Ann. Funct. Anal. 1 (2010), no. 1, 105–122
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

POLYNOMIAL IDENTIFICATION IN UNIFORM AND OPERATOR ALGEBRAS

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Communicated by P. Šemrl

ABSTRACT. Let $\mathcal{A}$ be a unital Banach algebra, and denote the spectral radius of $f \in \mathcal{A}$ by $\rho(f)$. If $\mathcal{A}$ is a uniform algebra and $\rho(fh+1) = \rho(gh+1)$ for all $h \in \mathcal{A}$, then it can be shown that $f = g$, a result that also carries in algebras of bounded linear operators on Banach spaces. On the other hand $\rho(fh) = \rho(gh)$ does not imply $f = g$ in any unital algebra, marking a distinction between the polynomials $p(z,w) = zw + 1$ and $p(z,w) = zw$. Such results are known as *spectral identification lemmas*, and in this work we demonstrate first- and second-degree polynomials of two variables that lead to identification via the spectral radius, peripheral spectrum, or full spectrum in uniform algebras and in algebras of bounded linear operators on Banach spaces. The primary usefulness of identification lemmas is to determine the injectivity of a class of mappings that preserve portions of the spectrum, and results corresponding to the given identifications are also presented.

1. BACKGROUND, NOTATION, AND PRELIMINARIES

Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$. The collection of multiplicatively invertible elements of $\mathcal{A}$ is denoted $\mathcal{A}^{-1}$, and the spectrum of $f \in \mathcal{A}$ is $\sigma(f) = \{\lambda \in \mathbb{C} : f - \lambda \notin \mathcal{A}^{-1}\}$. The *peripheral spectrum* of f is the set of spectral values of maximum modulus and is denoted by

$$\sigma_\pi(f) = \left\{ \lambda \in \sigma(f) : |\lambda| = \max_{z \in \sigma(f)} |z| \right\},$$

Date: Received: 11 July 2010; Revised: 12 October 2010; Accepted: 21 October 2010.
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2010 *Mathematics Subject Classification.* Primary 46J10; Secondary 47L10, 47A65, 46J20, 46H20, 47C05.

Key words and phrases. Uniform algebras, standard operator algebras, polynomial identification, peripheral spectrum, spectral preserver problems.

and the spectral radius of f is $\rho(f) = \max_{z \in \sigma(f)} |z|$. It is well-known that the spectrum of an element of a unital Banach algebra is a compact subset of the complex plane, so the spectral radius is finite (see e.g. [18]).

Throughout, K is a compact Hausdorff space, and $C(K)$ is the space of complex-valued, continuous functions on K. A *uniform algebra* on K is a subalgebra $\mathcal{A} \subset C(K)$ that contains the constant functions, separates points, and is complete with respect to the uniform norm.

We denote by X a Banach space over $\mathbb{C}$ and by $B(X)$ the Banach algebra of bounded linear operators from X to itself. A subalgebra $\mathcal{A} \subset B(X)$ is called a *standard operator algebra* if it contains the rank-one operators.

1.1. Spectral Identification and a Characterization of the Main Results.

There are different approaches to determining when two elements of a Banach algebra $\mathcal{A}$ are the same. One natural way to identify elements is by analyzing how they interact with other elements of the algebra. For example, $f, g \in \mathcal{A}$ are clearly the same if $fh = gh$ for even a single invertible element $h \in \mathcal{A}$. We can ask for less than knowing what the products of elements are, such as merely knowing what the spectrum or spectral radii of the products are. It is straightforward to see that $\rho(fh) = \rho(gh)$ does not ensure that $f = g$, since $\rho(-h) = \rho(h)$ for all $h \in \mathcal{A}$, and yet $-1 \neq 1$. If we ask for slightly more, however, then this multiplicative combination is enough in some algebras. For uniform algebras, we have the following:

Lemma 1.1 ([14], Lemma 3). *If $\mathcal{A} \subset C(K)$ is a uniform algebra, $f, g \in \mathcal{A}$, and*

$$\sigma_\pi(fh) = \sigma_\pi(gh) \tag{1.1}$$

for all $h \in \mathcal{A}$, then $f = g$.

This is an example of what has become known as a *spectral identification lemma*, because it allows to identify f and g via a criterion concerning the spectrum. In fact, it is not necessary that (1.1) hold for all $h \in \mathcal{A}$, but rather only for all h in a particular subset of $\mathcal{A}$.

Despite the fact that the spectral radius could not ensure identification via products, it is enough for certain other combinations.

Lemma 1.2 ([13], Lemma 2.1). *If $\mathcal{A} \subset C(K)$ is a uniform algebra, $f, g \in \mathcal{A}$, and*

$$\rho(fh + 1) = \rho(gh + 1) \tag{1.2}$$

for all $h \in \mathcal{A}$, then $f = g$.

It is again not necessary that (1.2) hold for all $h \in \mathcal{A}$, but it turns out that the subset of $\mathcal{A}$ needed in this case is different from the subset required for Lemma 1.1.

These two results show a contrast in two-variable polynomials. If $p(z, w) = zw + 1$, then $\rho(p(f, h)) = \rho(p(g, h))$ for all $h \in \mathcal{A}$ is enough to ensure that $f = g$, whereas if $q(z, w) = zw$, it is necessary that $\sigma_\pi(q(f, h)) = \sigma_\pi(q(g, h))$ to ensure that $f = g$. One goal is to characterize which polynomials will lead to identifications in uniform algebras via the spectral radius and which via the peripheral spectrum.

Given the results for uniform algebras, it is also natural to ask if such results hold in other unital Banach algebras. The answer is affirmative, and results similar to those given above for uniform algebras have been proven in algebras of bounded linear operators on Banach spaces.

Lemma 1.3 ([19], Lemma 1)**.** *If $\mathcal{A}$ is a subalgebra of $B(X)$ that contains the rank-one operators; $A, B \in \mathcal{A}$; and*

$$\sigma_\pi(AT) = \sigma_\pi(BT)$$

for all $T \in \mathcal{A}$ of rank one, then $A = B$.

This is the operator analogue of Lemma 1.1, which gives rise to several immediate questions. Firstly, can the first- and second-degree polynomials be classified according to their spectral identification properties as with uniform algebras? Secondly, is the classification the same for operator algebras as it is for uniform algebras?

It is immediate that the answer to the second question – whether the classification is the same for uniform and operator algebras – must be no, since the non-commutativity of operator algebras implies that there are many more polynomials of two variables than there are for commutative algebras. Nonetheless, for each polynomial characterized in the commutative case, there is a corresponding polynomial which can be classified in the non-commutative case. The real complication for algebras of operators, however, is not that they are noncommutative but that they are not semi-simple – i.e. that many elements in an operator algebra can have spectral radius 0 – and even the first-degree polynomial case is significantly more complicated for operators than for functions in uniform algebras.

Section 2 begins with characterizations of polynomials of two variables that cannot lead to identification in *any* unital Banach algebra. Uniform algebras are studied in Section 3, where we characterize first- and second-degree polynomials in two variables that lead to identification, including polynomials that lead to identification via the spectral radius, via the peripheral spectrum, and via the full spectrum. Moreover, several results are given demonstrating which subsets of a uniform algebra $\mathcal{A}$ are needed to guarantee identification on all elements. In Section 4, we classify spectral identifications in standard operator algebras via first-degree polynomials and give several results for second-degree polynomials, including an operator algebra analogue to Lemma 1.2 and a generalization of Lemma 1.3.

