)

$$\Theta_{E_t(f)} = e^{tf(1)} \begin{pmatrix} 1 & 0 \\ tf(p) & 1 \end{pmatrix} \text{ on } \mathbb{C}^2_{A_o}.$$

Proof. The proof of (6.2) is directly from (6.1). □

The formula (6.2) shows that, whenever a Krein-space operator $\Theta_{E_t(f)}$ is fixed on $\mathbb{C}^2_{A_o}$, there exists $h \in \mathfrak{A}_p$ (or $h \in \mathcal{A}$), such that: (i) h is unital in the sense that: $h(1) = 1$, (ii) $h(p) = tf(p)$, and (iii)

$$\Theta_{E_t(f)} = e^{tf(1)}\ \Theta_h \text{ on } \mathbb{C}^2_{A_o}.$$

Remark that such an element h is unique in $\mathfrak{A}_p$ (under the quotient on $\mathcal{A}$).

By (6.1) and (6.2), one can get that:

Proposition 6.2. *Let $E_t(f),\, E_s(h) \in (\mathfrak{A}_p,\, g_p)$, for $f,\, h \in \mathfrak{A}_p$, and $t,\, s \in \mathbb{R}$, and let $\Theta_{E_t(f)}$ and $\Theta_{E_s(h)}$ be corresponding Krein-space operators on $\mathbb{C}^2_{A_o}$. Then*

(6.3)

$$\Theta_{E_t(f)}\ \Theta_{E_s(h)} = e^{tf(1)+sh(1)} \begin{pmatrix} 1 & 0 \\ tf(p) + sh(p) & 1 \end{pmatrix},$$

on $\mathbb{C}^2_{A_o}$.

Proof. Note that

$$\Theta_{E_t(f)}\Theta_{E_s(h)} = \Theta_{E_t(f)*E_s(h)}.$$

Thus, it is identical to

$$\Theta_{E_1(tf+sh)} = \begin{pmatrix} (E_1(tf+sh))\,(1) & 0 \\ (E_1(tf+sh))\,(p) & (E_1(tf+sh))\,(1) \end{pmatrix}$$

$$= e^{tf(1)+sh(1)} \begin{pmatrix} 1 & 0 \\ tf(p) + sh(p) & 1 \end{pmatrix}.$$

□

7. Dynamical Systems on $\mathfrak{A}_p$

In this section, we act the flow $\mathbb{R} = (\mathbb{R},\, +)$ on the p-Banach algebra $\mathfrak{A}_p$, for a fixed prime p. In particular, we identify the flow $\mathbb{R}$ as its isomorphic group Γ_{f_0}, for some $f_0 \in \mathfrak{A}_p \setminus \{0_{\mathfrak{A}_p}\}$ (See (5.9)). Remark that, for any $h \in \mathfrak{A}_p \setminus \{0_{\mathfrak{A}_p}\}$, two subgroups Γ_h and Γ_{f_0} of the Lie group Γ_+ are group-isomorphic from each other, because

$$\Gamma_h \overset{\text{Group}}{=} \mathbb{R},$$

by (5.13), whenever h is a nonzero element of $\mathfrak{A}_p$. It means that: (i) we are free from the choice of f_0 to construct subgroups Γ_{f_0} in $\mathfrak{A}_p$, and (ii) Γ_+ has all isomorphic filters $\{\Gamma_h\}_{h\in\mathfrak{A}_p}$.

As in Section 6, one may understand $\mathfrak{A}_p$ as a Banach algebra $\mathbb{A}_p = \Theta(\mathfrak{A}_p)$ realized on the Krein space $\mathbb{C}^2_{A_o}$. i.e., one can identify $\mathfrak{A}_p$ as

$$\mathbb{A}_p = \{\Theta_f \in B(\mathbb{C}^2_{A_o}) : f \in \mathfrak{A}_p\}.$$

So, similarly, one may understand Γ_{f_0} as the subgroup

$$\left(\{\Theta_{E_t(f_0)} : t \in \mathbb{R}\},\ \cdot\right),$$

of $\mathbb{A}_p$. We denote the above group in $\mathbb{A}_p$ again by Γ_{f_0}.

Notation From now on, if there is no confusion, then denote $E_t(f_0) \in \Gamma_{f_0}$ simply by E^t, for a fixed $f_0 \in \mathfrak{A}_p \setminus \{0_{\mathfrak{A}_p}\}$. Also, denote the quantities $f_0(1)$ and $f_0(p)$ by w_1 and w_p, respectively. Further, let $u_j = \mathrm{Re}(w_j)$, for $j = 1, p$, where $\mathrm{Re}(z)$ means the *real part of* z, for all $z \in \mathbb{C}$. □

Define now an action α^{f_0} of the flow $\mathbb{R} \overset{\text{Group}}{=} \Gamma_{f_0}$ acting on the p-Banach algebra $\mathbb{A}_p$ by

(7.1)

$$\alpha_t^{f_0}(\Theta_f) \overset{def}{=} \Theta_{E^t}\Theta_f\Theta^*_{E^t}, \text{ for all } f \in \mathfrak{A}_p,$$

for all $t \in \mathbb{R}$.

By the very definition (7.1), each morphism $\alpha_t^{f_0}$ is a well-defined function on $\mathbb{A}_p$. And it satisfies that:

$$\begin{aligned}\left(\alpha_t^{f_0} \circ \alpha_s^{f_0}\right)(\Theta_{f_0}) &= \alpha_t^{f_0}\left(\alpha_s^{f_0}(\Theta_f)\right)\\ &= \alpha_t^{f_0}\left(\Theta_{E^s}\Theta_f\Theta^*_{E^s}\right) = \Theta_{E^t}\Theta_{E^s}\Theta_f\Theta^*_{E_s}\Theta^*_{E_t}\\ &= \Theta_{E^t * E^s}\Theta_f\Theta_{E_s^* * E_t^*} = \Theta_{E^{t+s}}\Theta_f\Theta_{(E^t*E^s)^*}\end{aligned}$$

since $\mathfrak{A}_p$ is commutative under $(*)$

$$= \Theta_{E^{t+s}}\Theta_f\Theta_{(E^{t+s})^*}$$

(7.2)

$$= \alpha_{t+s}^{f_0}(\Theta_f),$$

for all $t, s \in \mathbb{R}$.

Proposition 7.1. *The morphism α^{f_0} of (7.1) is a well-defined group action of the flow $\mathbb{R} = \Gamma_{f_0}$ acting on the Banach algebra $\mathbb{A}_p$, with*

$$\alpha_0^{f_0} = 1_{B(\mathbb{C}^2_{A_o})} = 1_{\mathbb{A}_p}, \text{ on } \mathbb{A}_p,$$

equivalently, $\alpha_t^{f_0}$ has its inverse $\alpha_{-t}^{f_0}$ on $\mathbb{A}_p$, for all $t \in \mathbb{R}$.

Proof. As we discussed in the above paragraph, each $\alpha_t^{f_0}$ is a well-defined function on $\mathbb{A}_p$, for all $t \in \mathbb{R}$, and the morphism α^{f_0} satisfies that

$$\alpha_t^{f_0} \circ \alpha_s^{f_0} = \alpha_{t+s}^{f_0}, \text{ for all } t, s \in \mathbb{R},$$

on $\mathbb{A}_p$, by (7.2). Therefore, indeed, the morphism α^{f_0} is a group action of Γ_{f_0}, which is group-isomorphic to the flow $\mathbb{R}$, acting on $\mathbb{A}_p$.

Let $t = 0$. Then, for any $\Theta_f \in \mathbb{A}_p$, one has that

$$\begin{aligned}\alpha_0^{f_0}(\Theta_f) &= \Theta_{E^0}\Theta_f\Theta^*_{(E^0)} = \Theta_{1_\mathcal{A}}\Theta_f\Theta^*_{1_\mathcal{A}} \\ &= 1_{\mathbb{A}_p}\Theta_f 1_{\mathbb{A}_p} = \Theta_f,\end{aligned}$$

by Section 5. i.e., $\alpha_0^{f_0} = 1_{\mathbb{A}_p}$, on $\mathbb{A}_p$.

It also demonstrates that each operator $\alpha_t^{f_0}$ on $\mathbb{A}_p$ has its inverse $\alpha_{-t}^{f_0}$, by (7.2), for all $t \in \mathbb{R}$. □

By (5.1) and by the fact; $(E^t)^* = E_t(f_0^*)$, one obtains that:

$$\alpha_t^{f_0}(f) = \Theta_{E^t}\Theta_f\Theta^*_{E^t} = \Theta_{E^t}\Theta_f\Theta_{E_t(f_0^*)}$$

$$= \left(e^{tf_0(1)}\begin{pmatrix} 1 & 0 \\ tf_0(p) & 1 \end{pmatrix}\right)\begin{pmatrix} f(1) & 0 \\ f(p) & f(1) \end{pmatrix}\left(e^{t\overline{f_0(1)}}\begin{pmatrix} 1 & 0 \\ t\overline{f_0(p)} & 1 \end{pmatrix}\right)$$

$$= e^{t\left(f_0(1)+\overline{f_0(1)}\right)}\begin{pmatrix} 1 & 0 \\ tf_0(p) & 1 \end{pmatrix}\begin{pmatrix} f(1) & 0 \\ f(p) & f(1) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t\overline{f_0(p)} & 1 \end{pmatrix}$$

$$= e^{t\mathrm{Re}(f_0(1))}\begin{pmatrix} f(1) & 0 \\ tf_0(p)f(1) + f(p) + tf(1)\overline{f_0(p)} & f(1) \end{pmatrix}$$

$$= e^{t\mathrm{Re}(f_0(1))}\begin{pmatrix} f(1) & 0 \\ tf(1)\mathrm{Re}\,(f_0(p)) + f(p) & f(1) \end{pmatrix}.$$

$$= e^{t\mathrm{Re}(w_1)}\begin{pmatrix} f(1) & 0 \\ tf(1)\,(\mathrm{Re}(w_p)) + f(p) & f(1) \end{pmatrix}.$$

$$= e^{tu_1}\begin{pmatrix} f(1) & 0 \\ tu_pf(1) + f(p) & f(1) \end{pmatrix}.$$

The following proposition is obtained by the above computation.

Proposition 7.2. *Let α^{f_0} be a group action (7.1) of the flow $\mathbb{R}$ acting on $\mathbb{A}_p$. Then*

(7.3)

$$\alpha_t^{f_0}(\Theta_f) = e^{tu_1}\begin{pmatrix} f(1) & 0 \\ tu_pf(1) + f(p) & f(1) \end{pmatrix} \text{ in } \mathbb{A}_p,$$

for all $\Theta_f \in \mathbb{A}_p$. □

Since we took f_0 in $\mathfrak{A}_p \setminus \{0_{\mathfrak{A}_p}\}$,

either $w_1 \neq 0$, or $w_p \neq 0$.

