ON n-NORMS AND BOUNDED n-LINEAR FUNCTIONALS IN A HILBERT SPACE

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ABSTRACT. In this paper we discuss the concept of n-normed spaces. In particular, we show the equality of four different formulas of n-norms in a Hilbert space. In addition, we study the notion of bounded n-linear functionals on an n-normed space and present some results on it.

1. INTRODUCTION AND PRELIMINARIES

Let X be a real vector space with $\dim(X) \geq 2$. A real-valued function $\|\cdot, \cdot\|$: $X \times X \longrightarrow \mathbb{R}$ is called a *2-norm* on X if the following conditions hold:

A1. $\|x, y\| = 0$ if and only if x, y are linearly dependent;
A2. $\|x, y\| = \|y, x\|$ for all $x, y \in X$;
A3. $\|\alpha x, y\| = |\alpha| \, \|x, y\|$ for all $\alpha \in \mathbb{R}, \ x, y \in X$;
A4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

The pair $(X, \|\cdot, \cdot\|)$ is then called a *2-normed space*.

The notion of 2-normed spaces has been extended to that of n-normed spaces. So, from now on, let n be a nonnegative integer and X be a real vector space of dimension $d \geq n$. A real-valued function $\|\cdot, \ldots, \cdot\|$ on $X^n = X \times \cdots \times X$ (n factors) satisfying the following four properties:

B1. $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
B2. $\|x_1, \ldots, x_n\|$ is invariant under permutation for all $x_1, \ldots, x_n \in X$;
B3. $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in \mathbb{R}, \ x_1, \ldots, x_n \in X$;
B4. $\|x_0 + x_1, x_2, \ldots, x_n\| \leq \|x_0, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x_n\|$ for all $x_0, x_1, \ldots, x_n \in X$,

is called an *n-norm* on X, and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an *n-normed space*.

If X is a normed space with dual X', then — as formulated by Gähler [3] — we may define an n-norm on X by

$$\|x_1, \ldots, x_n\|^G := \sup_{f_j \in X', \ \|f_j\| \leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix} = \sup_{f_j \in X', \ \|f_j\| \leq 1} \det\left[f_j(x_i) \right].$$

Meanwhile, if X is equipped with an inner product $\langle \cdot, \cdot \rangle$, we can define the *standard n-norm* on X by

$$\|x_1, \ldots, x_n\|^S := \sqrt{\det\left[\langle x_i, x_j \rangle \right]}.$$

The value of $\|x_1, \cdots, x_n\|^S$ is nothing but the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$.

The concept of 2-normed spaces and, more generally, that of n-normed spaces, were initially developed by Gähler [2, 3, 4, 5] in the 1960's. Since then, many researchers have developed and obtained various results, see e.g. [7, 8, 10, 11, 14]. Related works can be found, for examples, in [1, 9, 13].

Our interest here is to study various formulas of n-norms, especially in a Hilbert space. So far, we already have two formulas of n-norms, Gähler's formula and the standard one. In the next section, we show that the two formulas are identical. As it turns out, we also have two more definitions of n-norms, which have different formulas but are actually the same with the previous two. Moreover, we study the notion of bounded n-linear functionals on n-normed spaces and present some facts, including that in the standard case.

2. Main results

2.1. n-norms in a Hilbert space.
Hereafter, let X be a real Hilbert space, unless otherwise stated. By Riesz Representation Theorem, each $f \in X'$ can be identified by an element $y \in X$ such that $f(x) = \langle x, y \rangle$ for all $x \in X$. Hence Gähler's formula on X reduces to

$$\|x_1, \ldots, x_n\|^G = \sup_{y_j \in X, \ \|y_j\| \leq 1} \det\left[\langle x_i, y_j \rangle \right].$$

As shown in [11], this formula is actually identical with the standard one. Indeed, applying the generalized Cauchy-Schwarz inequality [12],

$$\|x_1, \ldots, x_n\|^G \leq \sup_{y_j \in X, \ \|y_j\| \leq 1} \|x_1, \ldots, x_n\|^S \|y_1, \ldots, y_n\|^S.$$