1.2. **Motivation and Applications.** Spectral identification lemmas are useful in the study of *spectral preserver problems.* A spectral preserver problem is the study of a map $T\colon \mathcal{A} \to \mathcal{B}$ between Banach algebras that preserves some subset or property of the spectrum. The study of spectral preservers is quite old – going back at least to Fröbenius [2] – but was reinvigorated by Molnár in [15], who showed that a surjective map $T\colon C(K) \to C(K)$ – where K is a first-countable, compact Hausdorff space – that satisfied

$$\sigma(fg) = \sigma(T(f)T(g)) \tag{1.3}$$

for all $f, g \in \mathcal{A}$ is automatically linear. If, moreover, $T(1) = 1$, then T is also multiplicative. Fundamental to this study was that linearity of T was not assumed *a priori*, but that it was a consequence of (1.3).

The results of Molnár have been extended in several directions. Suppose that K_1 and K_2 are compact Hausdorff spaces (not assumed to be first-countable), that $\mathcal{A} \subset C(K_1)$ and $\mathcal{B} \subset C(K_2)$ are uniform algebras, and that $T\colon \mathcal{A} \to \mathcal{B}$ is a surjective map. Luttman and Tonev [14] and Hatori, Miura and Takagi [6, 7] generalized the work of Molnár and showed that if

$$\sigma_\pi(fg) = \sigma_\pi(T(f)T(g)) \tag{1.4}$$

for all $f, g \in \mathcal{A}$, then T is linear. Again, if $T(1) = 1$, then T is also multiplicative and thus an isometric algebra isomorphism. It is, therefore, not required to preserve the full spectra of products, but merely the peripheral spectra.

Among the first steps in proving that such a T is a linear bijection is proving that T is injective, and this is where the associated spectral identification lemma is particularly useful. Note that the conditions

$$\sigma_\pi(T(f)T(h)) = \sigma_\pi(fh) \quad \text{and} \quad \sigma_\pi(fh) = \sigma_\pi(gh)$$

are clearly related, as they both correspond to a spectral condition involving the polynomial $p(z, w) = zw$. Any T satisfying (1.4) must be injective, since $T(f) = T(g)$ implies

$$\sigma_\pi(fh) = \sigma_\pi(T(f)T(h)) = \sigma_\pi(T(g)T(h)) = \sigma_\pi(gh)$$

for all $h \in \mathcal{A}$. Lemma 1.1 then gives that $f = g$.

A related result in [13], proven independently also by Honma [8, 9], was that if $T\colon \mathcal{A} \to \mathcal{B}$ satisfies

$$\|fh + 1\| = \|T(f)T(h) + 1\|$$

for all $h \in \mathcal{A}$, then T is a generalized weighted composition operator. Here $\|f\|$ denotes the uniform norm, which is equal to the spectral radius, so this naturally corresponds to Lemma 1.2. Again, if $T(f) = T(g)$, then we have

$$\|fh + 1\| = \|T(f)T(h) + 1\| = \|T(g)T(h) + 1\| = \|gh + 1\|$$

for all $h \in \mathcal{A}$, which implies that $f = g$ by Lemma 1.2.

These examples demonstrate that a general classification of spectral identification will lead to a large-scale characterization of which spectral-preserver maps must be injective.

2. Preliminary Results on Arbitrary Unital Algebras

Before looking specifically at uniform algebras and algebras of operators on Banach spaces, we first present some preliminary results that hold in any unital algebra, regardless of whether it is normed or complete.

2.1. **Polynomials that do not Identify.** A first step in determining which polynomials $p(z,w)$ of two variables can lead to an identification lemma is determining which polynomials will *not* lead to identification. In particular, $p(z,w) = z^2w$ cannot give identification in any unital algebra $\mathcal{A}$, since

$$\sigma\left((1)^2h\right) = \sigma\left((-1)^2h\right)$$

for all $h \in \mathcal{A}$, but $1 \neq -1$. Note that since preservation of the full spectrum will not suffice, neither will preservation of the peripheral spectrum nor spectral radius. We formalize this example as a lemma.

Lemma 2.1. *Let $\mathcal{A}$ be a unital Banach algebra and $f, g \in \mathcal{A}$. If $p(z,w)$ is a polynomial and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$ are such that $p(\alpha, w) = p(\beta, w)$ for all $w \in \mathbb{C}$, then*

$$\sigma\left(p(\alpha, h)\right) = \sigma\left(p(\beta, h)\right)$$

for all $h \in \mathcal{A}$, and therefore $p(z,w)$ does not lead to identification via the spectrum.

Though this result is helpful for ruling out certain polynomials, it is not useful unless we can know specifically which polynomials will have the property $p(\alpha, w) = p(\beta, w)$ for some distinct α and β and all $w \in \mathbb{C}$. It is well-known that a polynomial $p(z)$ over $\mathbb{C}$ satisfies $p(\alpha) = p(\beta)$ if and only if

$$p(z) = (z-\alpha)(z-\beta)h(z) + z_0 \tag{2.1}$$

for some polynomial $h(z)$ and some complex number z_0. The "if" part is obvious and the "only if" part follows as $z-\alpha$ and $z-\beta$ must both divide $p(z) - p(\alpha)$. Thus $z_0 = p(\alpha)$, and $(z-\alpha)(z-\beta)h(z) = p(z) - p(\alpha)$.

More generally, a characterization analogous to (2.1) holds for polynomials of two-variables.

Lemma 2.2. *Let $p(z,w)$ be a complex-valued polynomial and $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq \beta$. Then $p(\alpha, w) = p(\beta, w)$ for all $w \in \mathbb{C}$ if and only if*

$$p(z,w) = (z-\alpha)(z-\beta)q(z,w) + s(w) \tag{2.2}$$

for some polynomials $q(z,w)$ and $s(w)$.

Proof. If $p(z,w)$ has the form (2.2), then clearly $p(\alpha, w) = p(\beta, w)$ for all $w \in \mathbb{C}$, so assume that $p(\alpha, w) = p(\beta, w)$ for all $w \in \mathbb{C}$. Then $p(z,w)$ can be written

$$p(z,w) = p_n(z)w^n + p_{n-1}(z)w^{n-1} + \ldots + p_1(z)w + p_0(z)$$

with $p_i(z) \in \mathbb{C}[z]$, $0 \leq i \leq n$. Since $p(\alpha, w) = p(\beta, w)$ for all $w \in \mathbb{C}$, $p_i(\alpha) = p_i(\beta)$, so, by (2.1), there exist polynomials $h_i(z) \in \mathbb{C}[z]$ such that $p_i(z) = (z-\alpha)(z-\beta)h_i(z) + p_i(\alpha)$. Thus

$$\begin{aligned} p(z,w) &= p_n(z)w^n + p_{n-1}(z)w^{n-1} + \ldots + p_1(z)w + p_0(z) \\ &= (z-\alpha)(z-\beta)\left(h_n(z)w^n + h_{n-1}(z)w^{n-1} + \ldots + h_0(z)\right) + p(\alpha, w). \end{aligned}$$

Setting $q(z,w) = \sum\limits_{i=0}^{n} h_i(z)w^i$ gives

$$p(z,w) = (z-\alpha)(z-\beta)q(z,w) + p(\alpha, w),$$

which proves the result. $\square$

This leads to some immediate examples of polynomials in two variables that cannot lead to identification in any unital algebra.

Example 2.3. Let $p(z,w) = z^3w + z^2w + w + 1 = z(z+1)(zw) + (w+1)$. This polynomial has the form (2.2), so Lemma 2.2 shows that it will not lead to an identification. This can be seen directly, as $p(0,w) = p(-1,w)$ for all $w \in \mathbb{C}$.

Example 2.4. The polynomial $p(z,w) = z^2 + w^2 + z + w + 1$ cannot identify, since $p(0,w) = w^2 + w + 1 = p(-1,w)$ for all $w \in \mathbb{C}$. It will be seen that this example characterizes the degree-two polynomials that fail to identify in unital algebras.