Suppose both $w_1 \neq 0$, and $w_p \neq 0$. Then clearly, $\alpha_t^{f_0}(\Theta_f)$ satisfies the general expression (7.3);

$$\alpha_t^{f_0}(\Theta_f) = e^{tu_1}\begin{pmatrix} f(1) & 0 \\ tu_p f(1) + f(p) & f(1) \end{pmatrix}, \text{ in } \mathbb{A}_p.$$

Assume now that $w_1 = 0$, and $w_p \neq 0$. Then $u_1 = 0$, and $u_p = \mathrm{Re}(w_p)$ in $\mathbb{C}$. Thus, in such a case, the formula (7.3) goes to

$$\alpha_t^{f_0}(\Theta_f) = \begin{pmatrix} f(1) & 0 \\ tu_p f(1) + f(p) & f(1) \end{pmatrix}, \text{ in } \mathbb{A}_p$$

Let's assume $w_1 \neq 0$, and $w_p = 0$. Then $u_1 = \mathrm{Re}(w_1)$, and $u_p = 0$ in $\mathbb{C}$. So, in this case, the formula (7.3) becomes

$$\alpha_t^{f_0}(\Theta_f) = e^{tu_1}\begin{pmatrix} f(1) & 0 \\ f(p) & f(1) \end{pmatrix} = e^{tu_1}\Theta_f, \text{ in } \mathbb{A}_p.$$

More general to (7.3), we obtain the following computations.

Theorem 7.3. *Let α^{f_0} be the group action (7.1) of the flow $\mathbb{R} = \Gamma_{f_0}$ acting on $\mathbb{A}_p$. Then*

$$\alpha^{f_0}_{\Sigma_{j=1}^N t_j}(\Theta_f) = \left(\prod_{j=1}^{N} e^{t_j u_1}\right)\begin{pmatrix} f(1) & 0 \\ \sum_{j=1}^N t_j u_p f(1) + f(p) & f(1) \end{pmatrix},$$

and

$$\alpha_t^{f_0}\left(\prod_{j=1}^{N}\Theta_{f_j}\right) = e^{tu_1}\begin{pmatrix} k_1 & 0 \\ k_p & k_1 \end{pmatrix},$$

in $\mathbb{A}_p$, where

$$k_1 = \prod_{j=1}^{N} f_j(1),$$

and

$$k_p = tu_p\left(\prod_{j=1}^{N} f_j(1)\right) + \sum_{j=1}^N f_j(p)\left(\prod_{l\neq j\in\{1,\cdots,N\}} f_l(1)\right),$$

in $\mathbb{C}$, for all t, $t_1, \cdots, t_N \in \mathbb{R}$, and f, $f_1, \cdots, f_N \in \mathfrak{A}_p$, for all $N \in \mathbb{N}$.

Proof. By (7.3), if we let $t = \sum_{j=1}^N t_j$ in $\mathbb{R}$, then

$$\alpha^{f_0}_{\Sigma_{j=1}^N t_j}(\Theta_f) = \alpha_t^{f_0}(\Theta_f)$$

$$= e^{tu_1}\begin{pmatrix} f(1) & 0 \\ tu_p f(1) + f(p) & f(1) \end{pmatrix}$$

$$= \left(\prod_{j=1}^{N} e^{t_j u_1}\right)\begin{pmatrix} f(1) & 0 \\ \sum_{j=1}^N t_j u_p f(1) + f(p) & f(1) \end{pmatrix},$$

for all $t_1, \cdots, t_N \in \mathbb{R}$, for all $N \in \mathbb{N}$.

Also, one obtains that:

$$\alpha_t^{f_0}\left(\prod_{j=1}^{N}\Theta_{f_j}\right) = \alpha_t^{f_0}\left(\Theta_{\underset{j=1}{\overset{N}{*}} f_j}\right),$$

by (5.1) and (5.5)

$$= e^{tu_1}\begin{pmatrix} \left(\overset{N}{\underset{j=1}{*}} f_j\right)(1) & 0 \\ tu_p\left(\overset{N}{\underset{j=1}{*}} f_j\right)(1) + \left(\overset{N}{\underset{j=1}{*}} f_j\right)(p) & \left(\overset{N}{\underset{j=1}{*}} f_j\right)(1) \end{pmatrix}$$

by (7.3)

$$= e^{tu_1}\begin{pmatrix} \prod_{j=1}^{N}(f_j(1)) & 0 \\ tu_p\left(\prod_{j=1}^{N} f_j(1)\right) + g_p\left(\overset{N}{\underset{j=1}{*}} f_j\right) & \prod_{j=1}^{N}(f_j(1)) \end{pmatrix}$$

$$= e^{tu_1}\begin{pmatrix} k_1 & 0 \\ k_p & k_1 \end{pmatrix},$$

where

$$k_1 = \prod_{j=1}^{N}(f_j(1)),$$

and

$$k_p = tu_p\left(\prod_{j=1}^{N} f_j(1)\right) + \sum_{j=1}^{N} f_j(p)\left(\prod_{l\neq j\in\{1,\cdots,N\}} f_l(1)\right),$$

in $\mathbb{C}$, by (3.1.11), where $f_1, \cdots, f_N \in \mathfrak{A}_p$, for $N \in \mathbb{N}$. □

By the well-defined homomorphism Θ from $\mathfrak{A}_p$ to $\mathbb{A}_p$, one can understand our flowed action α^{f_0} (acting on $\mathbb{A}_p$) as a flowed action (7.6) below, acting on $\mathfrak{A}_p$,

(7.6)

$$\alpha_t^{f_0}(h) = E_t * h * E_t^*, \text{ for all } h \in \mathfrak{A}_p,$$

for all $t \in \mathbb{R}$. Remark that (7.6) is identified with

$$\alpha_t^{f_0}(h) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^{n+k}}{n!k!}\left((f_0)^{(n)} * h * (f_0^*)^{(k)}\right),$$

for all $h \in \mathfrak{A}_p$.

Definition 7.4. Let α^{f_0} be the group action (7.6) of the flow $\mathbb{R}$ acting on the p-Banach algebra $\mathfrak{A}_p$. The mathematical triple $(\mathbb{R}, \mathfrak{A}_p, \alpha^{f_0})$ is called the p-prime Γ_{f_0}-dynamical system of $\mathbb{R}$ on $\mathfrak{A}_p$.

Let $(\mathbb{R}, \mathfrak{A}_p, \alpha^{f_0})$ be the p-prime Γ_{f_0}-dynamical system of $\mathbb{R} = \Gamma_{f_0}$ on $\mathfrak{A}_p$. Then one can construct the corresponding crossed product Banach algebra,

(7.7)

$$\mathfrak{X}_{f_0:p} \overset{def}{=} \mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R},$$

by the Banach algebra generated by $\mathfrak{A}_p$ and

$$\alpha^{f_0}(\mathbb{R}) = \{\Theta_{E^t} \in \mathbb{A}_p : t \in \mathbb{R}\},$$

satisfying the following formulae (7.8) and (7.9) below:

$$\begin{aligned}(f\Theta_{E^t})(h\Theta_{E^s}) &= f\ \Theta_{E^t} h\Theta_{E^s} \\ &= f\ \Theta_{E^t} h\left(1_{\mathbb{A}_p}\Theta_{E^s}\right) \\ &= f\ \Theta_{E^t}\ h\ (\Theta_{E^0}\Theta_{E^s}) \\ &= f\ \Theta_{E^t}\ h\ \Theta_{(E^t)^* * (E^{-t})^*}\ \Theta_{E^s}\end{aligned}$$

because

$$(E^t)^* * (E^{-t})^* = (E^t * E^{-t})^* = (E^0)^* = 1_{\mathfrak{A}_p}^* = 1_{\mathfrak{A}_p}$$

in $\mathfrak{A}_p$, and hence,

$$= f\ \Theta_{E^t}\ h\ \Theta_{(E^t)^*}\ \Theta_{(E^{-t})^*}\ \Theta_{E^s}$$
$$= f\ \left(\Theta_{E^t}\ h\ \Theta_{(E^t)^*}\right)\ \Theta_{(E^{-t})^*}\ \Theta_{E^s}$$

(7.8)

$$= \left(f * \left(\alpha_t^{f_0}(h)\right)\right)\ \Theta_{(E^{-t})^*}\ \Theta_{E^s}$$
$$= \left(f * \left(\alpha_t^{f_0}(h)\right)\right)\ \Theta_{(E^{-t})^* * E^s}$$
$$= \left(f * \left(\alpha_t^{f_0}(h)\right)\right)\ \Theta_{E_{-t}(f_0^*) * E_s(f_0)}$$
$$= \left(f * \left(\alpha_t^{f_0}(h)\right)\right)\ \Theta_{E_1\left(-tf_0^* + sf_0\right)},$$

for all $f,\ h \in \mathfrak{A}_p$, and $t,\ s \in \mathbb{R}$.

Also, we have that

$$(f\ \Theta_{E^t})^* = \Theta_{E^t}^*\ f^* = \Theta_{E_t}^*\ f^*\ \Theta_{E^0}$$
$$= \Theta_{E^t}^*\ f^*\ (\Theta_{E^t}\Theta_{E^{-t}})$$
$$= (\Theta_{E^t}^* f^* \Theta_{E^t})\ \Theta_{E^{-t}}$$

(7.9)

$$= \left(\left(\alpha_t^{f_0}\right)^* (f^*)\right)\ \Theta_{E^{-t}},$$

for all $f \in \mathfrak{A}_p$, and $t \in \mathbb{R}$.

i.e., the crossed product Banach algebra

$$\mathfrak{X}_{f_0:p} = \mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R}$$

induced by the p-prime Γ_{f_0}-dynamical system $(\mathbb{R},\ \mathfrak{A}_p,\ \alpha^{f_0})$ is the Banach subalgebra of the Banach tensor product algebra $\mathfrak{A}_p \otimes_{\mathbb{C}} \mathbb{A}_p$, satisfying:

(7.8) $$(f\Theta_{E^t})\,(h\Theta_{E^s}) = \left(f * \left(\alpha_t^{f_0}(h)\right)\right)\ \Theta_{(E^{-t})^*}\Theta_{E^s},$$

and

(7.9) $$(f\Theta_{E^t})^* = \left(\left(\alpha_t^{f_0}\right)^* (f^*)\right)\Theta_{E^{-t}},$$

for all $f\ \Theta_{E^t},\ h\ \Theta_{E^s} \in \mathfrak{X}_{f_0:p}$, with $f,\ h \in \mathfrak{A}_p$, and $t,\ s \in \mathbb{R}$.