Now, by Hadamard's inequality [6],

$$\|y_1, \ldots, y_n\|^S \leq \|y_1\| \cdots \|y_n\|,$$

and so we get

$$\|x_1, \ldots, x_n\|^G \leq \|x_1, \ldots, x_n\|^S.$$

Conversely, assuming that $x_1, \ldots, x_n$ are linearly independent, let $x'_1, \ldots, x'_n$ be the vectors obtained from $x_1, \ldots, x_n$ through the Gram-Schmidt orthogonalization process. Then

$$\|x_1, \ldots, x_n\|^S = \|x'_1\| \cdots \|x'_n\|.$$

Now, if $y_j = \frac{1}{\|x'_j\|} x'_j$, $j = 1, \ldots, n$, then by the properties of determinants, we have

$$\begin{vmatrix} \langle x_1, y_1 \rangle & \ldots & \langle x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \ldots & \langle x_n, y_n \rangle \end{vmatrix} = \frac{1}{\|x'_1\| \cdots \|x'_n\|} \begin{vmatrix} \langle x'_1, x'_1 \rangle & \ldots & \langle x'_1, x'_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x'_n, x'_1 \rangle & \ldots & \langle x'_n, x'_n \rangle \end{vmatrix}$$
$$= \|x'_1\| \cdots \|x'_n\|.$$

Hence we obtain

$$\|x_1, \ldots, x_n\|^G \geq \|x_1, \ldots, x_n\|^S.$$

Therefore, the Gähler's formula is identical with the standard one, as claimed.

Suppose that X is separable, and $\{e_1, e_2, \ldots\}$ is a complete orthonormal set in X. Then, any member x of X may be identified by the sequence $(\langle x, e_j \rangle)$, which is in ℓ^2. As in [7], we can define an n-norm on X by

$$\|x_1, \ldots, x_n\|_2 := \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} |\det [\alpha_{i j_k}]|^2 \right]^{1/2},$$

where $\alpha_{ij} := \langle x_i, e_j \rangle$. By Parseval's identity, properties of determinants, and limiting arguments [7], this n-norm may be derived from the standard one. Therefore, we have:

Proposition 2.1. *On a separable Hilbert space, the three formulas of n-norms, namely $\|\cdot, \ldots, \cdot\|^G$, $\|\cdot, \ldots, \cdot\|^S$ and $\|\cdot, \ldots, \cdot\|_2$ coincide.*

In addition to the above formulas, we shall introduce another formula of n-norm — which we state in the following proposition.

Proposition 2.2. *The following function*

$$\|x_1, \ldots, x_n\|^D := \sup_{\substack{y_1, \ldots, y_n \in X \\ \|y_1, \ldots, y_n\|^S \leq 1}} \begin{vmatrix} \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix}$$

defines an n-norm on X.

Proof. It is obvious that, if $x_1, \ldots, x_n$ are linearly dependent vectors, then we have $\|x_1, \ldots, x_n\|^D = 0$. Conversely, if $\|x_1, \ldots, x_n\|^D = 0$, then the rows of the matrix $[\langle x_i, y_j \rangle]$ are linearly dependent for all $y_1, \ldots, y_n \in X$ with $\|y_1, \ldots, y_n\|^S \leq 1$. This happens only if $x_1, \ldots, x_n$ are linearly dependent.

Next, by the properties of determinants, we have the invariance of $\|x_1, \ldots, x_n\|^D$ under permutation.

Furthermore, we have $\|\alpha x_1, x_2, \ldots, x_n\|^D = |\alpha| \|x_1, x_2, \ldots, x_n\|^D$ for all $\alpha \in \mathbb{R}$.

Finally, for arbitrary elements $x_0, x_1, \ldots, x_n$ in X, we have

$$
\begin{vmatrix} \langle x_0 + x_1, y_1 \rangle & \cdots & \langle x_0 + x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix} = \begin{vmatrix} \langle x_0, y_1 \rangle & \cdots & \langle x_0, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix} + \begin{vmatrix} \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix}.
$$

Taking the supremums of both sides, we obtain

$$
\|x_0 + x_1, \ldots, x_n\|^D \leq \|x_0, \ldots, x_n\|^D + \|x_1, \ldots, x_n\|^D.
$$

This completes the proof. $\qquad\square$

Regarding the last formula, we have the following proposition.

Proposition 2.3. *The two formulas* $\|\cdot, \ldots, \cdot\|^G$ *and* $\|\cdot, \ldots, \cdot\|^D$ *are identical.*

Proof. Let $y_1, \ldots, y_n$ be elements in X. Since $\|y_1, \ldots, y_n\|^S \leq \|y_1\| \cdots \|y_n\|$, we have $\|y_1, \ldots, y_n\|^S \leq 1$ whenever $\|y_j\| \leq 1$ for $j = 1, \ldots, n$. It thus follows that

$$
\|x_1, \ldots, x_n\|^G \leq \|x_1, \ldots, x_n\|^D.
$$

Conversely, if $0 < \|y_1, \ldots, y_n\|^S \leq 1$, then by the generalized Cauchy-Schwarz inequality we have

$$
\begin{aligned}
\det\left[\langle x_i, y_j \rangle\right] &\leq \sqrt{\det\left[\langle x_i, x_j \rangle\right]} \sqrt{\det\left[\langle y_i, y_j \rangle\right]} \\
&= \|x_1, \ldots, x_n\|^S \|y_1, \ldots, y_n\|^S \\
&\leq \|x_1, \ldots, x_n\|^S = \|x_1, \ldots, x_n\|^G
\end{aligned}.
$$