Example 2.5. The polynomial $p(z,w) = z^3 + z^2w + z + 1$ will not lead to identification, since $p(z,w) = (z^2+1)(z+w) + (-w+1)$, which is of the form (2.2).

More generally, Lemma 2.2 has consequences for certain polynomials that have degree higher than one in z.

Corollary 2.6. *Let $\mathcal{A}$ be a unital Banach algebra, and*

$$p(z,w) = q(z)r(z,w) + s(w) \tag{2.3}$$

for some polynomials $q(z)$, $r(z,w)$, and $s(w)$, where $r(z,w)$ has no factors of the form $az+b$ ($a,b \in \mathbb{C}$). If $q(z)$ has more than one (distinct) root, then there exist $\alpha, \beta \in \mathbb{C}$ (and thus in $\mathcal{A}$) with $\alpha \neq \beta$ such that $\sigma(p(\alpha,h)) = \sigma(p(\beta,h))$ for all $h \in \mathcal{A}$.

Thus $p(z,w)$ cannot lead to an identification lemma. Note that every polynomial in two variables can be written in the form (2.3), and it is only to check whether $q(z)$ has more than 1 distinct root.

In particular, Corollary 2.6 shows that, for polynomials of degree higher than 2 in z, the "cross terms" are essential.

Example 2.7. Suppose $\mathcal{A}$ is a unital Banach algebra and

$$p(z,w) = \alpha(z-\nu)^2 + \beta w^2 + \gamma w + \eta$$

for $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $\nu, \gamma, \eta \in \mathbb{C}$. Then $p(z,w)$ does not identify. For the same reason, $p(z,w) = q(z) + s(w)$ cannot identify whenever the degree of q is greater than or equal to 2, without any restriction on $s(w)$.

3. Identification in Uniform Algebras

Let K be a compact Hausdorff space and $C(K)$ the space of continuous, complex-valued functions on K. A uniform algebra $\mathcal{A}$ on K is a subalgebra of $C(K)$ that is complete with respect to the uniform norm, contains the constant functions, and separates points, i.e. for every pair $x, y \in K$ with $x \neq y$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. Throughout we denote by $\|f\|$ the uniform norm of f.

In this section we generalize the known results on polynomial spectral identification in uniform algebras. As well as determining which polynomials will lead to identification in uniform algebras, we also explore which functions in a uniform algebra are necessary to use. A subset of a uniform algebra $\mathcal{A}$ of particular interest is the set of peaking functions. A *peaking function* is a function $h \in \mathcal{A}$ such that $\sigma_\pi(h) = \{1\}$. That is, h is a peaking function if $|h(x)| \le 1$ for all $x \in K$ and $|h(x)| = 1$ if and only if $h(x) = 1$. The *peak set*, or *maximizing set*, of a peaking function $h \in \mathcal{A}$ is the set $M(h) = \{x \in K : h(x) = 1\}$. It was noted after the statement of Lemma 1.1 that $\sigma_\pi(fh) = \sigma_\pi(gh)$ need not hold for all $h \in \mathcal{A}$, but only for a subset, in order to ensure $f = g$. The peaking functions are one such sufficient subset.

A related set of functions are the $\mathbb{C}$-peaking functions, which are simply scalar multiples of the peaking functions, i.e. the collection of functions with singleton peripheral spectrum. An analysis of the proof of Lemma 1.2 given in [13] shows that the $\mathbb{C}$-peaking functions are those needed for Lemma 1.2 to hold, so we have the following restatement:

Lemma 3.1. *If $\mathcal{A}$ is a uniform algebra, $f, g \in \mathcal{A}$, and*

$$\|fh + 1\| = \|gh + 1\|$$

for all $\mathbb{C}$-peaking functions $h \in \mathcal{A}$, then $f = g$.

The reason that the peaking functions and $\mathbb{C}$-peaking functions are of particular interest is that they isolate the important points of the underlying domain K. A point $x \in K$ is called a *strong boundary point* (or weak peak point or p-point) if for every open neighborhood U of x there exists a peaking function $h \in \mathcal{A}$ such that $\{x\} \subset M(h) \subset U$. It is well-known that in a uniform algebra, the strong boundary points are exactly the points of the Choquet boundary [3], which we denote by $\delta\mathcal{A}$. Since $\delta\mathcal{A}$ is a boundary for $\mathcal{A}$, it is only necessary to show that $f(x) = g(x)$ for all $x \in \delta\mathcal{A}$ in order that they be equal at all points of K.

A classical result that is necessary for our analysis is Bishop's Lemma [1, Theorem 2.4.1]. Though there are several versions of this result, here we give a variation of the version given by Hatori et. al. [4, Proposition 2.2], by combining it with [13, Corollary 1.2]. We denote by $\exp(\mathcal{A})$ the *exponent* of the algebra $\mathcal{A}$, i.e. the collection of elements $h \in \mathcal{A}$ such that $h = \sum_{k=0}^{\infty} \frac{f^k}{k!}$ for some $f \in \mathcal{A}$. It is well-known that every such element is invertible.

Lemma 3.2 (Bishop's Lemma)**.** *Let $\mathcal{A}$ be a uniform algebra on a compact Hausdorff space K, and suppose that $f \in \mathcal{A}$ and $x_0 \in \delta\mathcal{A}$. If $f(x_0) \neq 0$, then there exists a peaking function $h \in \exp(\mathcal{A})$ such that $h(x_0) = 1$ and $\sigma_\pi(fh) = \{f(x_0)\}$. If $f(x_0) = 0$, then for every $\varepsilon > 0$ there exists a peaking function $h \in \exp(\mathcal{A})$ such that $h(x_0) = 1$ and $\|fh\| < \varepsilon$. In either case, if U is any open neighborhood of x_0, h can be chosen so that $M(h) \subset U$.*

Proof. If $f(x_0) = 0$ and $\varepsilon > 0$ is given, then [4, Proposition 2.2] gives the existence of a peaking function $h_1 \in \exp(\mathcal{A})$ with $h_1(x_0) = 1$ and $\|fh_1\| < \varepsilon$. Since x_0 is a strong boundary point, given any open neighborhood U of x_0 there exists a peaking function k such that $x_0 \in M(k) \subset U$. Now $h_2 = \exp(k - 1)$ is also a peaking function with $M(h_2) = M(k)$ (see [13, Corollary 1.2]), so $h_1 h_2$ is a peaking function contained in $\exp(\mathcal{A})$ with $x_0 \in M(h_1 h_2) \subset U$ and $\|fh_1 h_2\| < \varepsilon$.

If $f(x_0) \neq 0$, then [4, Proposition 2.2] gives the existence of a peaking function $h_1 \in \exp(\mathcal{A})$ such that $\sigma_\pi(fh_1) = \{f(x_0)\}$. If U is any open neighborhood of x_0, then, since x_0 is a strong boundary point, there exists a peaking function $k \in \mathcal{A}$ such that $\{x_0\} \subset M(k) \subset U$, and [13, Corollary 1.2] gives a peaking function $h_2 \in \mathcal{A}$ such that $M(h_2) = M(k) \subset U$, $h_2 \in \exp(\mathcal{A})$, and fh_2 attains its maximum modulus exclusively on $M(k) = M(h_2)$. Thus $h_1(x_0)h_2(x_0) = 1$, which implies that h_1h_2 is a peaking function with $h_1h_2 \in \exp(\mathcal{A})$. Moreover, since fh_2 attains its maximum modulus exclusively on $M(h_2)$, we also have that $M(fh_1h_2) \subset U$ and $\{f(x_0)\} = \sigma_\pi(fh_1) = \sigma_\pi(fh_1h_2)$. □

A related result has been proven more recently for additive combinations by Yates and Tonev (see e.g. [21, Lemma 3.4.5] or [20, Lemma 1]) and will also be used.