Definition 7.5. The crossed product Banach algebra $\mathfrak{X}_{f_0:p} = \mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R}$ induced by the p-prime Γ_{f_0}-dynamical system $(\mathbb{R},\ \mathfrak{A}_p,\ \alpha^{f_0})$ is called the p-prime Γ_{f_0}-dynamical Banach (sub)algebra (of $\mathfrak{A}_p \otimes_{\mathbb{C}} \mathbb{A}_p$).

The crossed product Banach algebra $\mathfrak{X}_{f_0:p}$ has its norm $N_{f_0:p}$, defined by

$$N_{f_0:p}\,(f\ \Theta_{E^t}) \overset{def}{=} \left\|f * E^t\right\|_p,$$

where $\|.\|_p$ is in the sense of (3.2.1) and (3.2.1)$'$, for all $f\ \Theta_{E^t} \in \mathfrak{X}_{f_p:p}$, with $f \in \mathfrak{A}_p$, and $t \in \mathbb{R}$. It is a well-defined norm on $\mathfrak{X}_{f_0:p}$.

By construction, $\mathfrak{X}_{f_0:p}$ forms a Banach algebra under $N_{f_0:p}$. Observe that:

$$N_{f_0:p}\,(f\ \Theta_{E^t}) = \left\|f * \Theta_{E^t}\right\|_p$$
$$= \left\|\left(\left(f * E^t\right)(1),\ (f * E^t)(p)\right)\right\|_2$$

where $\|.\|_2$ means the usual norm on $\mathbb{C}^2$

$$= \left\|\left(f(1)E^t(1),\ f(1)E^t(p) + f(p)E^t(1)\right)\right\|_2$$
$$= \left\|\left(e^{tf_0(1)}f(1),\ te^{tf_0(1)}f(1)f_0(p) + e^{tf_0(1)}f(p)\right)\right\|_2,$$

for all $f\ \Theta_{E^t} \in \mathfrak{X}_{f_0:p}$, with $f \in \mathfrak{A}_p$, and $t \in \mathbb{R}$.

Now, let $\mathcal{E}_{f_0}$ be a subset of $\mathbb{A}_p$,

$$\mathcal{E}_{f_0} \overset{def}{=} \{\Theta_{E^t},\, \Theta_{E^t}^* : t \in \mathbb{R}\}.$$

Recall that

$$\begin{aligned}\Theta_{E^t}^* &= \Theta_{E_t(f_0)}^* = \Theta_{E_t(f_0)^*} \\ &= \begin{pmatrix} \overline{(E_t(f_0))(1)} & 0 \\ \overline{(E_t(f_0))(p)} & \overline{(E_t(f_0))(1)} \end{pmatrix} \\ &= e^{t\overline{f_0(1)}} \begin{pmatrix} 1 & 0 \\ t\ \overline{f_0(p)} & 1 \end{pmatrix} = e^{tf_0^*(1)}\Theta_{h_t^*}\end{aligned}$$

for all $t \in \mathbb{R}$.

Construct a Banach subalgebra $\mathbb{E}_{f_0}$ of $\mathbb{A}_p$ generated by $\mathcal{E}_{f_0}$. i.e.,

(7.10)

$$\mathbb{E}_{f_0} \overset{def}{=} \overline{\mathbb{C}\,[\mathcal{E}_{f_0}]},$$

where $\overline{Y}$ mean the norm-completions of subsets Y of $\mathbb{A}_p$. Every element of $\mathbb{E}_{f_0}$ can be understood as a (limit of) linear combination of $\{\Theta_{E^t}\}_{t\in\mathbb{R}}$.

Define now a "conditional" tensor product algebra

(7.11)

$$\mathbb{X}_{f_0:p} \overset{def}{=} \mathfrak{A}_p \otimes_{\alpha^{f_0}} \mathbb{E}_{f_0}$$

by a Banach subalgebra of $\mathfrak{A}_p \otimes_{\mathbb{C}} \mathbb{A}_p$, with the operations satisfying (7.12) and (7.13) under linearity:

(7.12)

$$(f \otimes \Theta_{E^t})\,(h \otimes \Theta_{E^s}) = \Big(f\ \ast \alpha_t^{f_0}(h)\Big) \otimes \left(\Theta_{(E^{-t})^*}\Theta_{E^s}\right),$$

and

(7.13)

$$(f \otimes \Theta_{E^t})^* = \left(\left(\alpha_t^{f_0}\right)^* (f^*)\right) \otimes \Theta_{E^{-t}},$$

for all $f \in \mathfrak{A}_p$, and $t \in \mathbb{R}$.

Theorem 7.6. *Let $\mathfrak{X}_{f_0:p} = \mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R}$ be the p-prime Γ_{f_0}-dynamical Banach algebra, and let $\mathbb{X}_{f_0:p} = \mathfrak{A}_p \otimes_{\alpha^{f_0}} \mathbb{E}_{f_0}$ be the conditional tensor product algebra in the sense of (7.11) satisfying (7.12) and (7.13). Then two Banach algebras $\mathfrak{X}_{f_0:p}$ and $\mathbb{X}_{f_0:p}$ are isomorphic. i.e.,*

(7.14)

$$\mathfrak{X}_{f_p:p} = \mathfrak{A} \times_{\alpha^{f_0}} \mathbb{R} \overset{Banach\text{-}Algebra}{=} \mathfrak{A} \otimes_{\alpha^{f_0}} \mathbb{E}_{f_0} = \mathbb{X}_{f_0:p}.$$

Proof. Define a morphism

$$\Phi : \mathfrak{X}_{f_0:p} \to \mathbb{X}_{f_0:p}$$

by

$$\Phi \overset{def}{=} 1_{\mathfrak{A}_p} \otimes \Theta,$$

i.e., it is a linear transformation satisfying

$$\Phi\left(f\ \Theta_{E^t}\right) = f \otimes \Theta_{E^t}, \text{ for all } f \in \mathfrak{A}_p,\ t \in \mathbb{R}.$$

By the very definition, Φ is a generator-preserving bijective linear morphism. Also, it satisfies that:

$$\begin{aligned}(7.15)\quad & \Phi\left((f\ \Theta_{E^t})(h\ \Theta_{E^s})\right) \\ &= \Phi\left(\left(f * \alpha_t^{f_0}(h)\right)\ \Theta_{(E^{-t})^*}\Theta_{E^s}\right) \\ &= \left(f\ * \alpha_t^{f_0}(h)\right) \otimes \left(\Theta_{(E^{-t})^*}\Theta_{E^s}\right).\end{aligned}$$

Thus, this bijective linear transformation Φ satisfies the multiplicativity (7.15), i.e., the multiplication (7.8) of $\mathfrak{X}_{f_0:p}$ is preserved to the multiplication (7.13) of $\mathbb{X}_{f_0:p}$, by Φ. Therefore, it is an algebra-isomorphism.

The norm $N_{f_0:p}$ on $\mathfrak{X}_{f_0:p}$ and the norm $N^{f_0:p}$ on $\mathbb{X}_{f_0:p}$ are equivalent because they are generated by those of $\mathfrak{A}_p$ and $\mathbb{A}_p$, which are equivalent. Moreover,

$$N^{f_0:p}\left(\Phi(f\ \Theta_{E^t})\right) = N^{f_0:p}\left(f \otimes \Theta_{E^t}\right) = N_{f_0:p}(f\ \Theta_f),$$

for all $f \in \mathfrak{A}_p$, $t \in \mathbb{R}$. Therefore, Φ is an isometric bijective algebra-isomorphism. Equivalently, two Banach algebras $\mathfrak{X}_{f_0:p}$ and $\mathbb{X}_{f_0:p}$ are Banach-algebra-isomorphic. □

The above theorem characterize the p-prime Γ_{f_0}-dynamical Banach algebra, the crossed product Banach algebra, $\mathfrak{X}_{f_0:p} = \mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R}$ induced by the p-prime Γ_{f_0}-dynamical system $(\mathbb{R},\ \mathfrak{A}_p,\ \alpha^{f_0})$, as a conditional tensor product subalgebra $\mathbb{X}_{f_0:p} = \mathfrak{A}_p \otimes_{\alpha^{f_0}} \mathbb{E}_{f_0}$ of the tensor product Banach algebra $\mathfrak{A}_p \otimes_{\mathbb{C}} \mathbb{A}_p$.

8. Freeness on $\mathfrak{X}_{f_0:p}$

In this section, we study the p-prime Γ_{f_0}-dynamical Banach algebra $\mathfrak{X}_{f_0:p}$ more in detail, in particular, we establish free-probabilistic model on $\mathfrak{X}_{f_0:p}$.

In Section 7, we showed that two Banach algebras $\mathfrak{X}_{f_0:p}$ and $\mathbb{X}_{f_0:p}$ are isomorphic from each other, where $\mathbb{X}_{f_0:p}$ is in the sense of (7.11), satisfying (7.12) and (7.13). It means that the flowed dynamical systems acting on the p-Banach algebra $\mathfrak{A}_p$ is analyzed by elements of

$$\mathfrak{X}_{f_0:p} \overset{\text{Banach-Algebra}}{=} \mathbb{X}_{f_0:p},$$

by (7.14). From now on, understand $\mathfrak{X}_{f_0:0}$ and $\mathbb{X}_{f_0:p}$ alternatively.