Hence we obtain

$$
\|x_1, \ldots, x_n\|^D \leq \|x_1, \ldots, x_n\|^G.
$$

Therefore the two formulas are identical. $\qquad\square$

Corollary 2.4. *On a separable real Hilbert space, the four formulas of n-norms,* $\|\cdot, \ldots, \cdot\|^G$, $\|\cdot, \ldots, \cdot\|^S$, $\|\cdot, \ldots, \cdot\|_2$, *and* $\|\cdot, \ldots, \cdot\|^D$, *are identical.*

The last formula makes us realize that we can actually define an n-norm on the dual of an n-normed space. Recall that we can define a norm on the dual X' of a normed space $(X, \|\cdot\|)$ by the formula

$$
\|f\| := \sup_{\|x\| \leq 1} |f(x)|, \quad f \in X'.
$$

Now, if X is equipped with an n-norm $\|\cdot, \ldots, \cdot\|$ and X' denotes the dual of X, we would like to induce an n-norm on X' from the n-norm $\|\cdot, \ldots, \cdot\|$ on X. It turns out that it is possible to do so, as it is shown in the following proposition. The proof is similar to that of Proposition 2.2, and so we do not repeat it here.

Proposition 2.5. *Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space. Then, the function $\|\cdot, \ldots, \cdot\|' : (X')^n \to \mathbb{R}$ given by the formula*

$$\|f_1, \ldots, f_n\|' := \sup_{\substack{x_i \in X \\ \|x_1, \ldots, x_n\| \leq 1}} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on X'.

Remark. Through Proposition 2.5 we have shown that the dual of an n-normed space is also an n-normed space, whose n-norm is induced by the n-norm on the original space.

2.2. **Bounded n-Linear Functionals.** In this part we shall observe the notion of bounded n-linear functionals on n-normed spaces. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space. The function $F : X^n \to \mathbb{R}$ is called an *n-linear functional* on X if F is linear in each variable. The n-linear functional F is *bounded* if there exists k such that

$$|F(x_1, \ldots, x_n)| \leq k\|x_1, \ldots, x_n\|, \quad (x_1, \ldots, x_n) \in X^n.$$

If F is bounded, we define the *norm* of F by

$$\|F\| := \sup_{\|x_1, \ldots, x_n\| \neq 0} \frac{|F(x_1, \ldots, x_n)|}{\|x_1, \ldots, x_n\|},$$

or equivalently

$$\|F\| := \sup_{\|x_1, \ldots, x_n\| = 1} |F(x_1, \ldots, x_n)|.$$

The following proposition gives two alternative formulas for $\|F\|$.

Proposition 2.6. *Let F be a bounded n-linear functional on X. Then, we have*
$$\begin{aligned} \|F\| &= \inf\{k : |F(x_1, \ldots, x_n)| \leq k\|x_1, \ldots, x_n\|, \ (x_1, \ldots, x_n) \in X^n\} \\ &= \sup_{\|x_1, \ldots, x_n\| \leq 1} |F(x_1, \ldots, x_n)|. \end{aligned}$$

Proof. Let $K = \{k : |F(x_1, \ldots, x_n)| \leq k\|x_1, \ldots, x_n\|, \ (x_1, \ldots, x_n) \in X^n\}$. It is obvious that $\|F\| \in K$, and so $\inf K \leq \|F\|$. Conversely, for each $k \in K$, we have

$$\frac{|F(x_1, \ldots, x_n)|}{\|x_1, \ldots, x_n\|} \leq k$$

whenever $\|x_1, \ldots, x_n\| \neq 0$, so that $\|F\| \leq k$. But since this is true for all $k \in K$, we obtain $\|F\| \leq \inf K$. Therefore $\|F\| = \inf K$.