Lemma 3.3 (Additive Bishop's Lemma). *Let $\mathcal{A} \subset C(K)$ be a uniform algebra, $f \in \mathcal{A}$, $x_0 \in \delta\mathcal{A}$, and $r > 1$. Then there exists a $\mathbb{C}$-peaking function $h \in \mathcal{A}$ such that $x_0 \in M(h)$, $h(x_0) > 0$, $\|h\| = r\|f\|$,*

$$|f(x)| + |h(x)| < |f(x_0)| + |h(x_0)|$$

for all $x \in K \setminus M(h)$, and

$$|f(x)| + |h(x)| = |f(x_0)| + |h(x_0)|$$

for all $x \in M(h)$. In particular, $\||f| + |h|\| = |f(x_0)| + |h(x_0)|$, and, given any neighborhood U of x_0, h can be chosen so that $M(h) \subset U$.

3.1. **Degree-One Polynomials.** In light of Lemma 1.1, which shows that products of algebra elements carry information about the factors, it is natural to ask whether sums also carry such information, and, in general, the answer is yes.

Theorem 3.4. *Let $\mathcal{A} \subset C(K)$ be a uniform algebra, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{C}$, and $f, g \in \mathcal{A}$. If*

$$\|\alpha f + \beta h + \gamma\| = \|\alpha g + \beta h + \gamma\| \tag{3.1}$$

for all $\mathbb{C}$-peaking functions $h \in \mathcal{A}$, then $f = g$.

First we note that, if h were allowed to range over all elements of $\mathcal{A}$, then the result is trivial, taking $h = \frac{-1}{\beta}(\alpha f + \gamma)$. Allowing h to range only over the $\mathbb{C}$-peaking functions – the elements of $\mathcal{A}$ with singleton peripheral spectrum – is not so direct.

Proof. Note that we may assume $\beta = 1$ without loss of generality, since h is a $\mathbb{C}$-peaking function if and only if βh is a $\mathbb{C}$-peaking function for every $\beta \in \mathbb{C} \setminus \{0\}$. We begin by assuming that $\alpha = 1$ and $\gamma = 0$.

Let $f, g \in \mathcal{A}$, and suppose that $\|f + h\| = \|g + h\|$ for all $\mathbb{C}$-peaking functions $h \in \mathcal{A}$. Assume $\|g\| < \|f\|$, and let $\lambda \in \sigma_\pi(f)$. Then $h(x) = \frac{f(x)}{2} + \frac{f(x)^2}{2\lambda}$ is a $\mathbb{C}$-peaking function with $\lambda \in \sigma_\pi(h)$ that attains its maximum modulus whenever f takes the value λ. In particular, $M(h) \cap M(f) \neq \emptyset$, so

$$\|g + h\| \leq \|g\| + \|h\| < \|f\| + \|h\| = 2|\lambda| = \|f + h\|$$

– contrary to hypothesis – which implies $\|f\| = \|g\|$.

In fact, not only must f and g be equal in norm, they must be equal in absolute value at every point $x \in \delta\mathcal{A}$. Fix $x_0 \in \delta\mathcal{A}$; assume $|f(x_0)| > |g(x_0)|$; and let U be an open neighborhood of x_0 such that $|f(x_0)| > |g(y)|$ for all $y \in U$. Since x_0 is an element of the Choquet boundary, we may choose a peaking function $k \in \mathcal{A}$ with $x_0 \in M(k) \subset U$, and, by taking a high enough power of k, we may assume that $|k(x)| < \frac{|f(x_0)|}{2\|f\|}$ for all $x \in K \setminus U$. Then $h = \frac{\|f\|}{|f(x_0)|} f(x_0)k$ is a $\mathbb{C}$-peaking function such that $h(x_0) = \frac{\|f\|}{|f(x_0)|} f(x_0)$ and $|h(x)| < |f(x_0)|/2$ on $K \setminus U$. Therefore

$$\begin{aligned} |f(x_0)| + \|f\| = |f(x_0)| + \frac{\|f\|}{|f(x_0)|} |f(x_0)| &= \left| f(x_0) + \frac{\|f\|}{|f(x_0)|} f(x_0) \right| \\ &= |(f+h)(x_0)| \le \|f+h\|. \end{aligned}$$

Combining this result with the fact that $\|f\| = \|g\|$ gives

$$|g(x) + h(x)| < \|g\| + |f(x_0)|/2 = \|f\| + |f(x_0)|/2 < \|f\| + |f(x_0)| \le \|f+h\|$$

for all $x \in K \setminus U$. On the other hand, for $x \in U$,

$$|g(x) + h(x)| \le |g(x)| + |h(x)| < |f(x_0)| + |h(x_0)| = |f(x_0)| + \|f\| \le \|f+h\|,$$

recalling that $|g(x)| < |f(x_0)|$ for $x \in U$ and that $x_0 \in M(h)$. Therefore $\|g+h\| < \|f+h\|$, which contradicts (3.1), proving $|f(x)| = |g(x)|$ for all $x \in \delta\mathcal{A}$.

Lastly, we prove that f and g must coincide in value at each point of $\delta\mathcal{A}$. Let $x_0 \in \delta\mathcal{A}$, and assume $f(x_0) \neq g(x_0)$. Since $|g(x_0)| = |f(x_0)|$, there is an open neighborhood U of x_0 with $f(U) \cap g(U) = \emptyset$ and such that $\arg(f(x_0)) \notin \arg(g(U))$. By Lemma 3.3, there exists a $\mathbb{C}$-peaking function $h \in \mathcal{A}$ with $x_0 \in M(h) \subset U$ such that $h(x_0) > 0$, $\|h\| = 2\|f\|$, $|f(x)| + |h(x)| < |f(x_0)| + |h(x_0)|$ for $x \notin M(h)$ and $\||f| + |h|\| = |f(x_0)| + |h(x_0)|$. Set $h' = \frac{f(x_0)}{|f(x_0)|} h$, then h' is a $\mathbb{C}$-peaking function with $M(h') = M(h)$ and $|h'| = |h|$. Moreover

$$|f(x_0)| + |h(x_0)| = |f(x_0)| + |h'(x_0)| = |f(x_0)| + \left| \frac{f(x_0)}{|f(x_0)|} h(x_0) \right| = |f(x_0) + h'(x_0)|,$$

where the last equality holds due to the fact that $h(x_0)/|f(x_0)|$ is a strictly positive real number.

Since h' is a $\mathbb{C}$-peaking function, (3.1) gives that $\|f + h'\| = \|g + h'\|$, but for $x \notin M(h)$

$$|g(x) + h'(x)| \le |g(x)| + |h'(x)| = |f(x)| + |h(x)| < |f(x_0) + h'(x_0)|.$$

Since the argument of $h'(x_0) = 2\|f\| \frac{f(x_0)}{|f(x_0)|}$ is the same as the argument of $f(x_0)$ and $\arg(f(x_0)) \notin \arg(g(U))$, for $x \in M(h) \subset U$ we have

$$\begin{aligned} |g(x) + h'(x)| = |g(x) + h'(x_0)| &< |g(x)| + |h'(x_0)| = |f(x)| + |h'(x_0)| \\ &\le \||f| + |h'|\| = |f(x_0)| + |h(x_0)|. \end{aligned}$$

Therefore $|g(x) + h'(x)| < |f(x_0) + h'(x_0)|$ for all x, contradicting the hypothesis. This contradiction yields that $f(x) = g(x)$ for all $x \in \delta\mathcal{A}$, which implies that $f(x) = g(x)$ for all $x \in K$.