Define a morphism

$$\Omega_p : \mathbb{X}_{f_0:p} = \mathfrak{X}_{f_0:p} \to \mathfrak{A}_p$$

by a linear transformation satisfying that:

(8.1)

$$\Omega_p\left(f \otimes \Theta_{E^t}\right) = \delta_{t,0}\left(f \otimes 1_{\mathbb{E}_{f_0}}\right) = \delta_{t,0}\ f,$$

where $1_{\mathbb{E}_{f_0}} = 1_{\mathbb{A}_p} = \Theta_{E^0}$, and δ means the Kronecker delta. i.e.,

$$\begin{aligned}\Omega_p\left(\textstyle\sum_{j=1}^N r_j\ \left(f_j \otimes \Theta_{E^{t_j}}\right)\right) &= \textstyle\sum_{j=1}^N r_j \Omega_p\left(f_j \otimes \Theta_{E^{t_j}}\right) \\ &= \textstyle\sum_{j=1}^N r_j\ \delta_{t_j,0} f_j.\end{aligned}$$

Then it is a well-defined conditional expectation from $\mathfrak{X}_{f_0:p}$ onto $\mathfrak{A}_p$. Indeed, for all $f \in \mathfrak{A}_p$, equivalent to

$$f \otimes 1_{\mathbb{E}_{f_0}} \text{ in } \mathfrak{A}_p \otimes_{\mathbb{C}} \{1_{\mathbb{E}_{f_0}}\} \subset \mathfrak{X}_{f_0:p},$$

we have

$$\Omega_p\left(f \otimes 1_{\mathbb{E}_{f_0}}\right) = f, \text{ for all } f \in \mathfrak{A}_p,$$

and

$$\begin{aligned} &\Omega_p\left(\left(f_1 \otimes 1_{\mathbb{E}_{f_0}}\right)(f_2 \otimes \Theta_{E^t})\right) \\ &\quad = \Omega_p\left((f_1 * f_2) \otimes \Theta_{E^t}\right) = \delta_{t,0}\,(f_1 * f_2) \\ &\quad = f_1 * (\delta_{t,0} f_2) = f_1 * \left(\Omega_p\left(f_2 \otimes \Theta_{E^s}\right)\right), \end{aligned}$$

for all $f_1 \in \mathfrak{A}_p$, $f_2 \otimes \Theta_{E^t} \in \mathfrak{X}_{f_0:p}$. Also, by definition, this morphism Ω_p is bounded (or continuous). So, under linearity, Ω_p is a (Banach-algebra) conditional expectation from $\mathfrak{X}_{f_0:p}$ onto $\mathfrak{A}_p$.

Lemma 8.1. *Let $\Omega_p : \mathfrak{X}_{f_0:p} \to \mathfrak{A}_p$ be a morphism in the sense of (8.1). Then it is a well-defined conditional expectation.* □

Define a linear functional

$$\varphi_{f_0:p} : \mathfrak{X}_{f_0:p} \to \mathbb{C},$$

by the linear functional, satisfying that:

(8.2)

$$\varphi_{f_0:p} \overset{def}{=} g_p \circ \Omega_p.$$

Indeed, this function $\varphi_{f_0:p}$ is linear, since

$$\begin{aligned} &\varphi_{f_0:p}\left(t\,(f_1 \otimes \Theta_{E^{t_1}}) + s(f_2 \otimes \Theta_{E^{t_2}})\right) \\ &\quad = g_p\left(\Omega_p\left(t(f_1 \otimes \Theta_{E^{t_1}}) + s(f_2 \otimes \Theta_{E^{t_2}})\right)\right) \\ &\quad = g_p\left(t\delta_{t_1,0}\ f_1 + s\delta_{t_2,0}\ f_2\right) \\ &\quad = t\ g_p\left(\delta_{t_1,0} f_1\right) + s\ g_p\left(\delta_{t_2,0} f_2\right) \\ &\quad = t\ \varphi_{f_0:p}\left(f_1 \otimes \Theta_{E^{t_1}}\right) + s\ \varphi_{f_0:p}\left(f_2 \otimes \Theta_{E^{t_2}}\right). \end{aligned}$$

By the boundedness of Ω_p, it is bounded, too. So, $\varphi_{f_0:p}$ is a continuous linear functional on $\mathfrak{X}_{f_0:p}$.

Definition 8.2. Let $\mathfrak{X}_{f_0:p} = \mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R} = \mathfrak{A}_p \otimes_{\alpha^{f_0}} \mathbb{E}_{f_0} = \mathbb{X}_{f_0:p}$ be the p-prime Γ_{f_0}-Banach algebra, and let $\varphi_{f_0:p} = g_p \circ \Omega_p$ be the linear functional (8.2) on $\mathfrak{X}_{f_0:p}$, where Ω_p is the conditional expectation (8.1). The corresponding Banach probability space $(\mathfrak{X}_{f_0:p},\ \varphi_{f_0:p})$ is called the p-prime Γ_{f_0}-dynamical probability space.

Let $(\mathfrak{X}_{f_0:p},\ \varphi_{f_0:p})$ be the p-prime Γ_{f_0}-dynamical probability space, consisting of the p-prime Γ_{f_0}-Banach algebra $\mathfrak{X}_{f_0:p}$ and the linear functional $\varphi_{f_0:p}$ of (8.2). Now, we compute free moments of free random variables of $(\mathfrak{X}_{f_0:p},\ \varphi_{f_0:p})$.

Recall that:

(8.3)

$$\begin{aligned} (f_1\ \Theta_{E^{t_1}})(f_2\ \Theta_{E^{t_2}}) &= \left(f_1 * \alpha_{t_1}^{f_0}(f_2)\right)\Theta_{(E^{-t_1})^*}\Theta_{E^{t_2}} \\ &= \left(f_1 * \alpha_{t_1}^{f_0}(f_2)\right)\Theta_{E_1\left(-t_1 f_0^* + t_2 f_0\right)}, \end{aligned}$$

for $f_j\ \Theta_{E^{t_j}} \in \mathfrak{X}_{f_0:p}$, for $j = 1,\ 2$.

Notation For convenience, we write $\alpha_t^{f_0}(h)$ simply by $h_{(t)}$, for all $h \in \mathfrak{A}_p$ and $t \in \mathbb{R}$. i.e.,

(8.4)

$$h_{(t)} = \alpha_t^{f_0}(h) = E_t(f_0) * h * E_t(f_0)^* \text{ in } \mathfrak{A}_p,$$

realized by (7.3) in $\mathbb{A}_p$. One can understand $f\ \Theta_{E^t} \in \mathfrak{X}_{f_0:p}$ and $f \otimes \Theta_{E^t} \in \mathbb{X}_{f_0:p}$ as same (or equivalent) elements below. □

Observe that:

$$(f_1\ \Theta_{E^{t_1}})\,(f_2\ \Theta_{E^{t_2}})\,(f_3\ \Theta_{E^{t_3}})$$
$$= \left(f_1 * f_{2(t_1)}\ \Theta_{E_1(-t_1 f_0^* + t_2 f_0)}\right)(f_3\ \Theta_{E^{t_3}})$$
$$= \left(f_1 * f_{2(t_1)} * f_{3(t_1+t_2)}\right)\ \Theta_{E_1\left(-(-t_1 f_0^* + t_2 f_0)^* + t_3 f_0\right)}$$

(8.5)

$$= \left(f_1 * f_{2(t_1)} * f_{3(t_1+t_2)}\right)\ \Theta_{E_1\left(t_1 f_0 - t_2 f_0^* + t_3 f_0\right)},$$

for $f_j\ \Theta_{E^{t_j}} \in \mathfrak{X}_{f_0:p}$, for $j = 1, 2, 3$.

Inductively, one can get that:

Lemma 8.3. *Let $f_j\ \Theta_{E^{t_j}} \in \mathfrak{X}_{f_0:p}$, for $j = 1, \cdots, n$, for $n \in \mathbb{N}$. Then one can get that:*

(8.6)

$$\prod_{k=1}^{n} (f_k\ \Theta_{E^{t_k}}) = \left(\mathop{*}_{k=1}^{n} f_{k\left(\sum_{i=0}^{k-1} t_j\right)}\right) \Theta_{E_1\left(\sum_{k=1}^{n} (-1)^{n-k} t_k f_0^{[k]}\right)},$$

where $f_{k(s)} = (f_k)_{(s)}$ in the sense of (8.2), for $j = 1, \cdots, n$, and $s \in \mathbb{R}$, and

(8.6)′

$$([k])_{j=1}^{n} = ([1],\ [2], \cdots, [n])$$

$$= \begin{cases} (*,\ 1,\ *,\ 1, \cdots, *,\ 1) & \text{if } n \text{ is even} \\ (1,\ *,\ 1,\ *, \cdots, *,\ 1) & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$.

Proof. The proof is by (8.5) and by induction. □

Now, recall that:

(8.7)

$$E^0 = E_0(f_0) = E_1\,(0 f_0) = E_1(0_{\mathfrak{A}_p}) = 1_{\mathfrak{A}_p}.$$

Observe now that:

$$\Omega_p\left(\prod_{k=1}^{n} (f_k\ \Theta_{E^{t_k}})\right) = \Omega_p\left(\left(\mathop{*}_{k=1}^{n} f_{k\left(\sum_{i=1}^{j-1} t_i\right)}\right) \Theta_{E_1\left(\sum_{k=1}^{n} (-1)^{n-k} f_0^{[k]}\right)}\right)$$

by (8.6)

$$= \begin{cases} \left(\mathop{*}_{k=1}^{n} f_{k\left(\sum_{i=1}^{j-1} t_i\right)}\right) & \text{if } \sum_{k=1}^{n} (-1)^{n-k} f_0^{[k]} = 0_{\mathfrak{A}_p} \\ \\ 0_{\mathfrak{A}_p} & \text{otherwise,} \end{cases}$$

in $\mathfrak{A}_p$, by (8.7). So, one has the following lemma.

Lemma 8.4. *Let $f_k\ \Theta_{E^{t_k}} \in (\mathfrak{X}_{f_0:p}, \varphi_{f_0:p})$, for $k = 1, \cdots, n$, for $n \in \mathbb{N}$. Then*

(8.8)

$$\Omega_p\left(\prod_{k=1}^{n} f_k\ \Theta_{E^{t_k}}\right) = \begin{cases} \left(\underset{k=1}{\overset{n}{*}} f_{k\left(\sum_{i=1}^{j-1} t_i\right)}\right) & \text{if } \sum_{k=1}^{n}(-1)^{n-k} f_0^{[k]} = 0_{\mathfrak{A}_p} \\ 0_{\mathfrak{A}_p} & \text{otherwise,} \end{cases}$$

in $\mathfrak{A}_p$. □

By (8.6), (8.7) and (8.8), we obtain the following free moment computations on the p-prime Γ_{f_0}-dynamical probability space $(\mathfrak{X}_{f_0:p}, \varphi_{f_0:p})$.

Theorem 8.5. *Let $f_k\ \Theta_{E^k}$ be free random variables in the p-prime Γ_{f_0}-dynamical probability space $(\mathfrak{X}_{f_0:p}, \varphi_{f_0:p})$, for $k = 1, \cdots, n$, for $n \in \mathbb{N}$. Then*

(8.9)

$$\varphi_{f_0:p}\left(\prod_{k=1}^{n} f_k\ \Theta_{E^{t_k}}\right) = \begin{cases} \sum_{k=1}^{n} v_{k:p}\left(\prod_{l\in\{1,\cdots,n\},\, l\neq k} v_{k:1}\right) & \text{if } \sum_{k=1}^{n}(-1)^{n-k} f_0^{[k]} = 0_{\mathfrak{A}_p} \\ 0, & \text{otherwise,} \end{cases}$$

where

$$v_{k:p} = e^{u_1\left(\sum_{i=1}^{k} t_i\right)}\left(\left(\sum_{i=1}^{k} t_i\right) u_p f_k(1) + f_k(p)\right),$$

and

$$v_{k:1} = e^{u_1\left(\sum_{i=1}^{k} t_i\right)} f_k(1), \text{ in } \mathbb{C},$$

for all $k = 1, \cdots, n$.