Next, if $\|x_1, \ldots, x_n\| \leq 1$, then

$$|F(x_1, \ldots, x_n)| \leq \|F\| \, \|x_1, \ldots, x_n\| \leq \|F\|.$$

This implies that

$$\sup_{\|x_1, \ldots, x_n\| \leq 1} |F(x_1, \ldots, x_n)| \leq \|F\|.$$

Conversely, we have

$$\|F\| = \sup_{\|x_1, \ldots, x_n\| = 1} |F(x_1, \ldots, x_n)| \leq \sup_{\|x_1, \ldots, x_n\| \leq 1} |F(x_1, \ldots, x_n)|.$$

Therefore, $\|F\| = \sup\limits_{\|x_1,\ldots,x_n\| \leq 1} |F(x_1,\ldots,x_n)|.$ $\qquad\square$

To give an example, consider the n-normed space $(\mathbb{R}^n, \|\cdot,\ldots,\cdot\|^S)$ with the standard basis $\{e_1,\ldots,e_n\}$. Define F by

$$F(x_1,\ldots,x_n) := \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{vmatrix} = \det[\alpha_{ij}]$$

where $x_i = \sum_{j=1}^n \alpha_{ij} e_j$, $i = 1,\ldots,n$. Then, one may observe that F is a bounded n-linear functional on $\mathbb{R}^n$, with $\|F\| = 1$.

Moreover, we have the following result.

Proposition 2.7. *Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, which is also equipped with the standard n-norm $\|\cdot,\ldots,\cdot\|^S$. For fixed elements $y_1,\ldots,y_n$ in X, define F on X^n by*

$$F(x_1,\ldots,x_n) = \det[\langle x_i, y_j \rangle].$$

Then F is a bounded n-linear functional on X and $\|F\| = \|y_1,\ldots,y_n\|^S$.

Proof. Based on Proposition 2.6, the norm of F is given by

$$\|F\| = \sup\limits_{\|x_1,\ldots,x_n\|^S \leq 1} \left| \det[\langle x_i, y_j \rangle] \right|.$$

The generalized Cauchy-Schwarz inequality gives us

$$\|F\| \leq \sup\limits_{\|x_1,\ldots,x_n\|^S \leq 1} \|x_1,\ldots,x_n\|^S \|y_1,\ldots,y_n\|^S \leq \|y_1,\ldots,y_n\|^S.$$

Now, if we choose

$$x_i = \frac{y_i}{\sqrt[n]{\|y_1,\ldots,y_n\|^S}},$$

then we see that

$$F(x_1,\ldots,x_n) = \|y_1,\ldots,y_n\|^S.$$

Therefore, we conclude that $\|F\| = \|y_1,\ldots,y_n\|^S$. $\qquad\square$

Note that, on the space ℓ^2 (which is equipped with the n-norm $\|\cdot,\ldots,\cdot\|_2$), the above functional F in Proposition 2.7 may be rewritten by the formula

$$F(x_1,\ldots,x_n) := \frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \det[x_{ij_k}] \det[y_{ij_k}]$$

where $y_1,\ldots,y_n$ are fixed in ℓ^2.

This formula also defines a bounded n-linear functional F on ℓ^p $(1 \leq p < \infty)$, which is equipped with the n-norm $\|\cdot,\ldots,\cdot\|_p$, where

$$\|x_1,\ldots,x_n\|_p := \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} |\det[x_{ij_k}]|^p \right]^{\frac{1}{p}}$$

(see [7]). By Hölder's inequality, we have

$$\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \det [x_{ij_k}] \det [y_{ij_k}] \leq \|x_1, \ldots, x_n\|_p \|y_1, \ldots, y_n\|_q$$

for $\frac{1}{p} + \frac{1}{q} = 1$. Hence it follows that F is bounded, with $\|F\| \leq \|y_1, \ldots, y_n\|_q$.

2.3. **Concluding Remark.** We have introduced a new formula of n-norm in a real Hilbert space which can be viewed as a modification of Gähler's formula. Accordingly, in a separable Hilbert space, we have four formulas of n-norms which are identical. In addition, we have presented some results regarding the bounded n-linear functionals on n-normed spaces, especially on Hilbert spaces (equipped with the standard n-norm).

In connection with the theory of n-normed spaces, we have introduced a formula for the induced n-norm on the dual space of an n-normed space. The formula is defined in a natural way which is similar to that in the theory of normed spaces. As a consequence of our result, we know that if X is an n-normed space, then the dual X' is also an n-normed space. It should be interesting to study the relation between an n-normed space and its dual, which is also an n-normed space. In particular, one might inquire whether the following Riesz type representation theorem holds: Let $(X, \langle \cdot, \cdot \rangle)$ be a separable real Hilbert space, which is also equipped with the standard n-norm $\|\cdot, \ldots, \cdot\|^S$. If F is a bounded n-linear functional on X, then there exists unique fixed elements $y_1, \ldots, y_n \in X$ such that $F(x_1, \ldots, x_n) = \det[\langle x_i, y_j \rangle]$ for all $x_1, \ldots, x_n \in X$. The research on this problem is still ongoing at the present time.

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