Recalling that $\beta = 1$ without loss of generality, suppose $\alpha \in \mathbb{C} \setminus \{0\}$ and $\gamma \in \mathbb{C}$, and set $f' = \alpha f + \gamma$ and $g' = \alpha g + \gamma$. Then

$$\|\alpha f + h + \gamma\| = \|\alpha g + h + \gamma\|$$

for all $\mathbb{C}$-peaking functions h implies that

$$\|f' + h\| = \|g' + h\|$$

for all $\mathbb{C}$-peaking functions h, which implies $f' = g'$. Thus $\alpha f + \gamma = \alpha g + \gamma$, i.e. $f = g$. $\square$

In this case it is natural to ask whether it is possible to further restrict the set of required h's, for example by allowing only the true peaking functions rather than all scalar multiples of peaking functions. It turns out that using only the peaking functions is not necessarily sufficient, depending on the norms of f and g. In fact, even the peripheral spectrum is not sufficient to ensure identification when h is taken over any class of functions with uniformly bounded norm (which includes the peaking functions).

Proposition 3.5. *Let $\mathcal{A} \subset C(K)$ be a uniform algebra, $f, g \in \mathcal{A}$, and $d > 0$. If $f(x) = g(x)$ for all $x \in K$ such that $|f(x)| \geq \|f\| - 2d$ or $|g(x)| \geq \|g\| - 2d$, then*

$$\sigma_\pi(f + h) = \sigma_\pi(g + h)$$

for all $h \in \mathcal{A}$ with $\|h\| \leq d$. Thus $\sigma_\pi(f + h) = \sigma_\pi(g + h)$ for all $\|h\| \leq d$ is not sufficient to ensure that $f = g$.

Proof. Given $x_f, x_g \in K$ where f and g respectively attain their maximum modulus, $\|f\| = |f(x_f)| = |g(x_f)|$ and $\|g\| = |g(x_g)| = |f(x_g)|$. Thus $\|f\| \geq |f(x_g)| = \|g\|$ and $\|g\| \geq |g(x_f)| = \|f\|$, implying $\|f\| = \|g\|$.

Let $N = \{x \in K : |f(x)| \geq \|f\| - 2d\}$, and let $h \in \mathcal{A}$ with $\|h\| \leq d$. Observe that $|(f + h)(x_f)| \geq \|f\| - d$ by the triangle inequality, so $x_f \in N$. For any $y \notin N$, $|f(y)| < \|f\| - 2d$, so $|(f + h)(y)| < \|f\| - d$, which implies $f + h$ attains its maximum modulus exclusively on N.

Let $M = \{x \in K : |g(x)| \geq \|g\| - 2d\}$. Given $x_N \in N$, $|g(x_N)| = |f(x_N)| \geq \|f\| - 2d = \|g\| - 2d$, so $x_N \in M$. Similarly, given $x_M \in M$, $|f(x_M)| = |g(x_M)| \geq \|g\| - 2d = \|f\| - 2d$, so $x_M \in N$. Thus $M = N$ and $g + h$ also attains its maximum modulus only on N.

By hypothesis, $f|_N = g|_N$ which implies that $(f + h)|_N = (g + h)|_N$. Then $\sigma_\pi(f + h) = \sigma_\pi(g + h)$ since $f + h$ and $g + h$ attain their maximum modulus only on N. $\square$

If either $\|f\|$ or $\|g\|$ is strictly greater than 2, then $\sigma_\pi(f + h) = \sigma_\pi(g + h)$ for all peaking functions h is not sufficient to ensure that $f = g$. Allowing h to range over all $\mathbb{C}$-peaking functions, however, gives that $\rho(f + h) = \rho(g + h)$ implies $f = g$. We demonstrate with a simple example.

Example 3.6. Suppose $K = [0, 1]$, $\mathcal{A} = C([0, 1])$, and let

$$f(x) = 20(1 - x) \quad \text{and} \quad g(x) = \begin{cases} 20(1 - x) & 0 \leq x \leq \frac{1}{2} \\ 10 - 20x\left(x - \frac{1}{2}\right) & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then $f, g \in C([0,1])$, and $f \neq g$. If $\|h\| \leq 5$, then $\|f+h\| \geq 15$. Thus $x_0 \in M(f+h)$ implies $|(f+h)(x_0)| \geq 15$, so $|f(x_0)| \geq 10$, which means $x_0 \in \left[0, \frac{1}{2}\right]$. But $f(x) = g(x)$ for $x \in \left[0, \frac{1}{2}\right]$. The same holds for $g+h$, so $\sigma_\pi(f+h) = \sigma_\pi(g+h)$ for all $h \in \mathcal{A}$ with $\|h\| \leq 5$.

Thus all degree-one polynomials $p(z, w)$, in which both z and w appear, uniquely identify elements of uniform algebras via the spectral radius, as long as the functions h to which we compare range over all $\mathbb{C}$-peaking functions. If h is restricted in norm, then it is possible that even the peripheral spectrum condition is not sufficient to identify.

3.2. **Second-Degree Polynomials.** All degree-two polynomials in z and w are of the form

$$p(z, w) = z(\alpha z + \gamma w + \delta) + (\beta w^2 + \eta w + \nu) \tag{3.2}$$

for α, β, γ, δ, η, $\nu \in \mathbb{C}$, where at least one of α, β, and γ is not zero. This section characterizes identification in uniform algebras via such polynomials.

Recalling that Lemma 3.1 requires only the $\mathbb{C}$-peaking functions to ensure identification, we have an immediate extension.

Corollary 3.7. *Let $\mathcal{A} \subset C(K)$ and $f, g \in \mathcal{A}$. If $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$ and*

$$\|\alpha f h + \beta h + \gamma\| = \|\alpha g h + \beta h + \gamma\| \tag{3.3}$$

for all $\mathbb{C}$-peaking functions $h \in \mathcal{A}$, then $f = g$.

Proof. Since $\gamma \neq 0$, (3.3) gives

$$\begin{aligned} |\gamma| \left\| \frac{\alpha}{\gamma} f h + \frac{\beta}{\gamma} h + 1 \right\| &= \|\alpha f h + \beta h + \gamma\| = \|\alpha g h + \beta h + \gamma\| \\ &= |\gamma| \left\| \frac{\alpha}{\gamma} g h + h \frac{\beta}{\gamma} + 1 \right\| \end{aligned}$$

for all $\mathbb{C}$-peaking functions $h \in \mathcal{A}$. Therefore

$$\left\| \left(\frac{\alpha}{\gamma} f + \frac{\beta}{\gamma} \right) h + 1 \right\| = \left\| \left(\frac{\alpha}{\gamma} g + \frac{\beta}{\gamma} \right) h + 1 \right\|,$$

for all $\mathbb{C}$-peaking functions $h \in \mathcal{A}$, which, by Lemma 3.1, implies that $\frac{\alpha}{\gamma} f + \frac{\beta}{\gamma} = \frac{\alpha}{\gamma} g + \frac{\beta}{\gamma}$, i.e. $f = g$. ∎

It is clear that $p(z, w) = zw$ is not sufficient to ensure identification via the spectral radius in any unital Banach algebra, since $\rho(h) = \rho(-h)$ for all $h \in \mathcal{A}$, though $1 \neq -1$. Nonetheless, in Lemma 1.1 it was noted that if $\sigma_\pi(fh) = \sigma_\pi(gh)$ for all $h \in \mathcal{A}$, then $f = g$. Though it is perhaps surprising that multiplicatively preserving the peripheral spectrum is enough to ensure the equality of f and g, it turns out that this is not entirely necessary. In fact, it is only necessary that the peripheral spectra of all products intersect each other, and a small class of functions $h \in \mathcal{A}$ is enough. The following is a generalization of results given in [14] and [10]:

Theorem 3.8. *Let $\mathcal{A} \subset C(K)$ and $f, g \in \mathcal{A}$. If*

$$\sigma_\pi(fh) \cap \sigma_\pi(gh) \neq \emptyset \tag{3.4}$$

for all peaking functions $h \in \exp(\mathcal{A})$, then $f = g$.