Proof. By (8.6), (8.7) and (8.8), we have

(8.10)

$$\varphi_{f_0:p}\left(\prod_{k=1}^{n} f_k\ \Theta_{E^{t_k}}\right) = \begin{cases} g_p\left(\underset{k=1}{\overset{n}{*}} f_{k\left(\sum_{i=1}^{k-1} t_i\right)}\right) & \text{if } \sum_{k=1}^{n}(-1)^{n-k} f_0^{[k]} = 0_{\mathfrak{A}_p} \\ g_p\left(0_{\mathfrak{A}_p}\right) = 0 & \text{otherwise,} \end{cases}$$

in $\mathbb{C}$.

By (7.3), we have

$$f_{(t)}(1) = e^{tu_1} f(1),$$

and

$$f_{(t)}(p) = e^{tu_1}\left(tu_p f(1) + f(p)\right),$$

for all $t \in \mathbb{R}$, where

$$u_1 = \text{Re}(w_1) = \text{Re}(f_0(1)),$$

and

$$u_p = \mathrm{Re}(w_p) = \mathrm{Re}(f_0(p)).$$

Therefore,

$$f_{k\left(\sum_{i=1}^{k-1} t_i\right)}(1) = e^{u_1\left(\sum_{i=1}^{k} t_i\right)} f_k(1) \overset{denote}{=} v_{k:1},$$

and

$$f_{k\left(\sum_{i=1}^{k-1} t_i\right)}(p) = e^{u_1\left(\sum_{i=1}^{k} t_i\right)} \left(\left(\sum_{i=1}^{k} t_i\right) u_p f_k(1) + f_k(p)\right) \overset{denote}{=} v_{k:p},$$

in $\mathbb{C}$, for all $k = 1, \cdots, n$.

So, by (8.10) and (3.1.11), if nonzero, then one can get that:

$$\varphi_{f_0:p}\left(\prod_{k=1}^{n} f_k \ \Theta_{E^{t_k}}\right) = g_p\left(\underset{k=1}{\overset{n}{*}} f_{k\left(\sum_{i=1}^{k-1} t_i\right)}\right)$$

$$= \sum_{k=1}^{n} v_{k:p}\left(\prod_{l \in \{1,\cdots,n\},\, l \neq k} v_{k:1}\right),$$

where $v_{k:1}$ and $v_{k:p}$ are given as above, for all $k = 1, \cdots, n$. $\square$

Consider now the case where the above computation (8.9) is non-zero. By condition, one should have

$$\sum_{k=1}^{n} (-1)^{n-k} f_0^{[k]} = 0_{\mathfrak{A}_p},$$

to make (8.9) be non-zero.

Suppose first that f_0 is self-adjoint in $\mathfrak{A}_p$, in the sense that: $f_0^* = f_0$, equivalently, f_0 is $\mathbb{R}$-valued,

$$\overline{f_0(n)} = f_0(n) \text{ in } \mathbb{C}, \text{ for all } n \in \mathbb{N}.$$

Then one can conclude that the formula (8.9) goes to;

$$\varphi_{f_0:p}\left(\prod_{k=1}^{n} f_k \ \Theta_{E^{t_k}}\right)$$

$$= \begin{cases} \sum_{k=1}^{n} v_{k:p}\left(\prod_{l \in \{1,\cdots,n\},\, l \neq k} v_{k:1}\right) & \text{if } \left(\sum_{k=1}^{n} (-1)^{n-k}\right) f_0 = 0_{\mathfrak{A}_p} \\ 0, & \text{otherwise,} \end{cases}$$

by the assumption that: $f_0^* = f_0$ in $\mathfrak{A}_p$

$$= \begin{cases} \sum_{k=1}^{n} v_{k:p}\left(\prod_{l \in \{1,\cdots,n\},\, l \neq k} v_{k:1}\right) & \text{if } \left(\sum_{k=1}^{n} (-1)^{n-k}\right) = 0 \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \sum_{k=1}^{n} v_{k:p} \left(\underset{l \in \{1, \cdots, n\}, \, l \neq k}{\Pi} v_{k:1} \right) & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$. More precisely, we obtain the following corollary.

Corollary 8.6. *Under the same hypothesis with the above theorem, if $f_0(1)$, $f_0(p)$ $\in \mathbb{R}$, then*

(8.11)

$$\varphi_{f_0:p} \left(\prod_{k=1}^{n} f_k \ \Theta_{E^{t_k}} \right) = \begin{cases} \sum_{k=1}^{n} v_{k:p} \left(\underset{l \in \{1, \cdots, n\}, \, l \neq k}{\Pi} v_{k:1} \right) & \textit{if } n \textit{ is even} \\ 0, & \textit{if } n \textit{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$, where $v_{k:p}$ and $v_{k:1}$ are given as in the above theorem, for all k $= 1, \cdots, n$.

Proof. In [15, 16], we showed that Θ_{f_0} is self-adjoint in the sense that $\Theta_{f_0}^{*} = \Theta_{f_0}$, if and only if $f_0(1)$ and $f_0(p)$ are contained in $\mathbb{R}$. i.e., in $\mathbb{E}_{f_0}$, it is self-adjoint. By [11], one can understand f_0 as a self-adjoint element in $\mathfrak{A}_p$. Thus, by the discussion of the above paragraph, one can get (8.11). □

Also, by (8.9), we obtain that:

Corollary 8.7. *Let $f \ \Theta_{E^t} \in (\mathfrak{X}_{f_0:p}, \varphi_{f_0:p})$, with $f \in \mathfrak{A}_p$, $t \in \mathbb{R}$. Then*

(8.12)

$$\varphi_{f_0:p}\left((f \ \Theta_{E^t})^n\right) = \begin{cases} \sum_{k=1}^{n} v_{k:p} \left(\underset{l \in \{1, \cdots, n\}, \, l \neq k}{\Pi} v_{k:1} \right) & \textit{if } \sum_{k=1}^{n} (-1)^{n-k} f_0^{[k]} = 0_{\mathfrak{A}_p} \\ 0, & \textit{otherwise,} \end{cases}$$

where

$$v_{k:p} = e^{u_1 \left(\sum_{i=1}^{k} t_i \right)} \left(\left(\sum_{i=1}^{k} t_i \right) u_p f(1) + f(p) \right),$$

and

$$v_{k:1} = e^{u_1 \left(\sum_{i=1}^{k} t_i \right)} f(1), \ \textit{in } \mathbb{C},$$

for all $k = 1, \cdots, n$, for all $n \in \mathbb{N}$. □

Suppose $f \in \mathfrak{A}_p$, and $E^t \in \Gamma_{f_0}$, for $t \in \mathbb{R}$. By Section 3.1,

$$g_p\left(E^t\right) = E^t(p) = \left(E_t(f_0)\right)(p) = t w_p \ e^{t w_1},$$

and

$$E^t(1) = \left(E_t(f_0)\right)(1) = e^{t w_1},$$

in $\mathbb{C}$, where $w_1 = f_0(1)$, and $w_p = f_0(p)$.

It shows that $E^t(1) \neq 0$ in $\mathfrak{A}_p$, and $g_p(E^t) \neq 0$, whenever $t \neq 0$. (Notice that $g_p(E^t) = 0$, only when $t = 0$.)

By (3.1.13) and (3.1.14), one can verify the following freeness characterization on the p-prime Banach probability space $(\mathfrak{A}_p, \, g_p)$.

Proposition 8.8. *Two "nonzero" free random variables* f_1 *and* f_2 *are free in* $(\mathfrak{A}_p,\, g_p)$*, if and only if either*

(8.13) $f_1(p) = 0 = f_2(p)$*, with* $f_1(1) \neq 0$ *and* $f_2(1) \neq 0$*, or*

(8.14) $f_i(1) = 0 = f_j(p)$*, for* $i \neq j \in \{1,\, 2\}$*, with* $f_i(p) \neq 0$*, and* $f_j(1) \neq 0$*.*

Proof. The proof of the theorem is by the little modification of that of (3.1.13) and (3.1.14) in [11, 12]. By the very definition-and-construction of the Banach space $\mathfrak{A}_p$ under the equivalence relation $\mathcal{N}_p$ on $\mathcal{A}$ (See Sections 5 and 6), if $f \in \mathfrak{A}_p$ is nonzero, then either $f(1) \neq 0$, or $f(p) \neq 0$. So, f_1 and f_2 are free in $(\mathfrak{A}_p,\, g_p)$, if and only if either (8.13) or (8.14) holds. □

The above proposition implies that:

Theorem 8.9. *Let* $f \in (\mathfrak{A}_p,\, g_p)$ *be nonzero, and* $E^t \in \Gamma_{f_0}$*, for* $t \in \mathbb{R}$*. Then* f *and* E^t *are free in* $(\mathfrak{A}_p,\, g_p)$*, if and only if either*

(8.15) $t = 0$ *and* $f(p) = 0$*, if* $f(1) \neq 0$*, or*

(8.16) $t = 0$ *and* $f(1) = 0$*, if* $f(p) \neq 0$*.*

Proof. Suppose $t \neq 0$. Then both

$$E^t(1) = e^{tu_1} \neq 0, \text{ and } E^t(p) = tu_p e^{tu_1} \neq 0.$$

So, by (8.13) and (8.14), if $f \neq 0_{\mathfrak{A}_p}$, then f and E^t are not free in $(\mathfrak{A}_p,\, g_p)$.

Assume now that $t = 0$. Then $E^0 = 1_{\mathfrak{A}_p}$, the identity element of $\mathfrak{A}_p$.

$$E^0(1) = 1_{\mathfrak{A}_p}(1) = 1 \neq 0, \text{ and } E^0(p) = 1_{\mathfrak{A}_p}(p) = 0.$$

So, f and E^0 are free in $(\mathfrak{A}_p,\, g_p)$, if and only if $f(p) = 0$ (with $f(1) \neq 0$), to satisfy (8.13). Similarly, f and E^0 are free in $(\mathfrak{A}_p,\, g_p)$, if and only if $f(1) = 0$ (with $f(p) \neq 0$), to satisfy (8.14). □

The above theorem shows that, in general, if $f \neq 0_{\mathfrak{A}_p}$, then f and E^t are not free in $(\mathfrak{A}_p,\, g_p)$, whenever $t \neq 0$.

The following corollary is the direct consequence of the above theorem.

Corollary 8.10. *Let* $f \in (\mathfrak{A}_p,\, g_p)$*, and* $f_{(t)} = \alpha_t^{f_0}(f) \in (\mathfrak{A}_p,\, g_p)$*, for* $t \in \mathbb{R}$*.*

(8.17) f *and* $f_{(t)}$ *are not free in* $(\mathfrak{A}_p,\, g_p)$*,*

whenever $f \neq 0_{\mathfrak{A}_p}$ *in* $\mathfrak{A}_p$*.*

Proof. Assume first that $t = 0$ in $\mathbb{R}$. Then $f_{(0)} = f$ in $\mathfrak{A}_p$. Therefore, f and $f_{(0)}$ are not free in $(\mathfrak{A}_p,\, g_p)$. Suppose now that $t \neq 0$ in $\mathbb{R}$. Then, by (8.15) and (8.16), f and E^t are not free in $(\mathfrak{A}_p,\, g_p)$. Therefore, mixed free cumulants of

$$f \text{ and } f_{(t)} = E_t(f_0) * f * E_t(f_0^*)$$

do not vanish in general, because mixed free cumulants of f and $f_{(t)}$ can be understood as certain mixed free cumulants of f and E^t. So, f and $f_{(t)}$ are not free in $(\mathfrak{A}_p,\, g_p)$.

Indeed, one can get that:

$$\begin{aligned} k_2\left(f,\ f_{(t)}\right) &= k_2\left(f,\ E_t(f_0) * f * E_t(f_0^*)\right) \\ &= g_p\left(f * E_t(f_0) * f * E_t(f_0^*)\right) \\ &\qquad - \left(g_p(f)\right)\left(g_p\left(E_t(f_0) * f * E_t(f_0^*)\right)\right) \end{aligned}$$

by the Möbius inversion of Section 2

$$= g_p\left(f^{(2)} * E_t(f_0) * E_t(f_0^*)\right) - f(p)\left(g_p\left(f * E_t(f_0) * E_t(f_0^*)\right)\right)$$

by the commutativity of $(*)$ on $\mathfrak{A}_p$

$$= g_p\left(f^{(2)} * E_t(f_0 + f_0^*)\right) - f(p)\left(g_p\left(f * E_t(f_0 + f_0^*)\right)\right)$$

by (5.7)

$$= g_p\left(f^{(2)} * E_t(\mathrm{Re}f_0)\right) - f(p)\left(g_p\left(f * E_t(\mathrm{Re}f_0)\right)\right)$$

since $f_0 + f_0^* = \mathrm{Re}f_0$, with $\mathrm{Re}f_0(1) = u_1$, and $\mathrm{Re}f_0(p) = u_p$

$$= \left((f(1))^2\, tu_p e^{tu_1} + 2e^{tu_1} f(1)f(p)\right) - f(p)\left(f(1)tu_p e^{tu_1} + e^{tu_1} f(p)\right)$$

$$= (f(1))^2\, tu_p e^{tu_1} + 2e^{tu_1} f(1)f(p) - f(1)f(p)tu_p e^{tu_1} - e^{tu_1} f(p)^2$$

$$= e^{tu_1}\left((f(1))^2\, tu_p + 2f(1)f(p) - f(1)f(p)tu_p - f(p)^2\right).$$

It shows that

$$k_2(f,\, f_{(t)}) = 0, \text{ if and only if } f(1) = 0 = f(p),$$

equivalently, $f = 0_{\mathfrak{A}_p}$ in $\mathfrak{A}_p$, for all $t \in \mathbb{R}$.

Therefore, if $f \neq 0_{\mathfrak{A}_p}$, then f and $f_{(t)}$ are not free in $(\mathfrak{A}_p,\, g_p)$, for all $t \in \mathbb{R}$. $\square$

The above corollary shows that the family $\{f_{(t)}\}_{t\in\mathbb{R}}$ in $\mathfrak{A}_p$ forms a non-free family in $(\mathfrak{A}_p,\, g_p)$. We obtain the following generalization of the above corollary

Proposition 8.11. *Let $f_1,\, f_2 \in (\mathfrak{A}_p,\, g_p)$ be nonzero. Then f_1 and $f_{2(t)}$ are not free in $(\mathfrak{A}_p,\, g_p)$, for all $t \in \mathbb{R}$.*

Proof. If $f_1 = f_2$ in $\mathfrak{A}_p$, then it holds, by (8.17). Suppose that $f_1 \neq f_2$. Assume further that f_1 and f_2 are not free in $(\mathfrak{A}_p,\, g_p)$. Similar to the proof of (8.17), observe that:

$$k_2\left(f_1,\ f_{2(t)}\right) = k_2\left(f_1,\ E_t(f_0) * f_2 * E_t(f_0^*)\right)$$

$$= g_p\left(f_1 * E_t(f_0) * f_2 * E_t(f_0^*)\right) - \left(g_p(f_1)\right)\left(g_p\left(E_t(f_0 * f_2 * E_t(f_0^*)\right)\right)\right)$$

$$= g_p\left(f_1 * f_2 * E_t(\mathrm{Re}f_0)\right) - f_1(p)\left(g_p(f_2 * E_t(\mathrm{Re}f_0))\right)$$

$$= \left(f_1 * f_2(1)\right)\left(E_t(\mathrm{Re}f_0)\right)(p) + \left(f_1 * f_2\right)(p)\left(E_t(\mathrm{Re}f_0)(1)\right) - f_1(p)\left(f_2(p)E_t(\mathrm{Re}f_0)(1) + f_2(1)E_t(\mathrm{Re}f_0)(p)\right)$$

$$= (f_1(1))\,(f_2(1))\,(tu_p e^{tu_1}) + f_1(1)f_2(p)e^{tu_1} + f_1(p)f_2(1)e^{tu_1} - (f_1(p))\,(f_2(p))\,e^{tu_1} + f_2(1)\,(tu_p e^{tu_1})$$

(8.18)

$$= e^{tu_1}\left(tu_p f_1(1)f_2(1) + f_1(1)f_2(p) + f_1(p)f_2(1) - f_1(p)f_2(p) + tu_p f_2(1)\right).$$

By (8.13) and (8.14), the above second mixed free cumulant of f_1 and $f_{2(t)} = \alpha_t^{f_0}(f_2)$ vanishes only if either $f_1 = 0_{\mathfrak{A}_p}$ or $f_2 = 0_{\mathfrak{A}_p}$. So, f_1 and $f_{2(t)}$ are not free whenever f_1 and f_2 are not free in $(\mathfrak{A}_p,\, g_p)$.

Suppose now that f_1 and f_2 are free in $(\mathfrak{A}_p,\, g_p)$. Then, by (8.13) and (8.14), either (i) $f_1(p) = 0 = f_2(p)$ with $f_1(1) \neq 0$, and $f_2(1) \neq 0$, or (ii) say $f_1(1) = 0 = f_2(p)$, with $f_1(p) \neq 0$, and $f_2(1) \neq 0$.

Assume first that the condition (i) holds, for the freeness of f_1 and f_2. Then the mixed second free cumulant (8.18) of f_1 and $f_{2(t)}$ becomes that

(8.18)′

$$tu_p e^{tu_1} f_2(1) \left(f_1(1) + 1\right).$$

So, in general, the formula (8.18)′ does not vanish. It vanishes only when $t = 0$ in $\mathbb{R}$. In fact, it guarantees the third mixed free cumulant

$$k_3(f_1,\, f_{2(0)},\, f_1) \neq 0.$$

Now, assume that the condition (ii) holds. Then the above formula (8.18) becomes

(8.18)″

$$e^{tu_1} \left(f_1(p) f_2(1) + tu_p f_2(1)\right).$$

So, the formula (8.18)″ does not vanish.

The formulae (8.18)′ and (8.18)″ show that even though f_1 and f_2 are free in $(\mathfrak{A}_p,\, g_p)$, the elements f_1 and $f_{2(t)}$ are not free in $(\mathfrak{A}_p,\, g_p)$. □

By the above consideration, we obtain the following theorem characterizing the freeness on $\mathfrak{X}_{f_0:p}$.

Theorem 8.12. *Let $T_j = f_j \Theta_{E^{t_j}}$ be nonzero free random variables in the p-prime Γ_{f_0}-dynamical probability space $(\mathfrak{X}_{f_0:p},\, \varphi_{f_0:p})$, for $j = 1, 2$. They are free in $(\mathfrak{X}_{f_0:p},\, \varphi_{f_0:p})$, if and only if both (8.19) and (8.20) hold, where*

(8.19) $t_1 = 0 = t_2$,

(8.20) f_1 *and* f_2 *are free in* $(\mathfrak{A}_p,\, g_p)$.

Proof. (⇐) Assume the conditions (8.19) and (8.20) holds. By (8.19), it is not difficult to check that

$$k_n^{f_0:p}\left(T_{i_1}, \cdots, T_{i_n}\right) = k_n\left(f_{i_1}, \cdots, f_{i_n}\right),$$

for all mixed n-tuples $(i_1, \cdots, i_n) \in \{1, 2\}^n$, for all $n \in \mathbb{N} \setminus \{1\}$, where $k_n^{f_0:p}(\cdots)$ means the free cumulants on $\mathfrak{X}_{f_0:p}$, with respect to the linear functional $\varphi_{f_0:p}$. By (8.20), all mixed free cumulants of f_1 and f_2 vanish, and hence,

$$k_n^{f_0:p}\left(T_{i_1}, \cdots, T_{i_n}\right) = 0,$$

for all mixed n-tuples $(i_1, \cdots, i_n)$, for all $n \in \mathbb{N} \setminus \{1\}$. So, T_1 and T_2 are free in $(\mathfrak{X}_{f_0:p},\, \varphi_{f_0:p})$.