Proof. By Bishop's Lemma 3.2, if $x_0 \in \delta\mathcal{A}$, $f(x_0) \neq 0$, and $g(x_0) \neq 0$, then there exist peaking functions $h_1, h_2 \in \exp(\mathcal{A})$ such that $h_1(x_0) = h_2(x_0) = 1$, $\sigma_\pi(fh_1) = \{f(x_0)\}$ and $\sigma_\pi(gh_2) = \{g(x_0)\}$. Setting $h = h_1 h_2$ then gives $h \in \exp(\mathcal{A})$, $h(x_0) = 1$, $\sigma_\pi(fh) = \{f(x_0)\}$ and $\sigma_\pi(gh) = \{g(x_0)\}$, so (3.4) implies $f(x_0) = g(x_0)$.

If $f(x_0) = 0$, then Lemma 3.2 implies that there exists $h \in \exp(\mathcal{A})$ such that $h(x_0) = 1$ and $\|fh\| < \varepsilon$. Since $\sigma_\pi(fg) \cap \sigma_\pi(gh) \neq \emptyset$ implies $\|fh\| = \|gh\|$, it must be that $\|gh\| < \varepsilon$. In particular, $h(x_0) = 1$ implies $|g(x_0)| < \varepsilon$, which, by the liberty of the choice of ε gives that $g(x_0) = 0$. □

The above result shows that the zw term alone carries a great deal of information for second-degree identification. This is not surprising, as Example 4 demonstrated that the zw term is, in fact, essential in any second-degree classification.

3.3. **Further Identifications in Uniform Algebras.** As has been hinted at now several times, polynomial terms divisible by zw are essential to higher-order polynomial identification. In fact, given that powers of peaking functions are peaking functions, it should not be surprising that combinations of z with powers of w also identify. The results that follow are extensions of results in [14], and the proofs are similar.

Lemma 3.9. *Let $n \in \mathbb{N}$ and $f, g \in \mathcal{A}$. If $\|fh^n\| \leq \|gh^n\|$ for all peaking functions $h \in \mathcal{A} \subset C(K)$, then $|f(x)| \leq |g(x)|$ on $\delta\mathcal{A}$.*

Proof. Let $f, g \in \mathcal{A}$, and fix $n \in \mathbb{N}$. Suppose that $|f(x_0)| > |g(x_0)|$ for some $x_0 \in \delta\mathcal{A}$; let $\gamma > 0$ be such that $|f(x_0)| > \gamma > |g(x_0)|$; and let V be an open neighborhood of x_0 such that $|g(x)| < \gamma$ on V. Let $h \in \mathcal{A}$ be any peaking function such that $h(x_0) = 1$ and $M(h) \subset V$, then there exists a power $m \in \mathbb{N}$ of h such that $|g(x)h^m(x)| < \gamma$ for all $x \in K \setminus V$. Now $k := h^m$ is a peaking function as is k^n, which satisfies $|g(x)k^n(x)| < \gamma$ for all $x \in K \setminus V$. This inequality also holds on V, so

$$\|gk^n\| < \gamma < |f(x_0)k^n(x_0)| \leq \|fk^n\|. \tag{3.5}$$

Therefore $\|fh^n\| \leq \|gh^n\|$ for all peaking functions $h \in \mathcal{A}$ implies $|f(x)| \leq |g(x)|$ on $\delta\mathcal{A}$. □

This intermediate lemma gives the following corollary:

Corollary 3.10. *Let $n \in \mathbb{N}$, $f, g \in \mathcal{A} \subset C(K)$. If $\|fh^n\| = \|gh^n\|$ for all peaking functions $h \in \mathcal{A}$ then $|f(x)| = |g(x)|$ for all $x \in \delta\mathcal{A}$.*

The identification result follows immediately from the corollary.

Theorem 3.11. *Let $n \in \mathbb{N}$. If $f, g \in \mathcal{A}$ satisfy*

$$\sigma_\pi(fh^n) \cap \sigma_\pi(gh^n) \neq \emptyset \tag{3.6}$$

for all peaking functions $h \in \mathcal{A}$, then $f = g$.

Proof. The hypothesis immediately implies that $\|fh^n\| = \|gh^n\|$ for all peaking functions $h \in \mathcal{A}$, and it follows from Corollary 3.10 that $|f(x)| = |g(x)|$ for all $x \in \delta\mathcal{A}$. Let $x_0 \in \delta\mathcal{A}$. If $f(x_0) = 0$, then $|g(x_0)| = |f(x_0)| = 0$, hence $g(x_0) = 0$.

If $|f(x_0)| \neq 0$, by Bishop's Lemma 3.2, there exists a peaking function $h_1 \in \exp(\mathcal{A})$ such that $\sigma_\pi(fh_1) = \{f(x_0)\}$ and a peaking function $h_2 \in \exp(\mathcal{A})$ such that $\sigma_\pi(gh_2) = \{g(x_0)\}$, so set $h = h_1h_2$. Then $h \in \exp(\mathcal{A})$ is a peaking function with $h(x_0) = 1$, $\sigma_\pi(fh) = \{f(x_0)\}$ and $\sigma_\pi(gh) = \{g(x_0)\}$. Fix $n \in \mathbb{N}$, then $h^n \in \exp(\mathcal{A})$ is also a peaking function with $h^n(x_0) = 1$, $\sigma_\pi(fh^n) = \{f(x_0)\}$ and $\sigma_\pi(gh^n) = \{g(x_0)\}$, so (3.6) implies that $f(x_0) = g(x_0)$. □

4. Identification in Algebras of Bounded Linear Operators on Banach Spaces

We now transition to analyzing polynomial identification in algebras of bounded linear operators on Banach spaces. Algebras of operators differ from uniform algebras primarily in two regards. Firstly, such algebras are, in general, non-commutative. They are also not semi-simple. This means that many elements in an algebra of operators may have spectral radius 0, a phenomenon that does not occur in uniform algebras.

Throughout this section, X is a Banach space and $B(X)$ is the Banach algebra of bounded linear operators from X to itself. We denote by $B_1(X)$ the operators of rank one. It is well known that rank-one operators are simple tensors, which is to say that for any $T \in B_1(X)$, there exist $x \in X$ and $f \in X^*$ such that $T = x \otimes f$, i.e. $Ty = (x \otimes f)(y) = f(y)x$ for any $y \in X$. A *standard operator algebra* $\mathcal{A}$ is a subalgebra of $B(X)$ such that $B_1(X) \subset \mathcal{A}$. It is not assumed that standard operator algebras are complete nor that they are unital.

Note that, though a standard operator algebra $\mathcal{A} \subset B(X)$ need not be unital, we can nonetheless talk about the spectrum of $A \in \mathcal{A}$ in the context of $B(X)$. Thus we set $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda\mathcal{I} \notin B(X)^{-1}\}$, where $\mathcal{I}$ is the identity operator. There are several subsets of the spectrum of an operator that are of interest, and one that will be important for the proofs that follow is the *point spectrum*, which is the set

$$\sigma_p(A) = \{\lambda \in \sigma(A) :\ A - \lambda\mathcal{I} \text{ is not injective}\}.$$

If U is the open unit ball in X, then $T \in B(X)$ is *compact* if $T(U)$ is a relatively compact set in X. In particular, finite-rank operators are compact. It is well-known that if T is compact and $\lambda \in \sigma(T) \setminus \{0\}$, then $\lambda \in \sigma_p(T)$ (see [18, Theorem 4.25(b)]), so that $\sigma_p(T) \setminus \{0\} = \sigma(T) \setminus \{0\}$, which implies $\sigma_\pi(T) \subset \sigma_p(T)$ when $\sigma_p(T)$ is not empty. Specifically, $\sigma_\pi(T) \subset \sigma_p(T)$ for any rank-one operator T when the dimension of X is greater than 1.