(⇒) Suppose T_1 and T_2 are free in $(\mathfrak{X}_{f_0:p},\, \varphi_{f_0:p})$. Assume that either t_1 or t_2 are nonzero in $\mathbb{R}$. Say $t_1 \neq 0$. i.e., we assume the condition (8.19) does not hold. Consider the mixed second cumulant of T_1 and T_2;

$$\begin{aligned} & k_2^{f_0:p}\left(T_1,\, T_2\right) = k_2^{f_0:p}\left(f_1\Theta_{E^{t_1}},\, f_2\Theta_{E^{t_2}}\right) \\ & = \varphi_{f_0:p}\left(f_1\Theta_{E^{t_1}} f_2\Theta_{E^{t_2}}\right) - \varphi_{f_0:p}\left(f_1\Theta_{E^{t_1}}\right)\varphi_{f_0:p}\left(f_2\Theta_{E^{t_2}}\right) \\ & = \varphi_{f_0:p}\left(\left(f_1 * f_{2(t_1)}\right)\Theta_{E^{t_1+t_2}}\right) - 0 \end{aligned}$$

since $t_1 \neq 0$, by (8.9)

$$= \begin{cases} g_p\left(f_1 * f_{2(t_1)}\right) & \text{if } t_1 f_0 - t_2 f_0^* = 0_{\mathfrak{A}_p} \\ 0 & \text{otherwise,} \end{cases}$$

by (8.10)

$$= \begin{cases} f_1(1)f_2(1)te^{t_1 u_1}u_p + (f_1(1)f_2(p) + f_1(p)f_2(1))e^{t_1 u_1}, & \text{or} \\ 0 \end{cases}$$

It shows that, in general, if $t_1 \neq 0$, then $k_2^{f_0:p}(T_1,\, T_2) \neq 0$. For instance, if $f_0^* = f_0$, and $t_1 = t_2$ in $\mathbb{R} \setminus \{0\}$, then the second mixed free cumulant of T_1 and T_2 does not vanish. It contradicts our assumption that T_1 and T_2 are free in $(\mathfrak{X}_{f_0:p},\, \varphi_{f_0:p})$.

Assume now that f_1 and f_2 are not free in $(\mathfrak{A}_p,\, g_p)$. i.e., suppose the condition (8.20) does not hold. It suffices to consider the case where the condition (8.19) holds. It shows again that

$$k_n^{f_0:p}\left(T_{i_1}, \cdots, T_{i_n}\right) = k_n\left(f_{i_1}, \cdots, f_{i_n}\right),$$

for mixed n-tuples $(i_1, \cdots, i_n)$. It shows that there exists $n_0 \in \mathbb{N}$ and mixed n_0-tuple $(i_1, \cdots, i_{n_0})$, such that

$$k_{n_0}\left(f_{i_1}, \cdots, f_{i_{n_0}}\right) = k_{n_0}^{f_0:p}\left(T_{i_1}, \cdots, T_{i_{n_0}}\right) \neq 0.$$

This contradicts our assumption that T_1 and T_2 are free.

Therefore, if T_1 and T_2 are free in $(\mathfrak{X}_{f_0:p},\, \varphi_{f_0:p})$, then both (8.19) and (8.20) hold. □

The above theorem completely characterize the inner freeness of the p-prime Γ_{f_0}-dynamical Banach algebra $\mathfrak{X}_{f_0:p}$, in terms of a fixed prime p and the flow determined by a fixed element $f_0 \in \mathfrak{A}_p$. Under the linear functional $\varphi_{f_0:p}$, the freeness on $\mathfrak{X}_{f_0:p}$ is affected by that on $\mathfrak{A}_p$.

9. Equivalent Dynamical Systems with $\left(\Gamma_{f_0},\ \mathfrak{A}_p,\ \alpha^{f_0}\right)$

In Sections 7 and 8, we established a certain flowed dynamical system induced by the p-prime Banach algebra $\mathfrak{A}_p$ and the flow $\mathbb{R}$, via a group action α^{f_0} for a fixed "nonzero" element $f_0 \in \mathfrak{A}_p$, having $\Gamma_{f_0} = \mathbb{R}$, and studied the corresponding crossed product Banach algebra $\mathfrak{X}_{f_0:p}$ to investigate how this dynamical system works on arithmetic functions. In this section, we study systems of such dynamical systems.

9.1. **Group Dynamical Systems on $\mathfrak{A}_p$ Induced by $\Gamma_{f_1+f_2+\cdots+f_k}$.** Let p be a fixed prime, and let $(\mathfrak{A}_p,\, g_p)$ be the p-prime Banach probability space induced by the arithmetic p-prime probability space $(\mathcal{A},\, g_p)$ (under quotient and topology), and let $\mathfrak{X}_{f_0:p}$ be the crossed product Banach algebra $\mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R}$ induced by the p-prime Γ_{f_0}-dynamical system $(\mathbb{R},\, \mathfrak{A}_p,\, \alpha^{f_0})$. Then we obtain the p-prime Γ_{f_0}-dynamical probability space $(\mathfrak{X}_{f_0:p},\, \varphi_{f_0:p})$.

Now, let f_0 be fixed in $\mathfrak{A}_p$ as above, and assume $f_1, \cdots, f_k \in \mathfrak{A}_p$, satisfying that:

$$f_0 = \sum_{j=1}^{k} f_j \text{ in } \mathfrak{A}_p.$$

Then we have

$$E^t = E_t(f_0) = E_t\left(\sum_{j=1}^{k} f_j\right) = \underset{j=1}{\overset{k}{*}} E_t(f_j)$$

in $\mathfrak{A}_p$, by Section 5, for all $t \in \mathbb{R}$.

Thus, one can get that

$$\Theta_{E^t} = \Theta_{\underset{j=1}{\overset{k}{*}} E_t(f_j)} = \prod_{j=1}^{k} \Theta_{E_t(f_j)} \text{ in } B\left(\mathbb{C}^2_{A_o}\right),$$

for all $t \in \mathbb{R}$, by Section 6.

From now on, we restrict our interests to the case where $k = 2$. i.e.,

$$f_0 = f_1 + f_2 \text{ in } \mathfrak{A}_p,$$

so

$$E^t = E_t(f_1 + f_2) = E_t(f_1) * E_t(f_2),$$

and hence,

$$\Theta_{E^t} = \Theta_{E_t(f_1)}\Theta_{E_t(f_2)} \text{ on } \mathbb{C}^2_{A_o}.$$

Under above conditions, one can have that:

$$\begin{aligned} \alpha_t^{f_0}(h) &= E_t(f_0) * h * E_t(f_0)^* \\ &= E_t(f_0) * h * E_t(f_0^*) \\ &= E_t(f_1 + f_2) * h * E_t(f_1^* + f_2^*) \\ &= E_t(f_1) * E_t(f_2) * h * E_t(f_1^*) * E_t(f_2^*) \\ &= E_t(f_2) * (E_t(f_1) * h * E_t(f_1^*)) * E_t(f_2^*) \end{aligned}$$

since $(*)$ is commutative on $\mathfrak{A}_p$

$$= \alpha_t^{f_2}\left(E_t(f_1) * h * E_t(f_1^*)\right)$$

where $\alpha_t^{f_2}$ is in the sense of (7.6) (and (7.6)$'$) for f_2

$$= \alpha_t^{f_2}\left(\alpha_t^{f_1}(h)\right) = \left(\alpha_t^{f_2} \circ \alpha_t^{f_1}\right)(h)$$

for all $h \in \mathfrak{A}_p$, for all $t \in \mathbb{R}$. i.e.,

(9.1.1)

$$\alpha_t^{f_0} = \alpha_t^{f_1+f_2} = \alpha_t^{f_1} \circ \alpha_t^{f_2}, \text{ for all } t \in \mathbb{R}.$$

Inductively, we obtain that:

Lemma 9.1. *Let α^{f_0} be the group action of the flow $\mathbb{R} = \Gamma_{f_0}$ acting on $\mathfrak{A}_p$ in the sense of (7.6)$'$. If $f_0 = \sum_{j=1}^{k} f_j$ in $\mathfrak{A}_p$, for some $k \in \mathbb{N}$, then*

(9.1.2)

$$\alpha_t^{f_0} = \underset{j=1}{\overset{k}{\circ}} \alpha_t^{f_j}, \textit{ for all } t \in \mathbb{R},$$

where $(\circ)$ means the usual functional composition.

Proof. By (9.1.1), we have

$$\alpha_t^{f_1+f_2} = \alpha_t^{f_1} \circ \alpha_t^{f_2}, \text{ for all } t \in \mathbb{R},$$

and hence, inductively, we obtain

$$\begin{aligned}\alpha_t^{\Sigma_{j=1}^k f_j} &= \alpha_t^{f_1+\Sigma_{j=2}^k f_j} = \alpha_t^{f_1} \circ \alpha_t^{\Sigma_{j=2}^k f_j} \\ &= \alpha_t^{f_1} \circ \alpha_t^{f_2} \circ \alpha_t^{\Sigma_{j=3}^k f_j} \\ &= \cdots = \underset{j=1}{\overset{k}{\circ}} \alpha_t^{f_j}\end{aligned}$$

for all $t \in \mathbb{R}$. □

The above general formula (9.1.2) says that, if a fixed nonzero element $f_0 \in \mathfrak{A}_p$ is formed by a sum $\sum_{j=1}^k f_j$ of other elements $f_1, \cdots, f_k$ of $\mathfrak{A}_p$, for some $k \in \mathbb{N}$, then the group action α^{f_0} of the flow $\mathbb{R}$ is understood as a certain product of group actions $\alpha^{f_1}, \cdots, \alpha^{f_k}$ of the flow $\mathbb{R}$.

Define now a product group

$$\mathbb{R}^k = \underbrace{\mathbb{R} \ \times \ \cdots \ \times \ \mathbb{R}}_{k\text{-times}},$$

equipped with an operation $(+_k)$,

$$(t_1, \cdots, t_k) +_k (s_1, \cdots, s_k) = (t_1 + s_1, \cdots, t_k + s_k).$$

Then the algebraic structure $\mathbb{R}^k = (\mathbb{R}^k, +_k)$ is a well-defined group with its group identity

$$0_k = \left(\underbrace{0, 0, \cdots, 0}_{k\text{-rimes}}\right),$$

where each element $(t_1, \cdots, t_k)$ has its $(+_k)$-inverse,

$$-(t_1, \cdots, t_k) = (-t_1, \cdots, -t_k),$$

for all $k \in \mathbb{N}$.

Define now a subgroup Δ_k of $(\mathbb{R}^k, +_k)$ by

$$\Delta_k = \{(t, t, t, \cdots, t) \in \mathbb{R}^k : t \in \mathbb{R}\},$$

under the inherited operation $(+_k)$, for all $k \in \mathbb{N}$. It is easy to check that indeed Δ_k is a subgroup of $\mathbb{R}^k$, moreover, it is group-isomorphic to the flow $\mathbb{R}$. Indeed, there exists a well-defined group-isomorphism,

$$\Delta_k \ni (t, t, \cdots, t) \longmapsto t \in \mathbb{R}.$$

Thus, one can understand the subgroup Δ_k of $(\mathbb{R}^k, +_k)$ as the flow $\mathbb{R}$, for all $k \in \mathbb{N}$.

For fixed elements $f_1, \cdots, f_k \in \mathfrak{A}_p$, define a group action $\kappa^{f_1, \cdots, f_k}$ of Δ_k acting on $\mathfrak{A}_p$ by

(9.1.3)

$$\kappa_{(t,t,\cdots,t)}^{f_1,\cdots,f_k}(h) \overset{def}{=} \left(\alpha_t^{f_1} \ \circ \cdots \circ \ \alpha_t^{f_k}\right)(h),$$

for all $h \in \mathfrak{A}_p$, and $t \in \mathbb{R}$.