Despite the fact that algebras of operators are quite different from uniform algebras, particularly in that they are not commutative, many of the identification results proven in uniform algebras have analogues in algebras of operators. Before proving identification results, we characterize the peripheral spectra of certain operators.

Lemma 4.1. *If $A \in B(X)$, $T = x \otimes f \in B_1(X)$, and $\alpha \in \mathbb{C} \setminus \{0\}$, then each of the following holds:*

(1) *If $dim(X) = 1$, then $\sigma(\alpha T) = \sigma_p(\alpha T) = \{\alpha f(x)\}$. If $dim(X) > 1$, then $\sigma(\alpha T) = \sigma_p(\alpha T) = \{\alpha f(x), 0\}$. In either case, $\sigma_\pi(\alpha T) = \{\alpha f(x)\}$.*

(2) *If $dim(X) = 1$, then $\sigma(\alpha AT) = \sigma_p(\alpha TA) = \{\alpha f(Ax)\}$. If $dim(X) > 1$, then $\sigma(\alpha AT) = \sigma_p(\alpha TA) = \{\alpha f(Ax), 0\}$. In either case, $\sigma_\pi(\alpha AT) = \sigma_\pi(\alpha TA) = \{\alpha f(Ax)\}$.*

(3) *If $dim(X) = 1$, then $\sigma(AT + \alpha\mathcal{I}) = \sigma(TA + \alpha\mathcal{I}) = \{f(Ax) + \alpha\}$. If $dim(X) > 1$, then $\sigma(AT + \alpha\mathcal{I}) = \sigma(TA + \alpha\mathcal{I}) = \{f(Ax) + \alpha, \alpha\}$. In particular, $\sigma_p(AT + \alpha\mathcal{I}) = \sigma(TA + \alpha\mathcal{I}) = \sigma(AT + \alpha\mathcal{I})$.*

Proof. Choose $x \in X \setminus \{0\}$, $f \in X^* \setminus \{0\}$, and $\alpha \in \mathbb{C} \setminus \{0\}$. Since $T = x \otimes f$ is rank-one, it is invertible if and only if the dimension of X is 1, which is to say that $0 \notin \sigma(\alpha T)$ if and only if $\dim(X) = 1$. We also have that $(\alpha T - \alpha f(x)\mathcal{I})x = \alpha(f(x)x - f(x)x) = 0$, which implies that $\alpha T - \alpha f(x)\mathcal{I}$ is not injective, i.e. $\alpha f(x) \in \sigma_p(\alpha T)$.

If $\lambda \in \sigma_p(\alpha T) \setminus \{0\}$, then there exist distinct $y_1, y_2 \in X$ such that

$$(\alpha T - \lambda\mathcal{I})y_1 = (\alpha T - \lambda\mathcal{I})y_2,$$

which is to say that $\alpha f(y_1 - y_2)x = \lambda(y_1 - y_2)$. Thus $kx = y_1 - y_2$ for some $k \in \mathbb{C}$, i.e. $\alpha f(kx)x = \lambda kx$. Since f is linear, this implies that $\lambda = \alpha f(x)$, which – along with the fact that T is compact – further gives

$$\{\alpha f(x)\} = \sigma_p(\alpha T) \setminus \{0\} = \sigma(\alpha T) \setminus \{0\}.$$

Therefore $\sigma(\alpha T) = \sigma_p(\alpha T) = \{\alpha f(x), 0\}$ when $\dim(X) > 1$ and $\sigma(\alpha T) = \sigma_p(\alpha T) = \{\alpha f(x)\}$ when $\dim(X) = 1$. In either case, $\sigma_\pi(\alpha T) = \{\alpha f(x)\}$, proving part (1).

Part (2) follows immediately, noting that $AT = (Ax) \otimes f$ and $TA = x \otimes (f \circ A)$.

Suppose that $\lambda \in \sigma(AT + \alpha\mathcal{I})$. Then $AT - (\lambda - \alpha)\mathcal{I}$ is not invertible, i.e. $\lambda - \alpha \in \sigma(AT) = \sigma_p(AT)$. By Part (2), $\lambda - \alpha \in \{f(Ax), 0\}$, so $\lambda = f(Ax) + \alpha$ or $\lambda = \alpha$. In either case, $\lambda \in \sigma_p(AT + \alpha\mathcal{I})$, proving that $\sigma(AT + \alpha\mathcal{I}) \subset \sigma_p(AT + \alpha\mathcal{I})$. Moreover, $\lambda = \alpha$ occurs when $\dim(X) = 1$ only if $f(Ax) = \alpha$. A similar argument applies to $\sigma(TA + \alpha\mathcal{I})$. □

4.1. **Degree-One Polynomials.** The result of the degree-one polynomial case for unital standard operator algebras is similar to that for uniform algebras. Since $\mathcal{A} \subset B(X)$ is not a semi-simple algebra, the method of proof for uniform algebras does not carry.

Theorem 4.2. *Let X be a Banach space and $\mathcal{A} \subset B(X)$ a unital standard operator algebra. If $A, B \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{C}$, and*

$$\rho(\alpha A + \beta T + \gamma) = \rho(\alpha B + \beta T + \gamma) \tag{4.1}$$

for all $T \in \mathcal{A}$, then $A = B$.

Proof. Fix $A, B \in \mathcal{A}$, assume that $\alpha = \beta = 1$ and $\gamma = 0$, and suppose that (4.1) holds for all $T \in \mathcal{A}$ and that $A \neq B$. Set $S = T + A$, and note that S ranges over all operators in $\mathcal{A}$ as T does. Thus (4.1) can be rewritten

$$\rho(S) = \rho((B-A)+S) \tag{4.2}$$

for all $S \in \mathcal{A}$, which immediately implies that $\rho(A) = \rho(B)$ (choosing $S = A$) and $\rho(B-A) = 0$ (choosing $S = 0$).

Since we have assumed $A \neq B$, there exists $x_0 \in X$ such that $(B-A)x_0 \neq 0$. Moreover, since $\rho(B-A) = 0$, $(B-A)x_0$ is not a multiple of x_0 (in particular, x_0 is not an eigenvector of $B-A$). Therefore x_0 and $x_0 - (B-A)x_0$ are linearly independent, which – by the Hahn-Banach Theorem – means there exists $f \in X^*$ such that $f(x_0) = 1$ and $f(x_0 - (B-A)x_0) = 0$. Now set $S = (x_0 - (B-A)x_0) \otimes f$. By Lemma 4.1,

$$\rho(S) = \rho\left((x_0 - (B-A)x_0) \otimes f\right) = |f(x_0 - (B-A)x_0)| = 0,$$

but

$$\begin{aligned}[(B-A)+S]\,x_0 &= (B-A)x_0 + Sx_0 = (B-A)x_0 + f(x_0)\,[x_0 - (B-A)x_0] \\ &= (B-A)x_0 + x_0 - (B-A)x_0 = x_0,\end{aligned}$$

which shows that $\rho((B-A)+S) \geq 1$. Thus $\rho(S) < \rho((B-A)+S)$, contrary to hypothesis. This contradiction yields that $A = B$.

Lastly we remove the assumptions on α, β, and γ. Note that $\rho(\alpha A + \beta T + \gamma) = \rho(\alpha B + \beta T + \gamma)$ for all $T \in \mathcal{A}$ if and only if $\rho\left(A + \frac{\beta}{\alpha}T + \frac{\gamma}{\alpha}\right) = \rho\left(B + \frac{\beta}{\alpha}T + \frac{\gamma}{\alpha}\right)$ for all $T \in \mathcal{A}$ if and only if $\rho(A+T) = \rho(B+T)$ for all $T \in \mathcal{A}$. Thus α and β can be chosen arbitrarily in $\mathbb{C} \setminus \{0\}$ and γ may chosen arbitrarily in $\mathbb{C}$. $\square$

Given the degree-one classification, we can now classify which spectral preservers are injective on algebras of bounded linear operators.