Then one can check that $\kappa_{(t,\cdots,t)}^{f_1,\cdots,f_k}$ is a well-defined function on $\mathfrak{A}_p$, because $\alpha_t^{f_j}$ are well-defined functions on $\mathfrak{A}_p$, for all $j = 1, \cdots, k$, for all $t \in \mathbb{R}$. Furthermore,

$$\begin{aligned}\kappa_{(t,\cdots,t)+(s,\cdots,s)}^{f_1,\cdots,f_k}(h) &= \kappa_{(t+s,\cdots,t+s)}^{f_1,\cdots,f_k}(h) \\ &= \left(\alpha_{t+s}^{f_1} \circ \cdots \circ \alpha_{t+s}^{f_k}\right)(h)\end{aligned}$$

$$= \alpha_{t+s}^{\Sigma_{j=1}^{k} f_j}(h)$$

by (9.1.2)

$$= \left(\alpha_t^{\Sigma_{j=1}^{k} f_j} \circ \alpha_s^{\Sigma_{j=1}^{k} f_j}\right)(h)$$

since α^f are well-defined group actions (for all $f \in \mathfrak{A}_p$)

$$= \left(\alpha_t^{f_1} \circ \cdots \circ \alpha_t^{f_k} \circ \alpha_s^{f_1} \circ \cdots \circ \alpha_s^{f_k}\right)(h)$$

by (9.1.2)

$$= \left(\left(\alpha_t^{f_1} \circ \cdots \circ \ \alpha_t^{f_k}\right) \circ \left(\alpha_s^{f_1} \circ \cdots \circ \ \alpha_s^{f_k}\right)\right)(h)$$

$$= \left(\kappa_{(t,\cdots,t)}^{f_1,\cdots,f_k} \ \circ \ \kappa_{(s,\cdots,s)}^{f_1,\cdots,f_k}\right)(h),$$

for all $h \in \mathfrak{A}_p$, and $(t, \cdots, t)$, $(s, \cdots, s) \in \Delta_k$. i.e., we obtain that:

(9.1.4)

$$\kappa_{(t,\cdots,t)+(s,\cdots,s)}^{f_1,\cdots,f_k} = \kappa_{(t,\cdots,t)}^{f_1,\cdots,f_k} \circ \kappa_{(s,\cdots,s)}^{f_1,\cdots,f_k} \text{ on } \mathfrak{A}_p.$$

Therefore, $\kappa^{f_1,\cdots,f_k}$ is a well-defined group action of Δ_k acting on $\mathfrak{A}_p$. Since Δ_k is group-isomorphic to the flow $\mathbb{R}$, one has a flowed group-dynamical system (Δ_k, $\mathfrak{A}_p$, $\kappa^{f_1,\cdots,f_k}$).

By (9.1.2), (9.1.3) and (9.1.4), we obtain the following theorem.

Theorem 9.2. *Let $\mathfrak{X}_{f_0:p} = \mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R}$ be the p-prime Γ_{f_0}-dynamical Banach algebra. If $f_0 = \sum_{j=1}^{k} f_j$ in $\mathfrak{A}_p$, for $f_1, \cdots, f_k \in \mathfrak{A}_p$, then $\mathfrak{X}_{f_0:p}$ is isomorphic to the crossed product Banach algebra*

$$\mathfrak{A}_p \times_{\kappa^{f_1,\cdots,f_k}} \Delta_k$$

induced by the group dynamical system (Δ_k, $\mathfrak{A}_p$, $\kappa^{f_1,\cdots,f_k}$).

Proof. It is sufficient to show that the dynamical systems

$$(\mathbb{R} = \Gamma_{f_1+f_2+\cdots+f_k}, \mathfrak{A}_p, \alpha^{f_0}) \text{ and } (\Delta_k, \mathfrak{A}_p, \kappa^{f_1,..,f_k})$$

are equivalent. But, we showed that two groups $\mathbb{R} = \Gamma_{f_0}$ and Δ_k are group-isomorphic, moreover, group actions α^{f_0} and $\kappa^{f_1,\cdots,f_k}$ satisfy (9.1.3), i.e.,

$$\alpha_t^{f_0} = \kappa_{(t,\cdots,t)}^{f_1,\cdots,f_k} \text{ on } \mathfrak{A}_p, \text{ for all } t \in \mathbb{R}.$$

In other words, the above two dynamical systems are equivalent. Therefore they induce isomorphic crossed product Banach algebras

$$\mathfrak{X}_{f_0:p} = \mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R}, \text{ and } \mathfrak{A}_p \times_{\kappa^{f_1,\cdots,f_k}} \Delta_k,$$

respectively. □

If we denote the crossed product algebra $\mathfrak{A}_p \times_{\kappa^{f_1,\cdots,f_k}} \Delta_k$ by $\mathbb{X}_{f_1,..,f_k:p}$, then it has equivalent free probability with that of $\mathfrak{X}_{f_0:p}$ by the above theorem and by Section 8.

9.2. **Group Dynamical Systems on $\mathfrak{A}_p$ Induced by $\Gamma_{f_1} \times \Gamma_{f_2} \times \cdots \times \Gamma_{f_k}$.** Let Γ_{f_j} be the groups, isomorphic to the flow $\mathbb{R}$, in the sense of Section 6, for fixed $f_j \in \mathfrak{A}_p$, for $j = 1, \cdots, k$, for some $k \in \mathbb{N}$. Construct now the *product group*

(9.2.1)

$$\Gamma_{f_1,\cdots,f_k} \overset{def}{=} \prod_{j=1}^{k} \Gamma_{f_j},$$

equipped with the operation $(\cdot)$, such that:

$$\begin{aligned}
&\left(\Theta_{E_{t_1}(f_1)},\ \Theta_{E_{t_2}(f_2)}, \cdots, \Theta_{E_{t_k}(f_k)}\right) \cdot \left(\Theta_{E_{s_1}(f_1)},\ \Theta_{E_{s_2}(f_2)}, \cdots, \Theta_{E_{s_k}(f_k)}\right) \\
&\quad = \left(\Theta_{E_{t_1}(f_1)}\Theta_{E_{s_1}(f_1)}, \cdots, \Theta_{E_{t_k}(f_k)}\Theta_{E_{s_k}(f_k)}\right) \\
&\quad = \left(\Theta_{E_1(t_1 f_1 + s_1 f_1)}, \cdots, \Theta_{E_1(t_k f_k + s_k f_k)}\right) \\
&\quad = \left(\Theta_{E_{t_1+s_1}(f_1)}, \cdots, \Theta_{E_{t_k+s_k}(f_k)}\right).
\end{aligned}$$

Clearly, the algebraic structure $(\Gamma_{f_1,\cdots,f_k}, \cdot)$ forms a group, as the product group of $\Gamma_{f_1}, \cdots, \Gamma_{f_k}$.

Define a subgroup $D_{f_1,\cdots,f_k}$ of the group $\Gamma_{f_1,\cdots,f_k}$ of (9.2.1) by

(9.2.2)

$$D_{f_1,\cdots,f_k} \overset{def}{=} \left\{\left(\Theta_{E_t(f_1)},\ \Theta_{E_t(f_2)}, \cdots, \Theta_{E_t(f_k)}\right) | t \in \mathbb{R}\right\},$$

under the inherited operation $(\cdot)$ from $\Gamma_{f_1,\cdots,f_k}$.

Then the pair $(D_{f_1,\cdots,f_k}, \cdot)$ becomes a subgroup of $\Gamma_{f_1,\cdots,f_k}$ of (9.2.1), moreover, it is group-isomorphic to the subgroup Δ_k of the product group $\mathbb{R}^k$ of Section 9.1. Indeed, one can define a group-isomorphism,

$$\left(\Theta_{E_t(f_1)}, \cdots, \Theta_{E_t(f_k)}\right) \overset{\beta_k}{\longmapsto} (t, \cdots, t),$$

where β_k means the group-isomorphism between $D_{f_1,\cdots,f_k}$ and Δ_k.

Since Δ_k is group-isomorphic to the flow $\mathbb{R} = \Gamma_{f_0}$ whenever $f_0 = \sum_{j=1}^{k} f_j$, the above group $D_{f_1,\cdots,f_k}$ is group-isomorphic to the flow $\mathbb{R} = \Gamma_{f_0}$, too. So, one can define a group action $\gamma^{f_1,\cdots,f_k}$ of $D_{f_1,\cdots,f_k}$ acting on $\mathfrak{A}_p$ by

(9.2.3)

$$\gamma^{f_1,\cdots,f_k} \overset{def}{=} \kappa^{f_1,\cdots,f_k} \circ \beta_k.$$

Then it is a well-defined group action, moreover, we obtain that:

Theorem 9.3. *The group dynamical systems*

$$(\mathbb{R} = \Gamma_{f_0},\ \mathfrak{A}_p,\ \alpha^{f_0}) \text{ and } (D_{f_1,\cdots,f_k},\ \mathfrak{A}_p,\ \gamma^{f_1,\cdots,f_k})$$

are equivalent, whenever $f_0 = \sum_{j=1}^{k} f_j$ *in* $\mathfrak{A}_p$.

Proof. We showed that two group dynamical systems $(\mathbb{R}, \mathfrak{A}_p, \alpha^{f_0})$ and $(\Delta_k, \mathfrak{A}_p, \kappa^{f_1,\cdots,f_k})$ are equivalent, whenever $f_0 = \sum_{j=1}^{k} f_j$ in $\mathfrak{A}_p$. By (9.2.2) and (9.2.3), it is not difficult to check that the group dynamical systems $(D_{f_1,\cdots,f_k}, \mathfrak{A}_p, \gamma^{f_1,\cdots,f_k})$ and $(\Delta_k, \mathfrak{A}_p, \kappa^{f_1,\cdots,f_k})$ are equivalent. Therefore, we obtain the desired consequence. □

The following corollary is the direct consequence of the above theorem.

Corollary 9.4. *Let* $f_0 = \sum_{j=1}^{k} f_j$ *in* $\mathfrak{A}_p$. *Then the Banach algebras*

$$\mathfrak{X}_{f_0:p} = \mathfrak{A}_p \times_{\alpha^{f_0}} \mathbb{R} \text{ and } \mathcal{X}_{f_1,\cdots,f_k} = \mathfrak{A}_p \times_{\gamma^{f_1,\cdots,f_k}} D_{f_1,\cdots,f_k}$$

are isomorphic. □

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