Corollary 4.3. *Suppose that X and Y are Banach spaces. Let $\mathcal{A} \subset B(X)$ and $\mathcal{B} \subset B(Y)$ be unital standard operator algebras. If $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{C}$, and $\varphi \colon \mathcal{A} \to \mathcal{B}$ is any map that satisfies*

$$\rho(\alpha A + \beta T + \gamma \mathcal{I}) = \rho(\alpha\varphi(A) + \beta\varphi(T) + \gamma\varphi(\mathcal{I}))$$

for all $T \in \mathcal{A}$, then φ is injective.

The proof is identical to the arguments given above for proving injectivity.

4.2. **Second-Degree Polynomials.** It was noted in Lemma 1.3 that $\sigma_\pi(AT) = \sigma_\pi(BT)$ for all $T \in \mathcal{A}$ ensures that $A = B$. We generalize that result here.

Lemma 4.4. *Let $\mathcal{A}$ be a standard operator algebra. If $A, B \in \mathcal{A}$, $\alpha \in \mathbb{C} \setminus \{0\}$, and either*

$$\sigma_\pi(\alpha AT) = \sigma_\pi(\alpha BT) \qquad \text{or} \qquad \sigma_\pi(\alpha TA) = \sigma_\pi(\alpha TB)$$

for all $T \in B_1(X)$, then $A = B$.

Proof. Suppose that $\sigma_\pi(\alpha TA) = \sigma_\pi(\alpha TB)$ for all $T \in B_1(X)$. Note that, without loss of generality, we may assume that $\alpha = 1$, since identifying αA and identifying A are equivalent. Observe that by Lemma 4.1, $\sigma_\pi(TA) = \{f(Ax)\}$ and $\sigma_\pi(TB) = \{f(Bx)\}$. Thus $\sigma_\pi(TA) = \sigma_\pi(TB)$ for all $T \in B_1(X)$ implies $f(Ax) = f(Bx)$

for all $f \in X^*$ and all $x \in X$. By the Hahn-Banach Theorem, $Ax = Bx$ for all $x \in X$, which is to say that $A = B$.

If $\sigma_\pi(\alpha AT) = \sigma_\pi(\alpha BT)$ for all $T \in B_1(X)$, then $A = B$ by a similar proof (which can be found in [19, Lemma 1]). □

Corollary 4.5. *If $\alpha \neq 0$ and $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a surjective mapping between standard operator algebras that satisfies*

$$\sigma_\pi(\alpha AT) = \sigma_\pi(\alpha\varphi(A)\varphi(T)) \quad \text{or} \quad \sigma_\pi(\alpha TA) = \sigma_\pi(\alpha\varphi(T)\varphi(A))$$

for all $A \in \mathcal{A}$ and all $T \in B_1(X)$, then φ is injective.

It has not previously been shown that $\rho(AT + \mathcal{I}) = \rho(BT + \mathcal{I})$ – which is the operator algebra analogy to Lemma 1.2 – is sufficient to guarantee identification, so we show that here.

Theorem 4.6. *Let $\mathcal{A} \subset B(X)$ be a unital standard operator algebra, $A, B \in \mathcal{A}$, and $\alpha, \beta \in \mathbb{C}\backslash\{0\}$. If*

$$\rho(\beta AT + \alpha\mathcal{I}) = \rho(\beta BT + \alpha\mathcal{I}) \quad \text{or} \quad \rho(\beta TA + \alpha\mathcal{I}) = \rho(\beta TB + \alpha\mathcal{I})$$

for all $T \in B_1(X)$, then $A = B$.

Proof. Note that βT is rank-one if and only if T is rank-one, so it is without loss of generality that we may assume $\beta = 1$. Thus we assume $\alpha \in \mathbb{C}\setminus\{0\}$ and $A \neq B$, so there exists $x_0 \in X$ such that $Ax_0 \neq Bx_0$. If $x \in X \setminus \{0\}$ and $f \in X^* \setminus \{0\}$, then, for $T = x \otimes f$, Lemma 4.1 gives

$$\begin{aligned} \sigma(AT + \alpha\mathcal{I}) &= \sigma_p(AT + \alpha\mathcal{I}) \subset \{f(Ax) + \alpha, \alpha\} \\ \sigma(BT + \alpha\mathcal{I}) &= \sigma_p(BT + \alpha\mathcal{I}) \subset \{f(Bx) + \alpha, \alpha\}. \end{aligned}$$

Since $Ax_0 \neq Bx_0$, without loss of generality we may assume that $\|Ax_0\| \geq \|Bx_0\|$. By the Hahn-Banach Theorem there exists $f \in X^*$ such that $f(Ax_0) = \alpha$, and, moreover, f may be chosen so that $f(Bx_0) \neq \alpha$ and $|f(Bx_0)| \leq |\alpha|$. Then, setting $T = x_0 \otimes f$ gives $\rho(AT + \alpha\mathcal{I}) = |f(Ax_0) + \alpha| = 2|\alpha|$ and $\rho(BT + \alpha\mathcal{I}) = \max\{|f(Bx_0) + \alpha|, |\alpha|\} < 2|\alpha|$. Thus $\rho(AT + \alpha\mathcal{I}) \neq \rho(BT + \alpha\mathcal{I})$, so $\rho(AT + \alpha\mathcal{I}) = \rho(BT + \alpha\mathcal{I})$ for all $T \in B_1(X)$ implies that $A = B$.

The proof for $\rho(\beta TA + \alpha\mathcal{I}) = \rho(\beta TB + \alpha\mathcal{I})$ follows similarly. □

As with the previous results, this leads immediately to a corollary on the injectivity of related spectral preservers.

Corollary 4.7. *If $\alpha \neq 0$ and $\varphi\colon \mathcal{A} \to \mathcal{B}$ is a surjective mapping between unital, standard operator algebras and satisfies*

$$\rho(AT + \alpha\mathcal{I}) = \rho(\varphi(A)\varphi(T) + \alpha\mathcal{I}) \quad \text{or} \quad \rho(TA + \alpha\mathcal{I}) = \rho(\varphi(T)\varphi(A) + \alpha\mathcal{I})$$

for all $A \in \mathcal{A}$ and all $T \in B_1(X)$, then φ is injective.

Thus, for unital standard operator algebras, we have a complete classification of degree-one spectral identifications, as well as the operator algebra analogues of the original degree-two results shown for uniform algebras, giving the related characterizations on the injectivity of the associated spectral preservers.

5. Conclusions

Proving that mappings between uniform algebras – or algebras of bounded linear operators on a Banach space – that preserve spectral properties are injective often first requires a technical tool to determine when two elements of a single algebra must coincide. Such results are known as spectral identification lemmas, and in this work we have collected many of the previously known polynomial spectral identification lemmas and presented a class of new results. In particular, we have given a complete classification of the degree-one polynomial identification in uniform algebras, as well as sufficient conditions for degree-one classification in algebras of operators. We have also extended the class of degree-two polynomials that have been shown to identify in both uniform algebras and standard operator algebras.

Acknowledgements: The second author would like to thank Dr. Namyong Lee for his help and direction on this project. All three authors would like to thank Kristopher Lee and Dr. Scott Lambert for helpful comments on the results and manuscript, as well as the referee for suggesting improvements to the work. The first author's research was partially supported by the Clarkson University Honors Program. The second author's research was partially supported by the Clarkson University Department of Mathematics and Minnesota State University-Mankato.

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